Numerical analysis of the initial conditions in fractional systems

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A B S T R A C T

Fractional dynamics is a growing topic in theoretical and experimental scientific research. A classical problem is the initialization required by fractional operators. While the problem is clear from the mathematical point of view, it constitutes a challenge in applied sciences. This paper addresses the problem of initialization and its effect upon dynamical system simulation when adopting numerical approximations. The results are compatible with system dynamics and clarify the formulation of adequate values for the initial conditions in numerical simulations.

Keywords:
Fractional Calculus
Initial conditions
Numerical simulation

1. Introduction

Fractional Calculus (FC) was started by the ideas of Leibniz in 1695 and represents the generalization of standard differential calculus up to fractional or complex orders [1–7]. The area remained untouched by applied sciences during several centuries, but in the last years considerable progress took place in the areas of physics, engineering and biology [8–21]. One of the difficulties often found by researchers consists of the initialization of fractional differential equations. In fact, while classical integer order systems require a finite set of initial conditions, fractional operators have an intrinsic memory of the phenomena that is translated into the requirement for a proper initialization and, eventually, to an infinite set of initial conditions [22–38]. The problem becomes even more intricate when we verify that there are several possible definitions for the fractional operators, that may lead to the requirement either of integer or of fractional order initial conditions. Furthermore, a pure mathematical perspective may be too far from the scenario found in the applied world, and analytical approaches may not totally describe the algorithms and restrictions implemented by numerical approximations.

This paper focus the effect of initial conditions upon the response of fractional order dynamical systems (FODS) to be simulated computationally. In this perspective mathematical discussions about integer versus fractional formulations, or physical considerations concerning possible correlations between the values adopted for the initial conditions are not addressed. FODS are simulated by means of algorithms that implement several approximations and, consequently, such issues must be decoupled in order to clarify the effect of each one separately.

Having these ideas in mind, this paper is organized as follows. Section 2 presents the fundamentals of fractional operators and their numerical approximations, formulates the procedure for studying the effect of initial conditions and analyses the results. Finally, Section 3 outlines the main conclusions.

2. Fundamental concepts

There are several definitions of fractional derivatives and integrals of order α [39]. The Riemann–Liouville, Caputo and Grünwald-Letnikov left-sided derivatives are given by:

$$ {}_a^{RL}\mathcal{D}_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \ t > a, \ \mathrm{Re}(\alpha) \in]n-1, n[\tag{1} $$

$$ {}_a^{C}\mathcal{D}_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \ t > a, \ n-1 < \alpha < n \tag{2} $$

$$ {}_a^{GL}\mathcal{D}_t^{\alpha} f(t) = \lim_{h \to 0} \left[\frac{1}{h^{\alpha}} \sum_{k=0}^{\left[\frac{t-a}{h}\right]} (-1)^k \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha-k+1)} f(t-kh) \right], \ t > a, \ \alpha > 0 \tag{3} $$

where $\Gamma(\cdot)$ is Euler's gamma function, $[x]$ means the integer part of x, and h is the step time increment.

The distinct formulations of fractional derivatives and integrals lead to several proposals for the interpretation of the generalized operator [40–54]. In spite of this diversity it is clear that FODS require an initialization procedure taking into account the history of the dynamical phenomenon that is not asked for in the classical integer order approach. This problematic has been the subject of debate during the last years, but a clear analysis of the impact of the initialization during numerical simulations is still lacking and further study is necessary.

In the numerical algorithm for calculating $\mathcal{D}_t^{\alpha}$ often is adopted the Grünwald–Letnikov formulation where the time increment h is approximated by the sampling period T_s and the series is truncated at the rth term yielding:

$$ \mathcal{D}^{\alpha}[f(t)] \approx \frac{1}{T_s^{\alpha}} \sum_{k=0}^{r} (-1)^k \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha-k+1)} f(t-kT_s) \tag{4} $$

Other techniques adopt the Laplace or the Z transforms and lead to rational fractions in the s or z domains. These approximations produce recursive expressions in discrete time, involving both input and output variables, but are not adopted in this study in order to tackle more clearly the initialization variables [55–57,54]. The objective of this paper is to investigate numerically the effect of the initial conditions upon the time responses. For that purpose three test FODS of order $0 < \alpha < 2$ are considered:

$$ \mathcal{D}^{\alpha} x + ax = u \tag{5} $$

$$ \mathcal{D}^{\alpha} x + ax + bx^3 = u \tag{6} $$

$$ \mathcal{D}^{\alpha} x + a \sin(x) = u \tag{7} $$

These expressions correspond in the mechanical domain to a fractance and a linear spring ($\mathcal{FL}$), a fractance and a nonlinear spring ($\mathcal{FN}$) and a fractional pendulum ($\mathcal{FP}$).

The FODS are excited by a unit impulse

$$ u(t) = \begin{cases} 1 & t = 0 \\ 0 & t \neq 0 \end{cases} \tag{8} $$

For analysing the effect of initial conditions is considered a statistical approach, that can be outlined as follows:

1. Definition of the parameters of the FODS and values of α
2. Repeat the following two steps for $i = 1, \ldots, N$ independent experiments
   (a) Generation of the initial conditions $x_i(t), t < 0$, in series (4), obeying a given probability distribution function (pdf)
   (b) Numerical simulation of the FODS
3. Calculation of standard deviation σ_x versus t of the dynamical responses $x_i(t), \ t \geqslant 0, \ i = 1, \ldots, N$

It should be noted that the standard deviation $\sigma_x(t)$ of the dynamic responses measures the variation between the numerical solutions, themselves, due to the different initial conditions. The index does not captures the deviation between the analytical solution and the numerical approximations. Therefore, numerical errors are not included in the comparisons.

In the sequel we adopt $N = 500, \ T_s = 0.001$ and $r = 1000$. For $0 < \alpha < 1$ a fractional integration of order α based on the Grünwald–Letnikov truncated series is adopted. On the other hand, for $1 < \alpha < 2$ a standard integration of order 1 followed by the numerical approximation of order $\alpha - 1$ is used.

We start by "stimulating" the initial conditions with a Gaussian pdf with zero average and standard deviation σ_0. This procedure does not addresses issues of physical meaning and has in mind solely producing a sample to be described statistically. By other words, considerations about the time correlation of the successive numerical values adopted for the initial

conditions are not included in the tests. This approach may be, in one hand, somewhat unrealistic and may introduce values considerably noisy, but, on the other hand, guarantees independence from additional considerations of physical or numerical nature.

Figs. 1 and 2 show the locus of standard deviation σ_x versus (α, t) of the numerical solution of $\mathcal{FL}$ (5), $a = 1$, for $\sigma_0 = 0.01$ and $\sigma_0 = 0.1$, $t < 0$, in the initialization pdf, to be denoted cases 1 and 2, respectively. Figs. 3 and 4, to be denoted cases 3 and 4, repeat the experiment with the $\mathcal{FN}$ (6), $(a, b) = (1, 1)$, and the $\mathcal{FP}$ (7), $a = 1$, respectively, for $\sigma_0 = 0.1$ and $t < 0$ in the initialization pdf. We observe that, as one should expect, the integer orders are not sensitive to the initial conditions contrary to the fractional case, and that the effect of the perturbations tend to dissipate as time evolves. Furthermore, we verify that

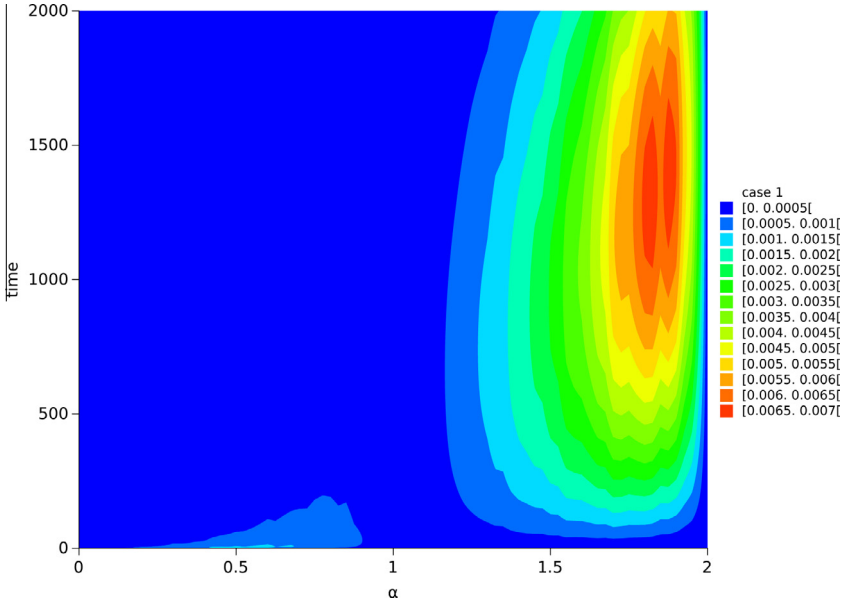

Fig. 1. Locus of σ_x versus (α, t) for the numerical solution of $\mathcal{FL}$ (5) and $\sigma_0 = 0.01$, $t < 0$, $r = 1000$ (case 1).

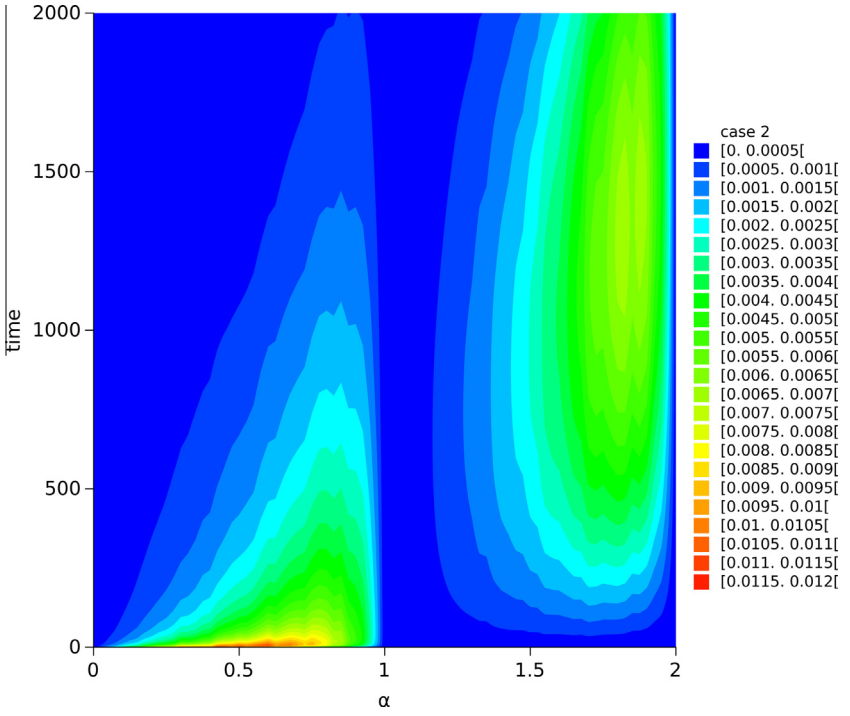

Fig. 2. Locus of σ_x versus (α, t) for the numerical solution of $\mathcal{FL}$ (5) and $\sigma_0 = 0.1$, $t < 0$, $r = 1000$ (case 2).

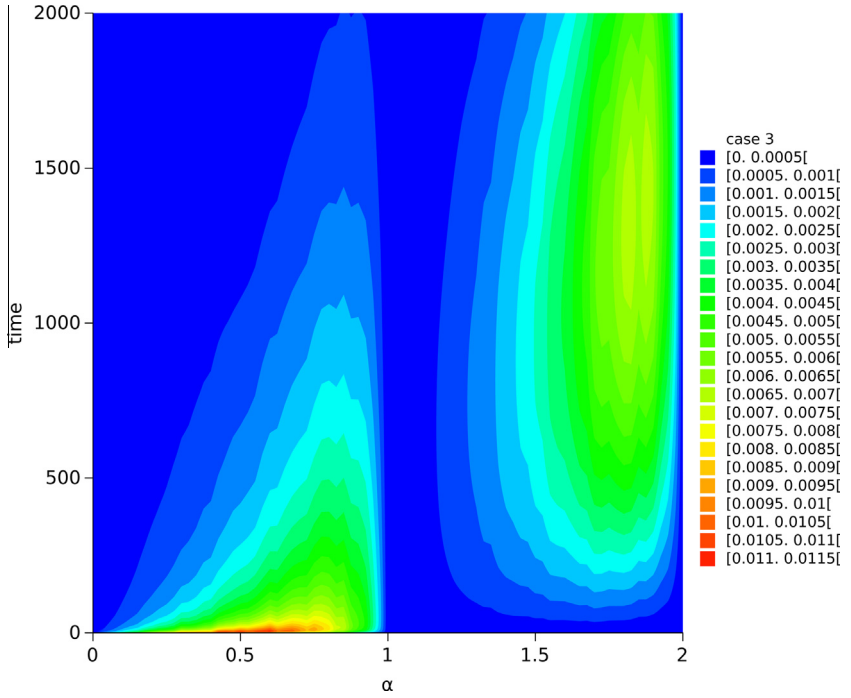

Fig. 3. Locus of σ_x versus (α, t) for the numerical solution of $\mathcal{FN}$ (6) and $\sigma_0 = 0.1,\ t < 0,\ r = 1000$ (case 3).

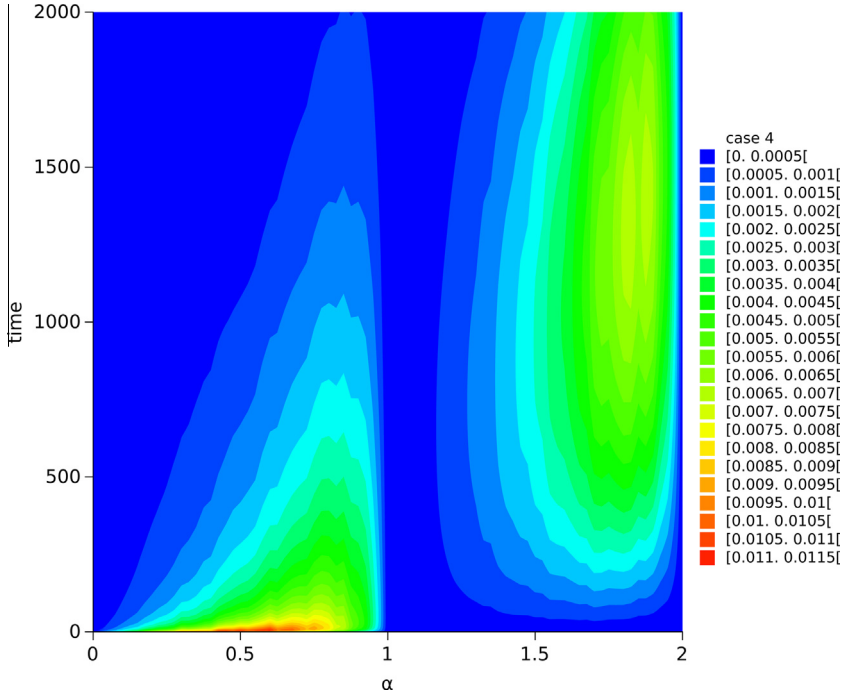

Fig. 4. Locus of σ_x versus (α, t) for the numerical solution of $\mathcal{FP}$ (7) and $\sigma_0 = 0.1,\ t < 0,\ r = 1000$ (case 4).

the time evolution of σ_x is distinct for $0 < \alpha < 1$ and $1 < \alpha < 2$. This is due to the range of α that leads to decaying response, in the first case, and oscillatory behaviour, in the second. On the other hand, there is no significant difference between the behaviour for linear and the non-linear systems in what concerns the sensitivity to the initial conditions. Experiments with other values of b in (6) lead to similar conclusions. It is interesting to note that the higher amplitudes of σ_x occur for $t \approx 0$ and $\alpha = 0.5$; however, the dissipation is slower for systems with $0.5 < \alpha < 1$ which leads to a slower decaying in σ_x.

Two other experiments were now designed for testing the effect of the initial conditions. For feeding the initial conditions in (4) is considered the Gaussian pdf, with zero average, but having a linear variation of the standard deviation $\sigma_0(t)$ between the extreme instants $t = -rT_s$ and $t = 0$ of the series. For the analysis is tested the linear system $\mathcal{FL}$ (5) and $\sigma_0(t) = 0.1\frac{t}{rT_s}$, or $\sigma_0(t) = 0.1\left(\frac{t}{rT_s} + 1\right),\ t < 0$, for increasing, or decreasing, amplitudes of the standard deviation (to be denoted as cases 5 and 6). Figs. 5 and 6 depict the corresponding results.

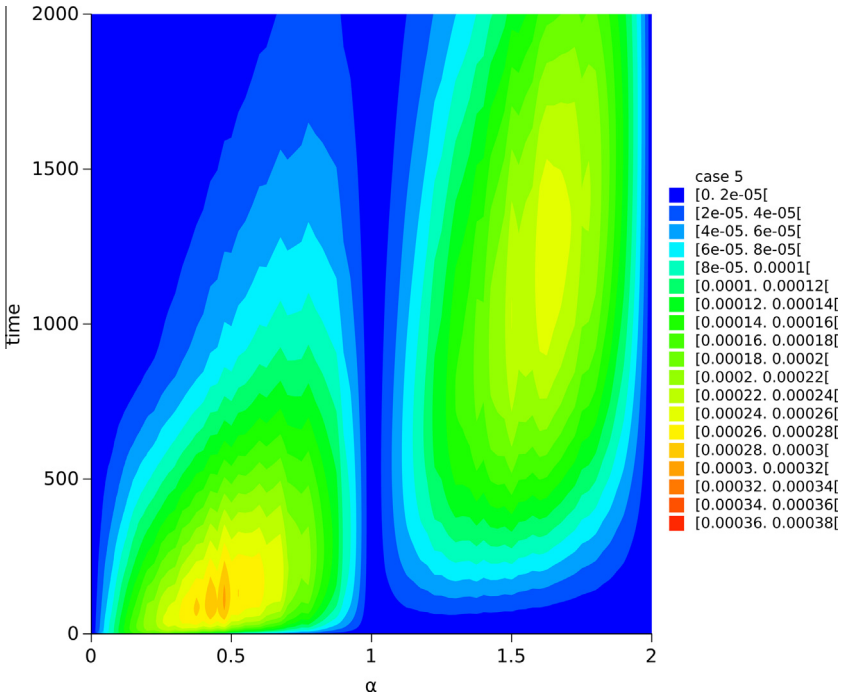

Fig. 5. Locus of σ_x versus (α, t) for the numerical solution of $\mathcal{FL}$ (5) and $\sigma_0(t) = 0.1\frac{t}{rT_s},\ t < 0,\ r = 1000$ (case 5).

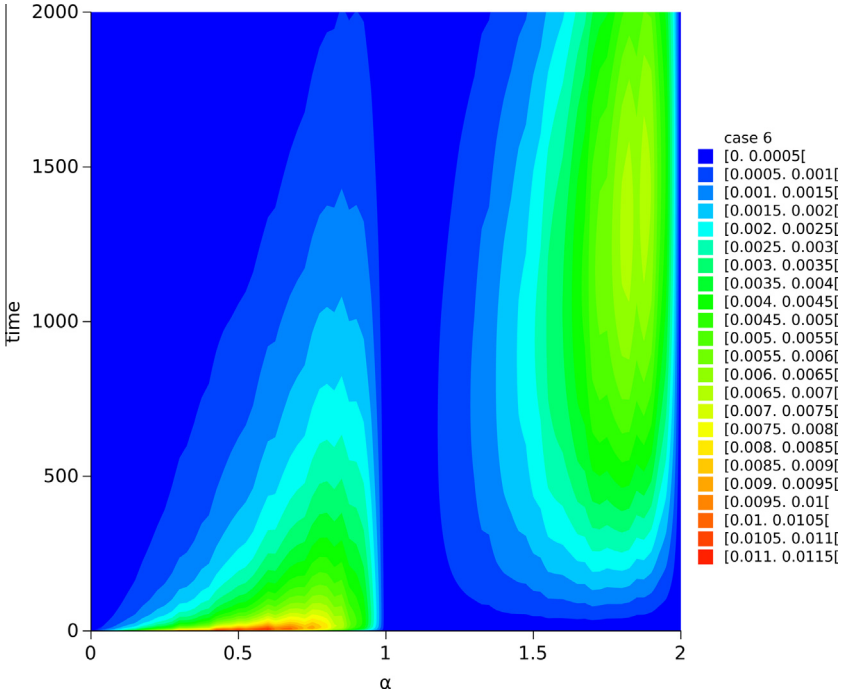

Fig. 6. Locus of σ_x versus (α, t) for the numerical solution of $\mathcal{FL}$ (5) and $\sigma_0(t) = 0.1\left(\frac{t}{rT_s} + 1\right),\ t < 0,\ r = 1000$ (case 6).

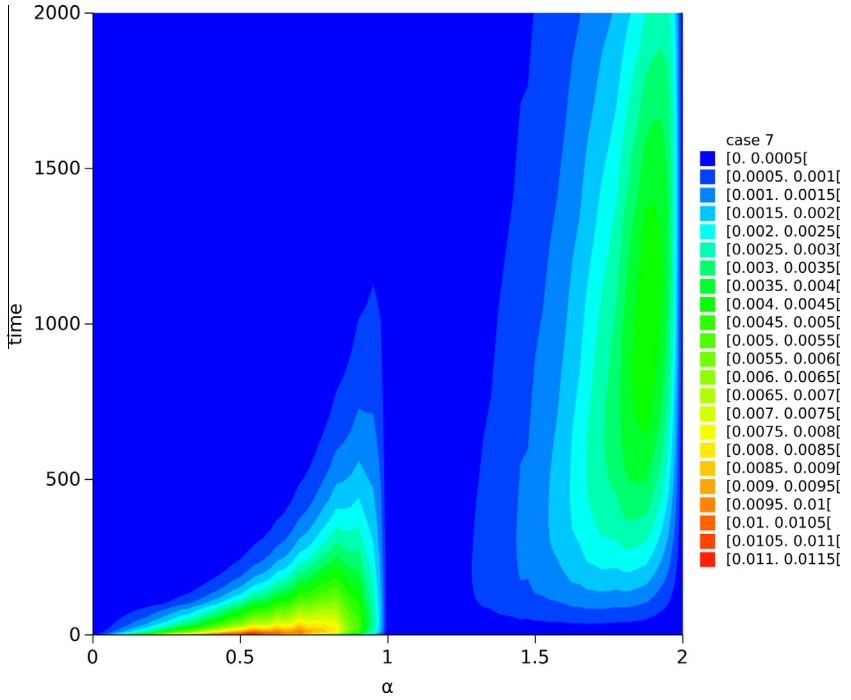

Fig. 7. Locus of σ_x versus (α, t) for the numerical solution of and the $\mathcal{FL}$ (5) and $\sigma_0 = 0.1,\ t < 0,\ r = 100$ (case 7).

We observe that case 5 produces deviations σ_x close to case 1, with fixed σ_0, while case 6 leads to deviations inferior to all previous experiments. These results demonstrate that the "recent" initial conditions (i.e., near to $t = 0$) have an higher effect upon the responses $x(t)$ than the "old" ones (i.e., near the region $t = -rT_s$).

Another aspect to be discussed consists on the effect of the truncation order r in (4) since it is the numerical representation of the memory effect. Therefore, the experiment of case 2 is repeated for a smaller truncation order, namely for the and the $\mathcal{FL}$ (5) and $r = 100$ (to be denoted as case 7), producing the locus depicted in Fig. 7. We verify that a lower value of r means having a smaller memory content and, consequently, a smaller effect over σ_x.

In conclusion, the proposed method of analysis leads to results consistent with intuitive knowledge in dynamical systems. There is no mathematical proof for all possible cases of fractional order systems when adopting experiments restricted to a finite set of numerical tests. Nevertheless, the scheme is straightforward to implement and, therefore, can by applied with different dynamical systems for evaluating the effect of approximations in the initialization.

3. Conclusions

In this paper the problem of initial conditions from the viewpoint of numerical approximations was analysed. While the majority of research as focused on the interpretation, or the analytical derivation of initial conditions for fractional differential equations, the present study embraces the computational implementation by means of truncated series. In this line of thought experiments may prove to be too stressful for dynamical system simulation, since no physical restrictions were considered for number generation, but experiments constitute critical benchmarks that encompass practical cases. The numerical tests were performed in a statistical perspective and demonstrated that the concepts usually adopted in fractional dynamics are valid and useful.

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