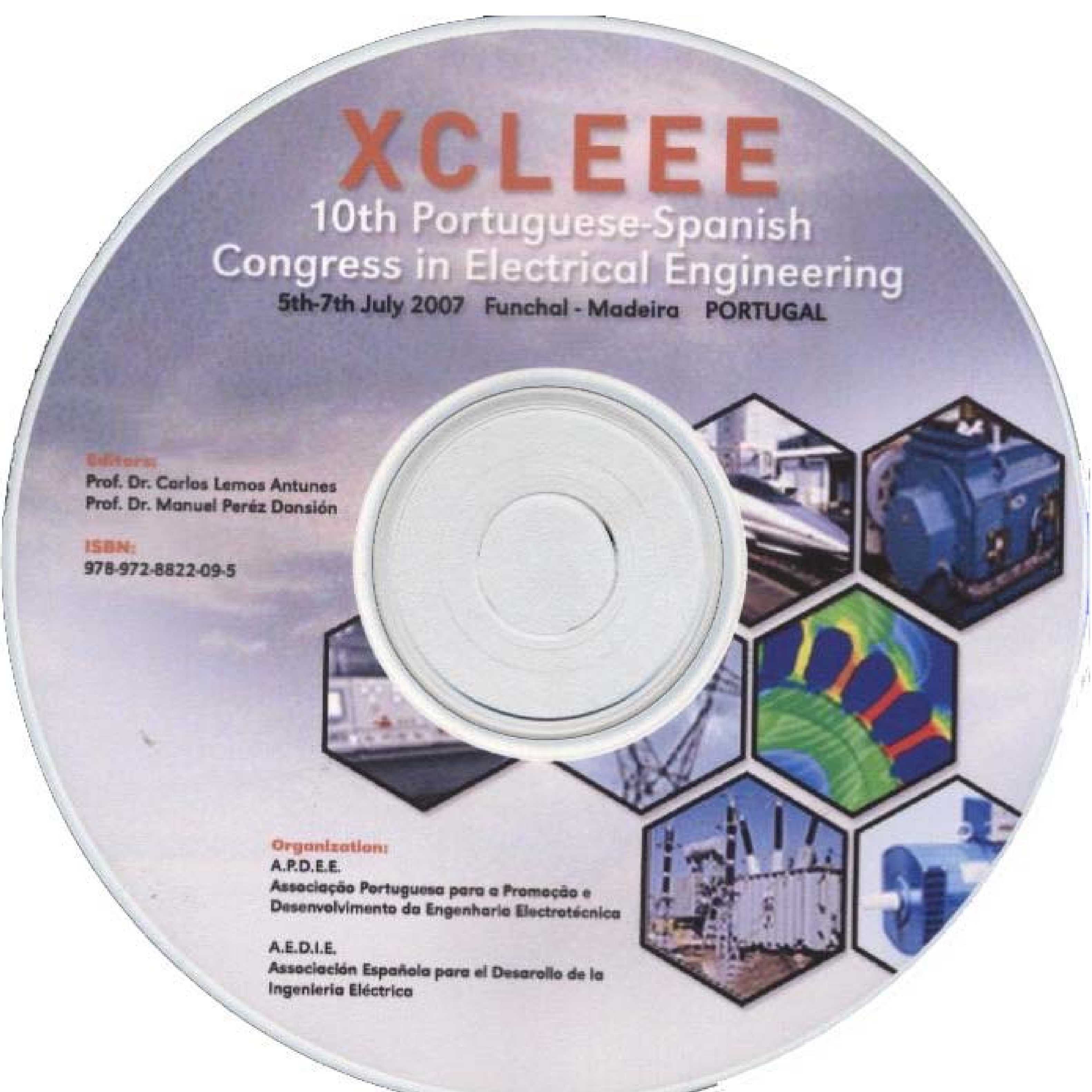

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Application of Fractional Calculus in Control and Electromagnetism

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Abstract

This article illustrates two applications of fractional calculus (FC) in engineering. It has been recognized the advantageous use of this mathematical tool in the modeling and control of many dynamical systems. In this perspective, this paper investigates the use of FC in PID tuning and electrical potential.

Keywords: Fractional Calculus, PID Tuning, Fractional Potential.

1. Introduction

In recent years fractional calculus (FC) has been a fruitful field of research in science and engineering [1-2]. In fact, many scientific areas are currently paying attention to the FC concepts and we can refer its adoption in viscoelasticity and damping, diffusion and wave propagation, electromagnetism, chaos and fractals, heat transfer, biology, electronics, signal processing, system identification, percolation, modelling and identification, chemistry, irreversibility, physics, control systems, economy and finance [1-2].

The FC deals with derivatives and integrals to an arbitrary order (real or, even, complex order). The mathematical definition of a derivative/integral of fractional order has been the subject of several different approaches [1-2]. For example, in (1) we represent the Laplace and the Grünwald-Letnikov definitions of a fractional derivative/integral of a signal $x(t)$:

$$D^{\alpha}x(t) = L^{-1}\left\{ s^{\alpha}X(s) - \sum_{k=0}^{n-1} s^{k} D^{\alpha-k-1}x(t)\Big|_{t=0} \right\}, \quad n-1 < \alpha \leq n, \quad \alpha > 0 \tag{1a}$$

$$D^{\alpha}x(t) = \lim_{h \to 0}\left[\frac{1}{h^{\alpha}} \sum_{k=0}^{\infty} (-1)^{k} \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)} x(t-kh) \right], \quad \alpha \in \Re \tag{1b}$$

where Γ is the Gamma function and h is the time increment. Expression (1) shows that fractional-order operators are “global” operators having a memory of all past events, making them adequate for modeling memory effects in most materials and systems.

Bearing these ideas in mind, sections 2 and 3 present two applications of FC concepts in PID tuning and electrical potential, respectively. Finally, in section 4 we draw the main conclusions.

2. Tuning of PID controllers based on Bode's ideal transfer function

The PID controllers are the most commonly used control algorithms in industry. Among the various existent schemes for tuning PID controllers, the Ziegler-Nichols (Z-N) method is the most popular and is still extensively used for the determination of the PID parameters. It is well known that the compensated systems, with controllers tuned by this method, have generally a step response with a high percent overshoot. Moreover, the Z-N heuristics are only suitable for plants with monotonic step response.

In this section we study a novel methodology for tuning PID controllers such that the response of the compensated system has an almost constant overshoot defined by a prescribed value. The proposed method is based on the minimization of the integral of square error (ISE) between the step responses of a unit feedback control system, whose open-loop transfer function *L*(*s*) is given by a fractional-order integrator, and that of the PID compensated system.

The open-loop transfer function *L*(*s*) is defined as ($\alpha \in \Re^+$):

$$L(s) = \left(\omega_c / s \right)^{\alpha} \tag{2}$$

where ω_c is the gain crossover frequency, that is, $|L(j\omega_c)| = 1$. The parameter α is the slope of the magnitude curve, on a log-log scale, and may assume integer as well noninteger values. In this study we consider $1 < \alpha < 2$, such that the output response may have a fractional oscillation (similar to an underdamped second-order system). This transfer function is also known as the Bode's ideal loop transfer function since Bode studies on the design of feedback amplifiers in the 1940's.

The closed-loop transfer function with Bode's ideal transfer function *L*(*s*) in (2) is:

$$G(s) = \frac{L(s)}{1 + L(s)} = \frac{1}{\left(s / \omega_c \right)^{\alpha} + 1}, \quad 1 < \alpha < 2 \tag{3}$$

The unit step response of *G*(*s*) is given by the expression:

$$y_d(t) = L^{-1}\left\{ \frac{1}{s} G(s) \right\} = L^{-1}\left\{ \frac{\omega_c^{\alpha}}{s\left(s^{\alpha} + \omega_c^{\alpha}\right)} \right\} = 1 - \sum_{n=0}^{\infty} \frac{\left[-(\omega_c t)^{\alpha} \right]^n}{\Gamma(1 + \alpha n)} = 1 - E_{\alpha}\left[-(\omega_c t)^{\alpha} \right] \tag{4}$$

where $E_\alpha(x)$ is the one-parameter Mittag-Leffler function. This function is a generalization of the common exponential function since for $\alpha = 1$ we have $E_1(x) = e^x$.

For the tuning of PID controllers we address the fractional-order transfer function (3) as the reference system [3]. With the order α and the crossover frequency ω_c we can establish the overshoot and the speed of the output response, respectively. For that purpose we consider the closed-loop system shown in Fig. 1, where $G_c(s)$ and $G_p(s)$ are the PID controller and the plant transfer functions, respectively.

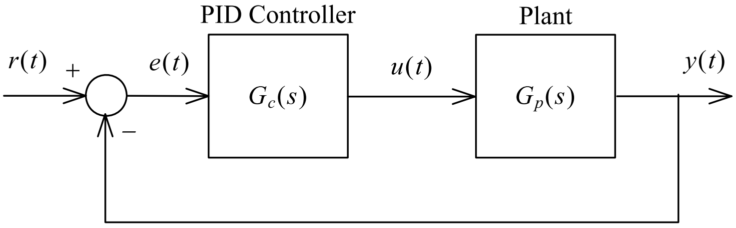

Fig. 1 – Closed-loop control system with PID controller $G_c(s)$.

The transfer function of the PID controller is:

$$G_c(s) = \frac{U(s)}{E(s)} = K\left(1 + \frac{1}{T_i s} + T_d s\right) \tag{5}$$

where $E(s)$ is the error signal and $U(s)$ is the controller's output. The parameters K, T_i, and T_d are the proportional gain, the integral time constant and the derivative time constant of the controller, respectively.

The design of the PID controller will consist on the determination of the optimum PID set gains (K, T_i, T_d) that minimize J, the integral of the square error (ISE), defined as:

$$J = \int_0^\infty \left[y(t) - y_d(t)\right]^2 dt \tag{6}$$

where $y(t)$ is the step response of the closed-loop system with the PID controller (Fig. 1) and $y_d(t)$ is the desired step response of the fractional-order transfer function given by (4).

To illustrate the effectiveness of proposed methodology we consider the following third-order plant transfer function:

$$G_p(s) = \frac{K_p}{(s+1)^3} \tag{7}$$

with nominal gain $K_p = 1$. Fig. 2 shows the step responses and the Bode diagrams of phase of the closed-loop system with the PID for the transfer function $G_p(s)$ for gain variations around the nominal gain ($K_p = 1$) corresponding to $K_p = \{0.6, 0.8, 1.0, 1.2, 1.4\}$, that is, for a variation up to ±40% of its nominal value. The system was tuned for $\alpha = 3/2$ (PM = 45°), $\omega_c = 0.8$ rad/s. We verify that we get the same desired iso-damping property corresponding to the prescribed (α, ω_c)-values. In fact, we observe that the step responses have an almost constant overshoot independently of the variation of the plant gain around the gain crossover frequency ω_c. Therefore, the proposed methodology is capable of producing closed-loop systems robust to gain variations and step responses exhibiting an iso-damping property. The proposed method was tested on several cases studies revealing good results. It was also compared with other tuning methods showing comparable or superior results [4].

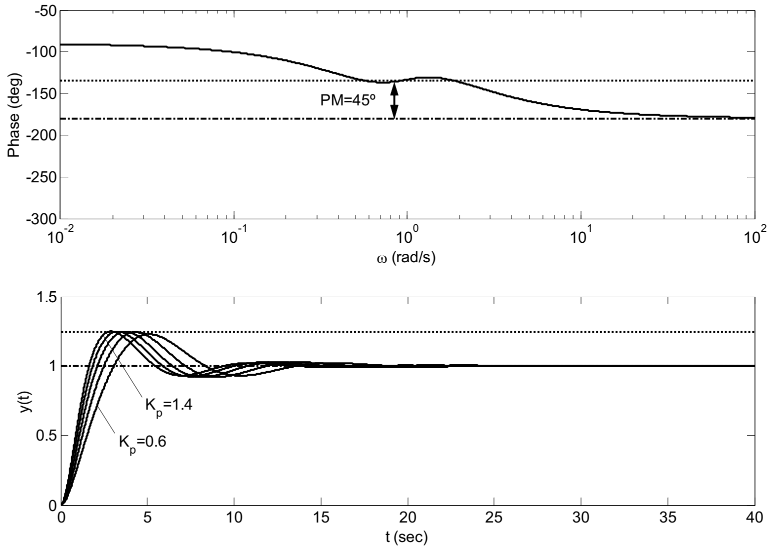

Fig. 2 – Bode phase diagrams and step responses for the closed-loop system with a PID controller for $G_p(s)$. The PID parameters are $K = 1.9158$, $T_i = 1.1407$ and $T_d = 0.9040$.

3. Implementation of the fractional potential

The classical expressions for the electrical potential φ of a single charge, a dipole, a quadrupole, an infinite filament carrying a charge λ per unit length, two opposite charged filaments, and a planar surface with charge density σ reveal the relationship $\varphi \sim r^{-3}, r^{-2}, r^{-1}, \ln r, r$ (where r is the distance to the measuring point) corresponding to an integer-order differential relationship. Such state of affairs, motivated several authors to propose its generalization to fractional multipoles that produce a potential $\varphi \sim r^{-\alpha}$, $\alpha \in \Re$. Nevertheless, besides the abstract manipulation of mathematical expressions, the truth is that there is no practical method, and physical interpretation, for establishing the fractional potential [5].

We start by re-evaluating the potential produced at point (x,y) by a straight filament with finite length l and charge q:

$$\varphi = \frac{1}{4\pi\varepsilon_0}\frac{q}{l}\ln\left\{\left[y+\frac{1}{2}l+\sqrt{x^2+\left(y+\frac{1}{2}l\right)^2}\right]\Big/\left[y-\frac{1}{2}l+\sqrt{x^2+\left(y-\frac{1}{2}l\right)^2}\right]\right\}+C,\ C\in\Re \tag{8}$$

It is well-known that for $x \to \infty$ we have $\varphi \to \left(q/4\pi\varepsilon_0\right)x^{-1} + C$ and, with $y = 0$, for $x \to 0$ we have $\varphi \to \left[q/\left(2\pi\varepsilon_0 l\right)\right]\ln\left(1/x\right) + C$. These limit cases correspond to a single charge and to an infinite filament.

We verify that expression (8) changes smoothly between the two limit cases. Therefore, we can have an intermediate fractional-order relationship as long as we restrict to a limited working range. This means that standard integer-order potential relationships have a *global* nature while fractional-order potentials have a *local* nature possible to capture only in a restricted region. This conclusion leads to an approximation scheme based on a recursive placement of integer-order functions.

In this line of thought, we developed a one-dimensional GA that places recursively n charges q_i ($i = 0, \ldots, (n-1)/2$, n – odd; $i = 1, \ldots, n/2$, n – even) at the symmetrical positions $\pm x_i$, with exception of $x_0 = 0$ that corresponds to the centre of the n-array of charges where there is a single charge q_0.

Our goal is to compare the desired reference potential $\varphi_{ref} = kx^{\alpha}$, with the approximate potential φ_{app}, resulting from a number n of charges q_i located at x_i, given by:

$$\begin{cases} \varphi_{app} = \dfrac{q_0}{|x|} + \displaystyle\sum_{i=1}^{\frac{n-1}{2}} \dfrac{q_i}{4\pi\varepsilon_0}\left(\dfrac{1}{|x-x_i|} + \dfrac{1}{|x+x_i|}\right), & n \text{ odd} \\ \varphi_{app} = \displaystyle\sum_{i=1}^{\frac{n}{2}} \dfrac{q_i}{4\pi\varepsilon_0}\left(\dfrac{1}{|x-x_i|} + \dfrac{1}{|x+x_i|}\right), & n \text{ even} \end{cases} \tag{9}$$

The experiments consist on executing the GA, for generating a combination of positions and charges that lead to an electrical potential with fractional slope similar to the desire reference potential. The values of GA parameters are: population number P = 40, crossover C(%) = 85.0%, mutation M(%) = 1.0%, elitist strategy ES(%) = 10.0% and a maximum number of generations G = 100. The optimization fitness function corresponds to the minimization of the error:

$$J = \sum_{k=1}^{m} \left(\ln\left|\varphi_{app}/\varphi_{ref}\right|\right)^2, \quad \min_i(J), \; i = 0, 1, \ldots, n-1 \tag{10}$$

where m is the number of sampling points along the x-axis.

In the present case, we consider a log-log perspective, but its modification for a lin-lin case is straightforward.

For example, Fig. 3 shows φ_{app} for an approximation with $n = 5$ charges, when $\varphi_{ref} = 1.0\ x^{-1.5}$ and $0.2 < x < 0.8$. After 32 iterations the GA leads to $q_{0A} = -0.489$ [volt], $q_{1A} = 0.920$ [volt] and $q_{2A} = -0.077$ [volt] (with scale factor $\times(4\pi\varepsilon_0)^{-1}$), at $x_{0A} = 0.0$ [m], $x_{1A} = \pm 0.147$ [m] and $x_{2A} = \pm 0.185$ [m], respectively.

The results show a good fit between the two functions. Executing the GA several times we verify that it is possible to find more than one ‘good’ solution. For a given application, a superior precision may be required and, in that case, a larger number of charges must be used. In this line of thought, we study the precision of this method for different number of charges, namely from $n = 1$ up to $n = 10$ charges.

Fig. 3 depicts the minimum, maximum and average of J *versus* n, to achieve a valid solution for a statistical sample of 10 GA executions. This chart confirms that we have a better precision the larger the number of charges. Also, the results reveal the requirement of a larger number of iterations when the number of charges increases, and consequently a larger calculation time.

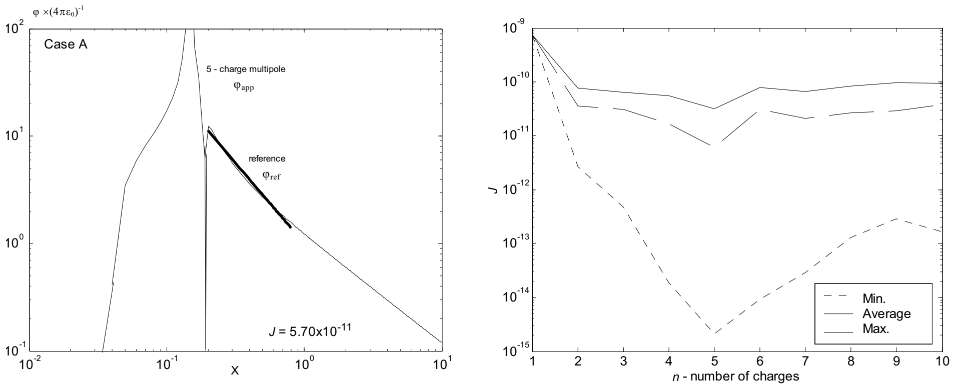

Fig.3 – Comparison of the electric potential φ_{app} and $\varphi_{ref} = 1.0 \ x^{-1.5}$ [volt] *versus x* for $0.2 < x < 0.8$ [m] and $n = 5$ (left) and approximation error *J vs.* number charges *n*, $\varphi_{ref} = 1.0 \ x^{-1.5}$ [volt], $0.2 < x < 0.8$ [m] (right).

We verify also that the position of the charges varies significantly with the number of charges used in the approximation. Therefore, pattern of the charge versus the location is not clear and its comparison with a fractal recursive layout is still under investigation.

4. Conclusions

We have presented two applications of the FC concepts. Firstly, we devise a novel approach to the design of PID controllers which gives closed-loop systems robust to gain variations. After, we analyze the electrical potential in a fractional calculus perspective. We demonstrate the advantages of using the FC theory in the area of electrical engineering.

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