Equilibria of Quantity Setting Differentiated Duopoly with Uncertainty

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Abstract: In this paper, we consider a Stackelberg duopoly competition with differentiated goods and with unknown costs. The firms' aim is to choose the output levels of their products according to the well-known concept of perfect Bayesian equilibrium. There is a firm (F_1) that chooses first the quantity q_1 of its good; the other firm (F_2) observes q_1 and then chooses the quantity q_2 of its good. We suppose that each firm has two different technologies, and uses one of them following a probability distribution. The use of either one or the other technology affects the unitary production cost. We show that there is exactly one perfect Bayesian equilibrium for this game. We analyse the advantages, for firms and for consumers, of using the technology with the highest production cost versus the one with the cheapest cost.
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1 Introduction

A game is a situation with any kind of interactions, and it has, by definition, participants who are called players. Bayesian games are games in which information about characteristics of the other players (i.e. payoffs) is incomplete. Following J. C. Harsanyi's framework, a Bayesian game can be modelled by assigning a random variable to each player which can take values for each type of player and associating probabilities or a probability density function to those types (see, for instance, [4, 5]). In a Bayesian game, the incompleteness of information means that at least one player is unsure about the type (and so the payoff function) of, at least, another player. In a non-Bayesian game, a strategy profile is a Nash equilibrium if every strategy in that profile is a best response to every other strategy in the profile, i.e. no player has anything to gain from changing his or her own strategy alone. In a Bayesian game, rational players are seeking to maximize their expected payoff, given their beliefs about the other players. A Nash equilibrium for a Bayesian dynamic game is called a perfect Bayesian equilibrium. The case that we will study belongs to this class of games, since there are market conflicts in which each firm knows its production costs, but does not know the production costs of the other firm.

Let F_1 and F_2 be two firms, each producing a differentiated product, and competing on quantities. Von Stackelberg [7] proposed a dynamic model of duopoly in which a dominant (leader) firm moves first and a subordinate (follower) firm moves second. In the case of complete information, it is well-known that the leading firm has advantages over the follower (see, for instance, [3]). The timing of the game is as follows: (i) The leading firm chooses a quantity level $q_1 \geq 0$; (ii) The follower observes

q_1, and then chooses a quantity level $q_2 \geq 0$. In § 2, we present the Stackelberg model with differentiated goods. In § 3, we study this model by considering that each firm has two different technologies, and uses one of them following a probability distribution. The use of either one or the other technology affects the unitary production cost. We suppose that firm F_1's unitary production cost is c_A with probability ϕ and c_B with probability $1-\phi$ (where $c_A > c_B$), and firm F_2's unitary production cost is c_H with probability θ and c_L with probability $1-\theta$ (where $c_H > c_L$). Both probability distributions of unitary production costs are common knowledge. In this work, we determine the quantities in the perfect Bayesian equilibrium for the above model, and we show that the follower firm may profit more than the leading (see § 3). We also analyse the variations of the prices and the profits over the parameters of the probability distributions, for some different degrees of product differentiation. In [1], it is studied the effect of the uncertain production costs on the model that considers that both firms choose their quantities simultaneously. In that case, and with homogeneous goods, if both firms have the same expected production costs, the firm with higher variance is the one that expects to have higher profits.

2 The model with complete information

We consider an economy with a monopolistic sector with two firms, F_1 and F_2. Firm F_i produces a differentiated product i at a constant marginal cost. We present the sequential-move model, with complete information, in which firms choose quantities. The timing of the game is as follows: (i) Firm F_1 (leader) chooses a quantity $q_1 \geq 0$ for its good; (ii) firm F_2 (follower) observes q_1 and then chooses a quantity $q_2 \geq 0$ for its good.

The inverse demands are

$$p_i = a - q_i - \gamma q_j,$$

provided that the quantities q_i are positive, with $i, j \in \{1,2\}$ and $i \neq j$, where $a > 0$ and $0 \leq \gamma \leq 1$. The parameter γ expresses the degree of product differentiation (see [6]), and since $\gamma \leq 1$, "cross effects" are dominated by "own effects" (see [2]). Moreover, if $\gamma = 1$, then the goods are homogeneous, and if $\gamma = 0$, then the goods are independent. Firm F_i's profit, π_i, is given by

$$\begin{aligned}\pi_i(q_i, q_j) &= q_i(p_i - c) \\ &= q_i(a - q_i - \gamma q_j - c),\end{aligned}$$

where $0 < c < a$ is the unitary production cost for both firms.

Now, we are going to compute the perfect Nash equilibrium (q_1^*, q_2^*) of this game. Using backwards-induction, we will first compute firm F_2's reaction, $q_2^*(q_1)$, to an arbitrary quantity q_1 fixed by the firm F_1. This quantity $q_2^*(q_1)$ is given by

$$q_2^*(q_1) = \arg\max_{q_2 \geq 0} \pi_2(q_1, q_2)$$
$$= \frac{a - c - \gamma q_1}{2}.$$

Firm F_1 can anticipate $q_2^*(q_1)$. Thus,

$$\pi_1(q_1, q_2^*(q_1)) = q_1(a - c - q_1 - \gamma q_2^*(q_1)) = q_1\left(a - c - q_1 - \gamma \cdot \frac{a - c - \gamma q_1}{2}\right).$$

Hence, the quantity q_1^* is given by

$$q_1^* = \arg\max_{q_1 \geq 0} q_1\left(a - c - q_1 - \gamma \cdot \frac{a - c - \gamma q_1}{2}\right)$$
$$= \frac{(2-\gamma)(a-c)}{2(2-\gamma^2)}.$$

Then,

$$q_2^*(q_1^*) = \frac{(4 - 2\gamma - \gamma^2)(a-c)}{4(2-\gamma^2)}.$$

So, the perfect Nash equilibrium (q_1^*, q_2^*) is equal to

$$\left(\frac{(2-\gamma)(a-c)}{2(2-\gamma^2)}, \frac{(4 - 2\gamma - \gamma^2)(a-c)}{4(2-\gamma^2)}\right).$$

Remark 1. Unless the goods are independent ($\gamma = 0$), the quantity of the good produced by the leading firm is higher than the quantity produced by the follower. In fact, we have that

$$q_1^* - q_2^* = \frac{\gamma^2(a-c)}{4(2-\gamma^2)} \geq 0.$$

Remark 2. Firm F_1's profit, π_1^*, at equilibrium is given by

$$\pi_1^* = \frac{(\gamma-2)^2(a-c)^2}{8(2-\gamma^2)};$$

and firm F_2's profit, π_2^*, at equilibrium is given by

$$\pi_2^* = \frac{(4 - 2\gamma - \gamma^2)^2(a-c)^2}{16(2-\gamma^2)^2}.$$

Then, we get

$$\pi_1^* - \pi_2^* = \frac{\gamma^3(4 - 3\gamma)(a-c)^2}{16(2-\gamma^2)^2} \geq 0,$$

which means that, unless the goods are independent ($\gamma = 0$), the leading firm has advantages over the follower.

3 The model with incomplete information

In this section, we consider the model presented in the previous section, but now with incomplete information. In a game of complete information the players' payoff functions are common knowledge. In a game of incomplete information, in contrast, at least one player is uncertain about, at least, another player's payoff function. These games are called Bayesian games. We suppose that each firm has two different technologies, and uses one of them following some probability distribution. The use of either one or the other technology affects the unitary production cost. The following probability distributions of unitary production costs are common knowledge among both firms:

$$C_1 = \begin{cases} c_A & \text{with probability } \phi \\ c_B & \text{with probability } 1-\phi \end{cases},$$

$$C_2 = \begin{cases} c_H & \text{with probability } \theta \\ c_L & \text{with probability } 1-\theta \end{cases}.$$

We suppose that $c_A > c_B$, $c_H > c_L$ and $c_A, c_B, c_H, c_L < a$. Firms' profits are given by

$$\pi_1(q_1(c_1), q_2(c_2)) = (a - q_1(c_1) - \gamma q_2(c_2) - c_1) q_1(c_1),$$

$$\pi_2(q_1(c_1), q_2(c_2)) = (a - q_2(c_2) - \gamma q_1(c_1) - c_2) q_2(c_2),$$

where c_i is firm F_i's unitary production cost, and the quantity $q_i(c_i)$ depends upon the unitary production cost c_i of firm F_i, for $i \in \{1,2\}$.

Firm F_1 should choose a quantity, $q_1^*(c_A)$ or $q_1^*(c_B)$, depending on its unitary production cost, to maximize its expected profit; and firm F_2 should choose a quantity, $q_2^*(c_H)$ or $q_2^*(c_L)$, depending on its unitary production cost, to maximize its expected profit.

Proposition 1. Let $E(C_2) = \theta c_H + (1-\theta) c_L$ be firm F_2's expected unitary production cost. For the Stackelberg model with uncertain costs considered above, the perfect Bayesian equilibrium is

$$\left((q_1^*(c_A), q_1^*(c_B)), (q_2^*(c_H \mid q_1^*(c_A)), q_2^*(c_H \mid q_1^*(c_B)), q_2^*(c_L \mid q_1^*(c_A)), q_2^*(c_L \mid q_1^*(c_B))) \right),$$

where

$$q_1^*(c_A) = \frac{(2-\gamma)a - 2c_A + \gamma E(C_2)}{2(2-\gamma^2)}, \tag{1}$$

$$q_1^*(c_B) = \frac{(2-\gamma)a - 2c_B + \gamma E(C_2)}{2(2-\gamma^2)}, \tag{2}$$

$$q_2^*(c_H \mid q_1^*(c_A)) = \frac{(4-2\gamma-\gamma^2)a + 2\gamma c_A - (4-2\gamma^2)c_H - \gamma^2 E(C_2)}{4(2-\gamma^2)}, \tag{3}$$

$$q_2^*(c_L \mid q_1^*(c_A)) = \frac{(4-2\gamma-\gamma^2)a + 2\gamma c_A - (4-2\gamma^2)c_L - \gamma^2 E(C_2)}{4(2-\gamma^2)}, \tag{4}$$

$$q_2^*(c_H \mid q_1^*(c_B)) = \frac{(4-2\gamma-\gamma^2)a + 2\gamma c_B - (4-2\gamma^2)c_H - \gamma^2 E(C_2)}{4(2-\gamma^2)}, \tag{5}$$

$$q_2^*(c_L \mid q_1^*(c_B)) = \frac{(4-2\gamma-\gamma^2)a + 2\gamma c_B - (4-2\gamma^2)c_L - \gamma^2 E(C_2)}{4(2-\gamma^2)}, \tag{6}$$

assuming that the parameters are such that all these quantities are non-negative.

Proof. Using backwards-induction, we will first compute firm F_2's reaction, $q_2^*(q_1)$, to an arbitrary quantity q_1 fixed by firm F_1. We point out that

$$q_2^*(c_2 \mid q_1^*) = \frac{a - c_2 - \gamma q_1^*}{2}, \tag{7}$$

with $c_2 \in \{c_H, c_L\}$, as under the complete information case.

Firm F_1 can anticipate $q_2^*(q_1)$ and then use this value to compute q_1^*. If firm F_1's unitary production cost is high, then

$$q_1^*(c_A) = \arg\max_{q_1 \geq 0}(\theta(a - q_1 - \gamma q_2^*(c_H \mid q_1(c_A)) - c_A)q_1 + (1-\theta)(a - q_1 - \gamma q_2^*(c_L \mid q_1(c_A)) - c_A)q_1);$$

and if it is low, then

$$q_1^*(c_B) = \arg\max_{q_1 \geq 0}(\theta(a - q_1 - \gamma q_2^*(c_H \mid q_1(c_B)) - c_B)q_1 + (1-\theta)(a - q_1 - \gamma q_2^*(c_L \mid q_1(c_B)) - c_B)q_1).$$

Hence,

$$q_1^*(c_A) = \frac{(2-\gamma)a - 2c_A + \gamma(\theta c_H + (1-\theta)c_L)}{2(2-\gamma^2)} \tag{8}$$

and

$$q_1^*(c_B) = \frac{(2-\gamma)a - 2c_B + \gamma(\theta c_H + (1-\theta)c_L)}{2(2-\gamma^2)}. \tag{9}$$

Replacing, in (7), q_1^* by $q_1^*(c_A)$ or $q_1^*(c_B)$, given, respectively, by (8) and (9), we get

$$q_2^*(c_2 \mid q_1^*(c_A)) = \frac{(4-2\gamma-\gamma^2)a + 2\gamma c_A - (4-2\gamma^2)c_2 - \gamma^2(\theta c_H + (1-\theta)c_L)}{4(2-\gamma^2)},$$

$$q_2^*(c_2 \mid q_1^*(c_B)) = \frac{(4-2\gamma-\gamma^2)a + 2\gamma c_B - (4-2\gamma^2)c_2 - \gamma^2(\theta c_H + (1-\theta)c_L)}{4(2-\gamma^2)},$$

with $c_2 \in \{c_H, c_L\}$.

Remark 3. Let $E(C_1) = \phi c_A + (1-\phi)c_B$ be firm F_1's expected unitary production cost, and let $E(C_2) = \theta c_H + (1-\theta)c_L$ be firm F_2's expected unitary production cost. The expected quantity, $E(q_1^*)$, produced by firm F_1 is given by

$$\begin{aligned} E(q_1^*) &= q_1^*(c_A)\phi + q_1^*(c_B)(1-\phi) \\ &= \frac{(2-\gamma)a - 2E(C_1) + \gamma E(C_2)}{2(2-\gamma^2)}, \end{aligned} \tag{10}$$

and the expected quantity, $E(q_2^*)$, produced by firm F_2 is given by

$$\begin{aligned} E(q_2^*) &= \left(q_2^*(c_H \mid q_1^*(c_A))\phi + q_2^*(c_H \mid q_1^*(c_B))(1-\phi)\right)\theta + \\ &\quad + \left(q_2^*(c_L \mid q_1^*(c_A))\phi + q_2^*(c_L \mid q_1^*(c_B))(1-\phi)\right)(1-\theta) \\ &= \frac{(4-2\gamma-\gamma^2)a + 2\gamma E(C_1) - (4-\gamma^2)E(C_2)}{4(2-\gamma^2)}. \end{aligned} \tag{11}$$

Therefore, the expected quantity produced by a firm is decreasing with its probability of using the more expensive technology and increasing with the probability that the other firm is using the more expensive technology.

Remark 4. The expected aggregate quantity, $E(Q^*)$, produced by both firms is given by

$$\begin{aligned} E(Q^*) &= E(q_1^*) + E(q_2^*) \\ &= \frac{(8-4\gamma-\gamma^2)a - 2(2-\gamma)E(C_1) - (4-2\gamma^2-\gamma^2)E(C_2)}{4(2-\gamma^2)}. \end{aligned}$$

Therefore, the aggregate quantity is increasing with the probabilities that the firms are using the less expensive technologies.

Remark 5. The expected market price, $E(p_1^*)$, of the good produced by firm F_1 is given by

$$\begin{aligned} E(p_1^*) &= a - E(q_1^*) - \gamma E(q_2^*) \\ &= \frac{(2-\gamma)a + 2E(C_1) + E(C_2)}{4}, \end{aligned}$$

and the expected market price, $E(p_2^*)$, of the good produced by firm F_2 is given by

$$\begin{aligned} E(p_2^*) &= a - \gamma E(q_1^*) - E(q_2^*) \\ &= \frac{(4-2\gamma-\gamma^2)a + 2\gamma E(C_1) + (4-3\gamma^2)E(C_2)}{4(2-\gamma^2)}. \end{aligned}$$

Therefore, the expected price of the good produced by a firm is increasing with the probabilities that the firms are using the more expensive technologies.

Remark 6. Let $V(C_i)$ be the variance of firm F_i's unitary production cost, for $i \in \{1,2\}$. Firm F_1's expected profit, $E(\pi_1^*)$, is given by

$$E(\pi_1^*) = \frac{\left((2-\gamma)a - 2E(C_1) + \gamma E(C_2)\right)^2}{8(2-\gamma^2)} + \frac{V(C_1)}{2(2-\gamma^2)},$$

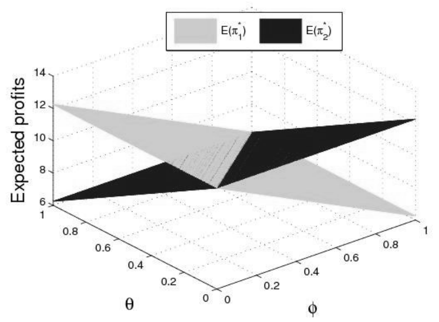
and firm F_2's expected profit, $E(\pi_2^*)$, is given by

$$E(\pi_2^*) = \frac{\left((4-\gamma-\gamma^2)a + 2\gamma E(C_1) - (4-\gamma^2)E(C_2)\right)^2}{16\left(2-\gamma^2\right)^2} + \frac{\gamma^2 V(C_1)}{4\left(2-\gamma^2\right)^2} + \frac{V(C_2)}{4}.$$

Therefore, the expected profit of the leading firm is increasing with the variance of its production costs, and the expected profit of the follower firm is increasing with both variance of its production costs and variance of leading firm's production costs.

The effect of the probabilities ϕ and θ over the firms' expected profits is shown in Figure 1 A for the case of independent goods ($\gamma = 0$), and in Figure 1 B for the case of an intermediate degree of differentiation of the goods ($\gamma = 0.8$), for some parameter region of the model. We see that the follower firm may expect to have higher profits than the leading. However, if both firms have the same expected production costs, then the leading firm has higher expected profits than the follower. We note that both firms profit more in the case of independent goods than in the case of differentiated goods.

A **B**



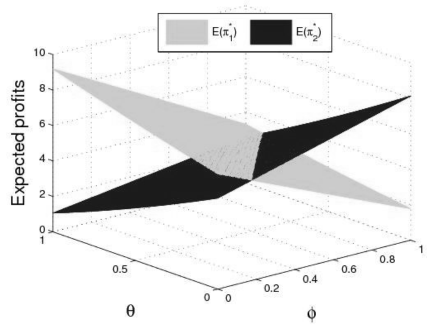


Figure 1: Firms' expected profits, $E(\pi_1^*)$ and $E(\pi_2^*)$, in the case of: (**A**) firms producing independent goods ($\gamma = 0$); and (**B**) firms producing differentiated goods with degree of differentiation $\gamma = 0.8$. Other parameters values: $a = 10$, $c_A = c_H = 5$ and $c_B = c_L = 3$.

4 Conclusion

For the Stackelberg model, we showed that in the case of uncertainty on the production costs, and in contrast to the case with complete information, the follower firm can profit more than the leading firm. We also showed that the expected profit of the leading firm increases with the variance of its production costs, and the expected profit of the follower firm increases with both variance of its production costs and variance of leading firm' production cost.

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