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Fractional Dynamics in Particle Swarm Optimization

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Abstract— **This paper studies the fractional dynamics during the evolution of a Particle Swarm Optimization (PSO). Some swarm particles of the initial population are randomly changed for stimulating the system response. After the result is compared with a reference situation. The perturbation effect in the PSO evolution is observed in the perspective of the time behavior of the fitness of the best individual position visited by the replaced particles. The dynamics is investigated through the median of a sample of experiments, while adopting the Fourier analysis for describing the phenomena. The influence of the PSO parameters upon the global dynamics is also analyzed by performing several experiments for distinct values.**

I. Introduction

Fractional Calculus (FC) is a natural extension of classical mathematics. Indeed, since the early stages of differential and integral calculus theory, several mathematicians investigated the calculation of non-integer order derivatives and integrals. Nevertheless, the application of FC has been scarce until recently, where the advances in the theory of chaos motivated a renewed interest in this field.

Evolutionary algorithms have been successfully applied to solve complex engineering problems which require to address some optimization algorithms. Together with genetic algorithms, the Particle swarm optimization (PSO) algorithm proposed recently by Kennedy and Ebelhart [1] has obtained considerable success in solving optimization problems. While the PSO algorithm and variants have been extensively studied, the influence of perturbation signals over the operations conditions of this algorithm is not yet well known.

Bearing these ideas in mind, this paper analyzes the system evolution and the fractional-order dynamics in the population of a PSO-based optimization. The article is organized as follows. Section II makes a briefly introduction to fractional calculus, PSO algorithm and the optimization function used during the evolution. Based on this formulation, section III presents the results for several simulations involving different working conditions, and studies the resultant dynamic phenomena. Finally, section IV outlines the main conclusions.

II. Fractional calculus and particle swarming optimization

This section studies the dynamical phenomena involved in the signal propagation within the PSO population. In this perspective, one particle is randomly replaced in the initial population of the PSO system and its influence upon the swarm fitness is evaluated. The experiments reveal a fractional-order dynamics capable of being described by systems theory tools.

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A. Introduction to fractional calculus

Since the foundation of the differential calculus the generalization of the concept of derivative and integral to a non-integer order α has been the subject of distinct approaches. Due to this reason there are several alternative definitions of fractional derivatives. For example, the Laplace definition of a derivative of fractional order $\alpha \in \mathbb{C}$ of the signal $x(t)$, $D^\alpha[x(t)]$, is a 'direct' generalization of the classic integer-order scheme yielding to equation (1). This means that frequency-based analysis methods have a straightforward adaptation (for zero initial conditions):

$$\mathscr{L}\{D^\alpha[x(t)]\} = s^\alpha X(s) \tag{1}$$

An alternative approach, based on the concept of fractional differential, is the Grünwald-Letnikov definition given by equation (2):

$$D^\alpha\left[x(t)\right] = \lim_{h\to 0}\left[\frac{1}{h^\alpha}\sum_{k=0}^{+\infty}\frac{(-1)^k\Gamma(\alpha+1)x(t-kh)}{\Gamma(k+1)(\alpha-k+1)}\right] \tag{2}$$

An important property revealed by equation (2) is that while an integer-order derivative implies just a finite series, the fractional-order derivative requires an infinite number of terms. This means that integer derivatives are 'local' operators in opposition with fractional derivatives which have, implicitly, a 'memory' of all past events.

The characteristics revealed by fractional-order models make this mathematical tool well suited to describe phenomena such as irreversibility and chaos because of its inherent memory property. In this line of thought, the propagation of perturbations and the appearance of long-term dynamic phenomena in a population of individuals subjected to an evolutionary process configure a case where FC tools fit adequately.

B. Particle Swarm Optimization

The particle swarm optimization algorithm was proposed originally by Kennedy and Eberhart [1]. This optimization technique is inspired in the way swarms (*e.g.*, flocks of birds, schools of fishes, herds) elements behave and move in a synchronized way, as a tactic both for searching for food and as a defensive mechanism. An analogy is established between a particle and a swarm element. The particle movement is

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characterized by two vectors representing its current position x and velocity v. Since 1995 many techniques have been proposed to refine and/or complement the original PSO algorithm, namely regarding tuning parameters [2] and by considering hybridization with other evolutionary techniques [3].

In this study is considered a standard elementary PSO (see algorithm 1). The basic algorithm begins by initializing the swarm randomly in the search space. As it can be seen in algorithm 1, the position of each particle is changed during the iterations by adding a new velocity. This velocity is evaluated by summing an increment to the previous velocity value. The increment is a function of two components representing the cognitive and the social knowledge.

The cognitive knowledge of each particle is included by evaluating the difference between the current position x and its best position so far b. The social knowledge of each particle is incorporated through the difference between its current position x and the best swarm global position achieved so far g. The cognitive and social knowledge factors are multiplied by randomly generated terms ϕ_1 and ϕ_2, respectively. The velocity of particles are restricted, in order to keep velocities from exploding, through the inertia term I [4].

Initialize Swarm;
repeat
  forall *particles* **do**
    calculate fitness f
  end
  forall *particles* **do**
    $v_{t+1} = Iv_t + \phi_1(b - x) + \phi_2(g - x)$;
    $x_{t+1} = x_t + v_{t+1}$;
  end
until *stopping criteria* ;

Algorithm 1: Particle swarm optimization

C. The Optimization System

This section presents the problem used in the study of the PSO dynamic system. The objective function consists on minimizing the Bohachevsky function (3) [5]. This function has two parameters and the optimum has the value of $f(x_1, x_2)|_{\text{opt}} = 0.0$. The variables consist in $\{x_1, x_2\} \in [-50, 50]$ and the algorithm uses a real code to represent each potencial solution.

$$f(x_1, x_2) = x_1^2 + x_2^2 - 0.3\cos(2\pi x_1) - 0.4\cos(4\pi x_2) + 0.7 \quad (3)$$

A 50–population PSO is executed during 200 generations under $\{\phi_1, \phi_2\} \sim U[0, 1.5]$.

The influence of several factors can be analyzed in order to study the PSO dynamics, particulary the inertia factor I or the ϕ_i constants, $i = \{1, 2\}$. This effect can vary according to the type of population size, fitness function, generation number used in the PSO. In this work, it is changed randomly one particle of the initial population. The influence of the inertia parameter is studied by performing tests for the values $I = \{0.4, 0.5, \ldots, 0.8\}$. The fitness evolution of the best individual particle position (b) is taken as the output signal.

III. Evolution, Signal Propagation and Fractional Dynamics

This section studies the dynamical phenomena involved in the signal propagation through the PSO population. For a statistical sample of n independent cases, a particle is randomly initialized, in every experiment, and replaces the same particle of the initial population. The experiments reveal a fractional dynamics of the perturbation propagation during the evolution capable of being described by system theory tools.

A. The PSO dynamics

The PSO algorithm is considered to perform the Bohachevsky function optimization, and it is called simply the 'system'. Initially the system is executed without any perturbation signal, during $T_m = 5000$ iterations, for a predefined inertia weight value I. The data regarding this test is stored, namely the global particle fitness and stochastic parameters. This will serve as a reference test. The *optimization* system perturbation consists in replacing an initial particle of the stored reference swarm population, in every algorithm execution, by another particle randomly generated. Indeed, this stimulus to the system results in a swarm fitness modification δf which is evaluated. This perturbation test is repeated for $n = 5000$ cases. It is important to state that the remaining test conditions, namely the stochastic reference stored values, stay unchanged along the n experiments. Therefore, the variation of the resulting individual best particle fitness perturbation during the evolution, can be viewed as the output signal that varies during the successive iterations.

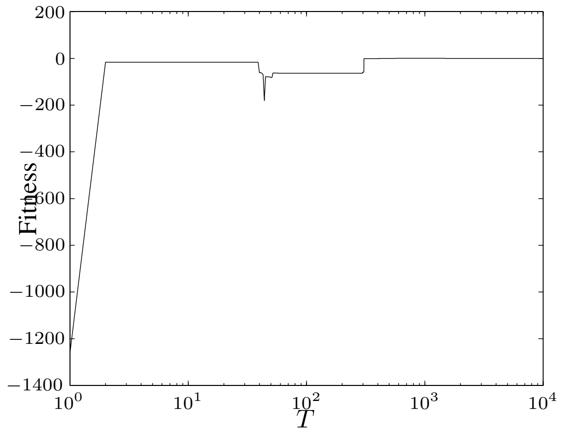
The output signal consists in the difference between the population fitness with and without the initial perturbation, that is, $\delta f(T) = f_{\text{pert}}(T) - f(T)$. Figure 1(a) shows the output signal $\delta f(T)$, for one particle replacement, in the iteration domain. To analyze the dynamics for each experiment is calculated the Fourier transform, $\mathscr{F}[\delta f(T)]$, of the signal perturbation.

Once having de Fourier description of the output signals it is possible to calculate the corresponding normalized transfer function (4):

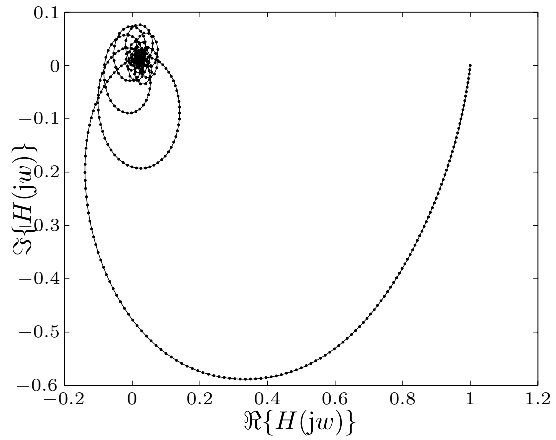
$$H(\mathrm{j}w) = \frac{\mathscr{F}\{\delta f(T)\}(\mathrm{j}w)}{\mathscr{F}\{\delta f(T)\}(w = 0)} \quad (4)$$

where w represents the frequency and T the time evolution and $\mathrm{j} = \sqrt{-1}$. The transfer function $H(\mathrm{j}w)$ for a single experiment is depicted in figure 1(b).

Finally it is obtained a 'representative' transfer function, by using the median of the statistical sample [6] of n experiments (see figure 2). Figure 3 shows the results for inertial values of $I = \{0.4, 0.5, \ldots, 0.8\}$. The medians of the transfer functions calculated previously (*i.e.*, for the real



(a) Iteration domain



(b) Polar diagram

Fig. 1. Output signal for an initial perturbation, 1 of n experiment with $I = 0.7$.

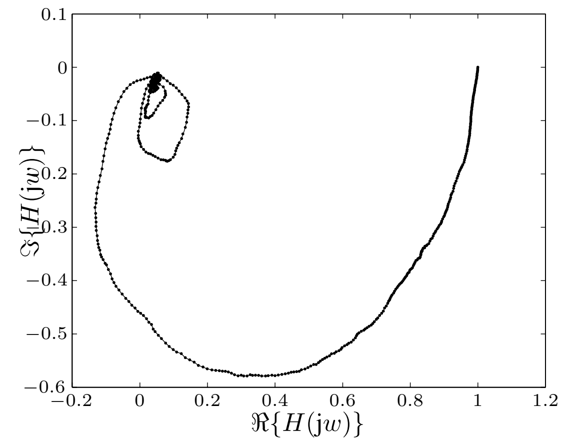


Fig. 2. Median transfer function $H(\mathrm{j}w)$ experiments an inertial term $I = 0.7$.

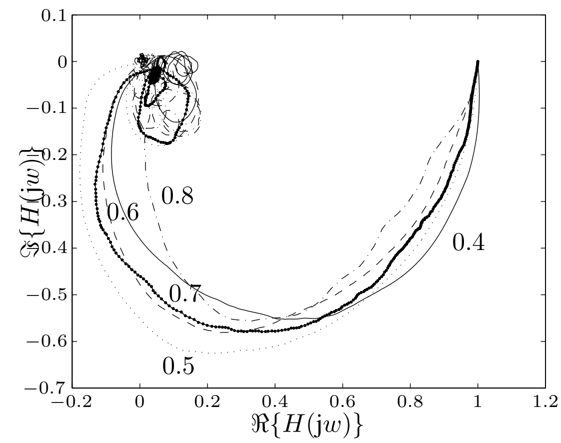


Fig. 3. Median transfer function $H(\mathrm{j}w)$, of the n experiments for $I = \{0.4, 0.5, \ldots, 0.8\}$.

and the imaginary parts for each frequency) are taken as the final result of the numerical transfer function $H(\mathrm{j}w)$.

The frequency response shows that the system converges. The polar diagrams confirm the existence of a time delay T_d, which represents the perturbation propagation in the swarm evolution. Moreover, the dynamics follows the behavior of a low-pass filter.

B. Transfer function identification

As explained previously, the *optimization* PSO has stochastic dynamics. Therefore, every time the PSO is executed with a different particle replacement, it leads to a slightly different transfer function. Consequently, in order to obtain numerical convergence [6] are performed $n = 5000$ perturbation experiments with different replacement particles (all the other particles remain unchanged). The *optimization* PSO dynamics transfer function is evaluated by computing the Fourier transform (FT) of the normalized signals (equation 4). The medians of the transfer functions calculated previously (*i.e.*, for the real and the imaginary parts for each frequency) are taken as the final result of the numerical transfer function $H(\mathrm{j}w)$.

In this section the median of the numerical transfer functions is approximated by analytical expressions with gain $\kappa \in \mathbb{R}^+$, one pole $a \in \mathbb{R}^+$ of fractional order $\alpha \in \mathbb{R}^+$, and a time delay T_d, given by equation (5):

$$G(\mathrm{j}w) = \frac{\kappa \mathrm{e}^{-T_d}}{\left(\frac{\mathrm{j}w}{a} + 1\right)^{\alpha}} \tag{5}$$

Since is used the normalized transfer function H, it yield's $\kappa = 1$. In order to estimate the transfer function parameters $\{a,\ \alpha,\ T_d\}$ another PSO algorithm is used, which is named the *identification* PSO. The *identification* PSO is executed during $T_{ide} = 200$ iterations with a 100 particle population. The PSO parameters are: $\{\phi_1, \phi_2\} \sim \mathrm{U}[0, 1.5]$, $I = 0.6$, and the transfer function parameters intervals are: $a \in [4 \times 10^{-3}, 15]$, $\alpha \in [0, 15]$ and $T_d \in [0, 50]$.

The fitness function f_{ide} measures the distance between the numerical median $H(\mathrm{j}w_k)$ and the analytical expression $G(\mathrm{j}w_k)$:

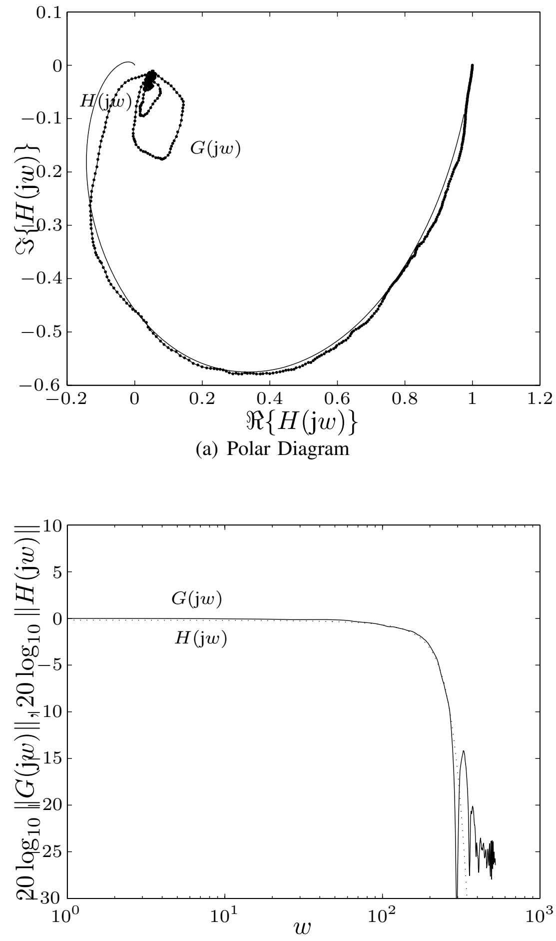
$$f_{\text{ide}} = \sum_{k=1}^{nf} \parallel H(\mathrm{j}w_k) - G(\mathrm{j}w_k) \parallel \qquad (6)$$

where nf is the total number of sampling points and w_k, $k = \{1, ..., nf\}$, is the corresponding vector of frequencies.

Figure 4 shows, superimposed, the normalized median transfer function $H(\mathrm{j}w)$ and the corresponding identified transfer function $G(\mathrm{j}w)$, both as polar (a) and amplitude (b) diagrams. As it can be observed from these figures the fractional order transfer function, identified by the PSO, captures the *optimization* PSO dynamics quite well, apart from the high frequency range.

For evaluating the influence of the inertia parameter I several simulations are performed ranging from $I = 0.4$ up to $I = 0.8$. The estimated parameters for $\{a, \alpha, T_d\}$ are depicted in figure 5.

We also conclude that, by enabling the zero/pole order to vary freely, we get non-integer values for α, while the adoption of an integer-order transfer function would lead to a larger number of zero/poles to get the same quality in the analytical fitting to the numerical values.



(a) Polar Diagram

(b) Module Logarithmic Diagram

Fig. 4. Polar and Amplitude Diagram of $H(\mathrm{j}w)$ and $G(\mathrm{j}w)$ for $I = 0.6$.

IV. Conclusions

This paper analyzed the signal propagation and the dynamic phenomena involved in the time evolution of a swarm. The study was established on the basis of a the Bohachevsky function optimization. While PSO schemes have been extensively studied, the influence of perturbation signals over the operating conditions is not well known.

Bearing these ideas in mind, the fractional calculus perspective calculus was introduced in order to develop simple, but comprehensive, approximating transfer functions of non-integer order.

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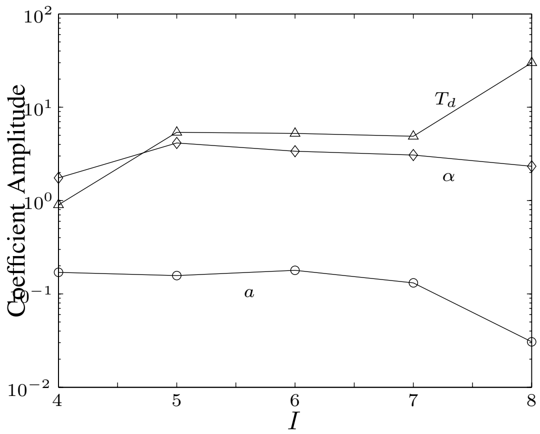

Fig. 5. Parameters $\{a, \alpha, T_d\}$ of $G(\mathrm{j}w)$, for $I = \{0.4, \ldots, 0.8\}$.

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