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Entropy Analysis of Fractional Derivatives and Their Approximation

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Abstract

Fractional derivatives (FDs) need to be calculated using series or rational fractions. The effect of such approximations are usually analysed by means of the frequency or the time responses. This paper takes advantage of a probabilistic interpretation of FDs and adopts the Shannon entropy for assessing the truncation effect produced by the approximations.

1 Introduction

Fractional calculus (FC) generalizes the standard differential calculus by considering integrals and derivatives of a non-integer order [1]– [4]. In the last decades FC emerged as a important mathematical tool for modelling and analyzing phenomena exhibiting nonlinear dynamics [5]– [12]. The practical evaluation of fractional derivatives (FDs) requires their approximation using either series or fractions and the truncation effects are usually analyzed with the frequency and the time responses. This paper takes advantage of a probability interpretation of FDs previously proposed [13]– [14] and adopts the entropy measure for analyzing the approximation errors.

Bearing these ideas in mind this paper is organized as follows. Section 2 presents the main mathematical concepts, namely the Grünwald-Letnikov definition of FDs, their interpretation in the perspective of probability theory, and the main aspects of entropy. Section 3 analyzes the approximation of FDs using entropy for capturing the effect of the truncation. Finally, section 4 outlines the main conclusions.

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2 Fundamental Concepts

The Grünwald-Letnikov definition of a FD of order $0 \leq \alpha \leq 1$ of a signal $x(t)$ is given by:

$$D^{\alpha}x(t) = \lim_{h \to 0} \frac{1}{h^{\alpha}} \sum_{k=0}^{\infty} \gamma(\alpha,k)\, x(t-kh) \tag{1}$$

$$\gamma(\alpha,k) = (-1)^k \frac{\Gamma(\alpha+1)}{k!\,\Gamma(\alpha-k+1)} \tag{2}$$

where Γ represents the gamma function and h the time increment.

It was observed [13]– [14] that

$$\gamma(\alpha,0) = 1 \tag{3}$$

$$-\sum_{k=1}^{\infty} \gamma(\alpha,k) = 1 \tag{4}$$

which lead to a probabilistic interpretation where $-\sum_{k=0}^{\infty} \gamma(\alpha,k)\, x(t-kh)$ is viewed as the calculation of the expected value of the random variable. Therefore, the past samples of $x(t)$ are weighted by mean of $-\gamma(\alpha,k)$, being the probability value higher the closer we get to the present sample.

Entropy is a well established index for characterizing probability distributions. The concept of entropy was introduced in the field of thermodynamics by Clausius and Boltzmann, and was later applied by Shannon and Jaynes in the area of information theory [15]– [20]. The so-called Shannon entropy S is defined by:

$$S = -\sum_{i=1}^{n} p_i \ln(p_i) \tag{5}$$

and represents the expected value of the information $-\ln(p_i)$. For the uniform probability distribution we have $p_i = \frac{1}{n}$ and the Shannon entropy takes its maximum value $S = \ln(n)$. Therefore, S measures the "disorder" and exhibits values closer to zero the smaller the uncertainty.

3 Entropy analysis

Grünwald-Letnikov definition (1) leads to a discrete-time calculation algorithm, based on replacing the time increment h by the sampling period T, truncating the series at the rth-term, yielding:

$$D^{\alpha}x(t) = \frac{1}{T^{\alpha}} \sum_{k=0}^{r} \gamma(\alpha,k)\, x(t-kh) \tag{6}$$

The effect of truncating the series can be measured in the viewpoint of truncating the probability distribution. Figure 1 depicts the coefficients $-\gamma(\alpha,k)$ versus k and α.

We verify that the coefficients diminish very slowly with k and that approximations may require a large number of terms. On the other hand, we observe that $-\gamma(\alpha,k)$ has smaller values near the integer orders $\alpha = 0$ and $\alpha = 1$, but that the chart is not symmetrical. The shape of the probability distribution can be quantitatively measured using entropy (6). Figure 2 shows S versus α for truncation order $r = 10^m$, $m = \{1, \cdots, 6\}$. We observe that, contrary to the standard approach that evaluates approximations in the middle range, that is, for $\alpha = 0.5$, we have worst approximations the lower the value of α. This result is due to the fact that the probability distribution has higher dispersion the closer α get to zero.

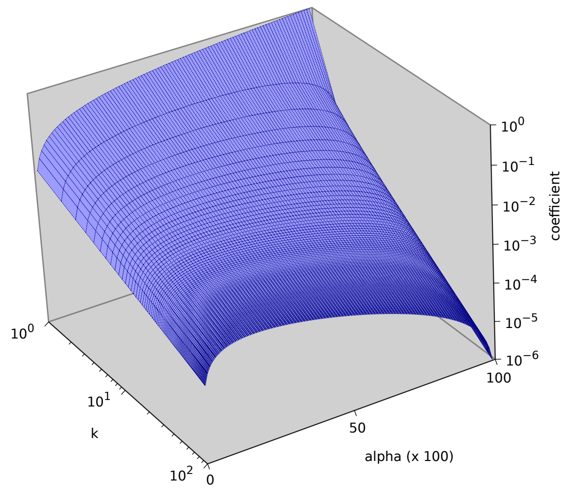

Fig. 1 Coefficients $-\gamma(\alpha,k)$ versus k and α.

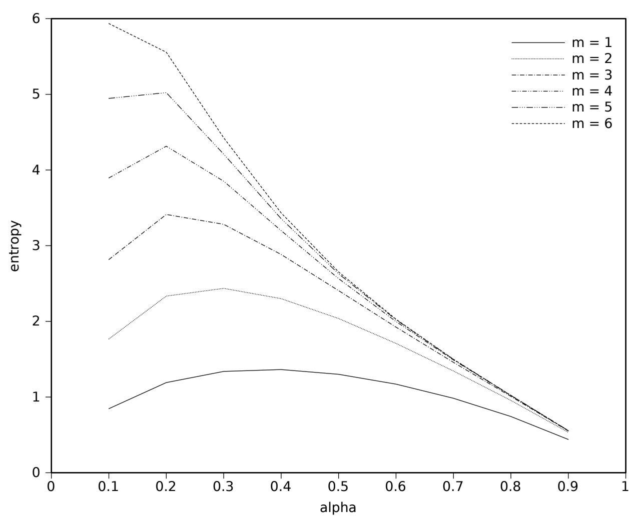

Fig. 2 Entropy S versus α for truncation orders $r = 10^m$, $m = \{1,\cdots,6\}$.

4 Conclusions

In the last years FC emerged as a useful mathematical tool for analysing phenomena in the areas of physics and engineering. The application of FDs requires approximations and their influence needs to be evaluated. This paper presented an approach based on the probability interpretation the Grünwald-Letnikov definition of FDs. The adoption of the Shannon entropy lead to an assertive measure for capturing the effect of truncation errors and revealed the influence of the fractional order upon the results.

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