Cantor exchange systems and renormalization

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Abstract

We prove a one-to-one correspondence between (i) C^{1+} conjugacy classes of C^{1+H} Cantor exchange systems that are C^{1+H} fixed points of renormalization and (ii) C^{1+} conjugacy classes of C^{1+H} diffeomorphisms f with a codimension 1 hyperbolic attractor Λ that admit an invariant measure absolutely continuous with respect to the Hausdorff measure on Λ. However, we prove that there is no $C^{1+\alpha}$ Cantor exchange system, with bounded geometry, that is a $C^{1+\alpha}$ fixed point of renormalization with regularity α greater than the Hausdorff dimension of its invariant Cantor set.

1. Introduction

The works of Masur [11], Penner and Harer [14], Thurston [27,28] and Veech [29] show a strong link between affine interval exchange maps and Anosov and pseudo-Anosov maps. We develop a smooth version of the above link proving that every C^{1+H} diffeomorphism f on a surface, with a codimension 1 hyperbolic attractor, induces a C^{1+H} Cantor exchange system Φ_f. E. Ghys and D. Sullivan (see E. Cawley [3]) observed that Anosov diffeomorphisms on the torus determine circle diffeomorphisms that have an associated renormalization operator. In the same direction, we prove that every C^{1+H} diffeomorphism f on a surface, with a codimension 1 hyperbolic attractor, determines a renormalization operator acting on the topological conjugacy class $[\Phi_f]_{C^0}$ of Φ_f. Then, we go one step further proving that every C^{1+} conjugacy classes of C^{1+H} Cantor exchange systems $\Phi \in [\Phi_f]_{C^0}$ that are C^{1+H} fixed points of renormalization, $[R_f \Phi]_{C^{1+H}} = [\Phi]_{C^{1+H}}$, determines a unique C^{1+H} diffeomorphism g, topologically conjugate to f, with an invariant measure absolutely continuous with respect to the Hausdorff measure on its invariant set. Furthermore, there is a Teichmüller space of solenoid functions (as introduced in [19]) which characterizes the set of all C^{1+} conjugacy classes of C^{1+H} Cantor exchange systems $\Phi \in [\Phi_f]_{C^0}$ that are C^{1+H} fixed points of renormalization $[R_f \Phi]_{C^{1+H}} = [\Phi]_{C^{1+H}}$. Denjoy [4] has shown the existence of upper bounds for the smoothness of Denjoy maps (see related re- sults of J. Harrison [8] and A. Norton [13]). We prove that there is no $C^{1+\alpha}$ Cantor exchange system $\Psi \in [\Phi_f]_{C^0}$, with bounded geometry, that is a $C^{1+\alpha}$ fixed point of renormalization with regularity α greater than the Hausdorff dimension of the Cantor invariant set of Ψ.

1.1. *Train tracks*

A *train track* $T = \bigsqcup_{j=1}^{n} \tilde{I}_j / \sim$ is the disjoint union of non-trivial compact intervals $\tilde{I}_j \subset \mathbb{R}$ with a given endpoints equivalence relation. Let $\bigsqcup_{j=1}^{n} \tilde{I}_j$ be a finite disjoint union of non-trivial compact intervals $\tilde{I}_j \subset \mathbb{R}$. An *endpoints equivalence relation* consists in fixing pairwise disjoint equivalence classes $E_1, \dots, E_i$ such that $\bigcup_{j=1}^{i} E_j$ is equal to the set of all endpoints of the intervals $\tilde{I}_1, \dots, \tilde{I}_n$, and any two endpoints x and y are equivalent if, and only if, they belong to a same set E_j. We allow the case where some equivalence classes are singletons.

A *parametrization* $\alpha : I \to T$ in T is the image of a non-trivial interval I in $\mathbb{R}$ by a homeomorphism onto its image. If I is closed (respectively, open), we say that $\alpha(I)$ is a closed (respectively, open) arc in T. A *chart* in T is the inverse of a parametrization. A *topological atlas* B on the train track T is a given set of charts $\{(j, J)\}$ on the train track covering locally every arc. A $C^{1+\alpha}$, with $\alpha > 0$, atlas B on the train track T is a topological atlas such that the overlap maps are $C^{1+\alpha}$ and have uniformly $C^{1+\alpha}$ bounded norm. A C^{1+H} atlas B is a $C^{1+\alpha}$ atlas, for some $\alpha > 0$.

1.2. *Cantor exchange systems*

A C^{1+H} *exchange system* $\Phi = \{\varphi_i;\ i = 1, \dots, n\}$ on a train track T_Φ, with a C^{1+H} atlas B_Φ, is a finite set of maps $\varphi_i : \tilde{I}_{\varphi_i} \to \tilde{J}_{\varphi_i}$ with the following properties:

(i) The sets $\tilde{I}_{\varphi_i}$ and $\tilde{J}_{\varphi_i}$ are closed intervals in the train track T_Φ, and, for some $\alpha > 0$, the maps $\varphi_i \in \Phi$ are $C^{1+\alpha}$ diffeomorphisms with respect to the charts in the atlas B_Φ;

(ii) If $\varphi_i : \tilde{I}_{\varphi_i} \to \tilde{J}_{\varphi_i}$ is in Φ, then there is $\varphi_j : \tilde{I}_{\varphi_j} \to \tilde{J}_{\varphi_j}$ in Φ such that $\tilde{I}_{\varphi_j} = \tilde{J}_{\varphi_i}$, $\tilde{J}_{\varphi_j} = \tilde{I}_{\varphi_i}$ and $\varphi_j = \varphi_i^{-1}$;

(iii) For every $x \in T_\Phi$, there exist at most two distinct intervals $\tilde{I}_{\varphi_i}$ and $\tilde{I}_{\varphi_j}$ containing x.

We note that condition (i) implies that the intervals $\tilde{J}_{\varphi_i}$ are also closed intervals. We say that a finite sequence $\{\varphi_{i_n} \in \Phi\}_{n=1}^{m}$ or an infinite sequence $\{\varphi_{i_n} \in \Phi\}_{n \geq 1}$ is *admissible* with respect to x, if $\varphi_{i_n} \circ \cdots \circ \varphi_{i_1}(x) \in \tilde{I}_{\varphi_{i_{n+1}}}$ and $\varphi_{i_n} \neq \varphi_{i_{n-1}}^{-1}$ for all $n > 1$. We define the invariant set Ω_Φ of Φ as being the set of all points $x \in T_\Phi$ for which there are two distinct infinite admissible sequences $\{\varphi^F_{i_n} \in \Phi\}_{n \geq 1}$ and $\{\varphi^B_{i_n} \in \Phi\}_{n \geq 1}$ with respect to x. The forward orbit $O^F(x)$ of a point $x \in \Omega_\Phi$ is the set $\{\varphi^F_{i_n}(x) : n \geq 1\}$, and the backward orbit $O^B(x)$ of x is the set $\{\varphi^B_{i_n}(x) : n \geq 1\}$. We will assume that the invariant set Ω_Φ is *minimal*, i.e., for every $x \in \Omega_\Phi$, the closure $\overline{O^F(x)}$ is equal to the invariant set Ω_Φ and that the closure $\overline{O^B(x)}$ is also equal to the invariant set Ω_Φ. Furthermore, without loss of generality, we will assume that there are inter- vals $I_{\varphi_1} \subset \tilde{I}_{\varphi_1}, \ldots, I_{\varphi_n} \subset \tilde{I}_{\varphi_n}$ such that the endpoints of the intervals $I_{\varphi_1}, \ldots, I_{\varphi_n}$ belong to the invariant set Ω_Φ and $\Omega_\Phi \subset \bigcup_{i=1}^{n} I_{\varphi_i}$. We denote the Hausdorff dimension of Ω_Φ by $HD(\Omega_\Phi)$. If $0 < HD(\Omega_\Phi) < 1$, we call Φ a C^{1+H} *Cantor exchange system*, which is the case studied in this paper.

We say that a Cantor exchange system Φ is determined by a map $\varphi : I_\varphi \to J_\varphi$ if all the maps $\varphi_i : I_{\varphi_i} \to J_{\varphi_i}$ contained in Φ are the restriction of the map φ or its inverse φ^{-1} to I_{φ_i}. In this case, we call φ a *Cantor exchange map*. We note that not all Cantor exchange systems are determined by a Cantor exchange map.

We say that two $C^{1+\alpha}$ Cantor exchange systems $\Phi = \{\varphi_i : \tilde{I}_{\varphi_i} \to \tilde{J}_{\varphi_i};\ i = 1, \ldots, n\}$ and $\Psi = \{\psi_i : \tilde{I}_{\psi_i} \to \tilde{J}_{\psi_i};\ i = 1, \ldots, n\}$, with $\alpha > 0$, are C^0 *conjugate* if there is a map $h : \Omega_\Phi \to \Omega_\Psi$, with a homeomorphic extension $\xi : T_\Phi \to T_\Psi$ to the train track T_Φ, such that $h \circ \varphi_i(x) = \psi_i \circ h(x)$ for all $x \in \Omega_\Phi$. We denote by $[\Phi]_{C^0}$ the set of all C^{1+H} Cantor exchange systems that are C^0 conjugate to Φ. By minimality of the invariant set Ω_Φ, the map h is unique (but its extension ξ is not necessarily unique). If h has a $C^{1+\alpha}$ diffeomorphic extension, with respect to the $C^{1+\alpha}$ atlas B_Φ, to the train track T_Φ, then we say that Φ and Ψ are $C^{1+\alpha}$ *conjugate*. We denote by $[\Phi]_{C^{1+\alpha}}$ the set of all $C^{1+\alpha}$ Cantor exchange systems that are $C^{1+\alpha}$ conjugate to Φ, and we denote by $[\Phi]_{C^{1+H}}$ the set $\bigcup_{\alpha>0} [\Phi]_{C^{1+\alpha}}$.

1.3. *Renormalization*

Let $\Phi = \{\varphi_i : \tilde{I}_{\varphi_i} \to \tilde{J}_{\varphi_i}:\ i = 1, \ldots, n\}$ and $\Psi = \{\psi_i : \tilde{I}_{\psi_i} \to \tilde{I}_{\psi_i}:\ i = 1, \ldots, m\}$ be C^{1+H} Cantor exchange systems. We say that Ψ is a *renormalization of* Φ if there is a *renormalization sequence set* $S = S(\Phi, \Psi) = \{s^1, \ldots, s^m\}$ with the following properties:

(i) For every $i \in \{1, \ldots, n\}$, we have that

$$\psi_i = \phi_{s^i_{k(\underline{s}^i)}} \circ \cdots \circ \phi_{s^i_1} | I_{\psi_i},$$

where $\underline{s}^i = s^i_{k(\underline{s}^i)} \ldots s^i_1 \in \mathcal{S}$. In particular, $\Omega_\Psi \subset \Omega_\Phi$ and $I_{\psi_i} \subset I_{\phi_{s^i_1}}$.

(ii) For every $x \in \Omega_\Phi \setminus \Omega_\Psi$, there are exactly two distinct sequences $\underline{s}^i, \underline{s}^j \in S$ with the property that there are points $y_i \in I_{\psi_i}$, $y_j \in I_{\psi_j}$ such that

$$x = \phi_{s^i_{k(x,i)}} \circ \cdots \circ \phi_{s^i_1}(y_i) \quad \text{and} \quad x = \phi_{s^j_{k(x,j)}} \circ \cdots \circ \phi_{s^j_1}(y_j),$$

for some $0 < k(x, i) < k(\underline{s}^i)$ and $0 < k(x, j) < k(\underline{s}^j)$.

If Ψ is a renormalization of Φ, with renormalization sequence set $S(\Phi, \Psi)$, then there is a unique renormalization operator $R = R_{S(\Phi,\Psi)} : [\Phi]_{C^0} \to [\Psi]_{C^0}$ defined as follows: Let $\underline{\Phi}$ be a C^{1+H} Cantor exchange system topologically conjugate to Φ. Let $\xi : T_\Phi \to T_\Psi$ be a homeo- morphic extension to the train track T_Φ of the topological conjugacy $h : \Omega_\Phi \to \Omega_\Psi$. Since h is unique, by minimality of Ω_Φ, for every $s^i \in S$, $\xi(I_{\psi_i})$ and $\xi(J_{\psi_i})$ are the smallest closed arcs containing $h(I_{\psi_i})$ and $h(J_{\psi_i})$, respectively, and, so, are uniquely determined. Define the C^{1+H} Cantor exchange system $\underline{\Psi}$ by

$$\underline{\Psi} = \{\underline{\psi}_i = \underline{\phi}_{s^i_{k(\underline{s}^i)}} \circ \cdots \circ \underline{\phi}_{s^i_1} : \xi(\tilde{I}_{\psi_i}) \to \xi(\tilde{J}_{\psi_i}),\ \text{for every}\ \underline{s}^i \in \mathcal{S}(\Phi, \Psi)\}.$$

By construction, $\underline{\Psi}$ is topologically conjugate to Ψ and does not depend on the extension ξ of h considered in the sets $\xi(I_{\psi_1}), \ldots, \xi(I_{\psi_n})$. Furthermore, $\underline{\Psi}$ is a C^{1+H} Cantor exchange system that is a renormalization of $\underline{\Phi}$ with respect to the renormalization sequence set $S(\underline{\Phi}, \underline{\Psi}) = S(\Phi, \Psi)$. Hence, the renormalization operator R is well defined by $\underline{R\Phi} = \underline{\Psi}$.

Definition 1.1. Let $R : [\Phi]_{C^0} \to [\Psi]_{C^0}$ be a renormalization operator. We say that a $C^{1+\alpha}$ Cantor exchange system $\Gamma \in [\Phi]_{C^0}$ is a $C^{1+\alpha}$ *fixed point of the renormalization operator* R, if $R\Gamma$ is $C^{1+\alpha}$ conjugated to Γ, i.e., $[R\Gamma]_{C^{1+\alpha}} = [\Gamma]_{C^{1+\alpha}}$. We say that a C^{1+H} Cantor exchange system $\Gamma \in [\Phi]_{C^0}$ is a C^{1+H} *fixed point of the renormalization operator* R, if Γ is $C^{1+\alpha}$ fixed point of the renormalization operator R, for some $\alpha > 0$.

1.4. *Codimension 1 attractors*

Throughout this paper, (f, Λ, M) is a C^{1+H} diffeomorphism f with a codimension 1 hyperbolic attractor Λ and with a Markov partition M on Λ satisfying the disjointness property as we pass to describe.

We say that (f, Λ) is a C^{1+H} *diffeomorphism f with a codimension* 1 *hyperbolic attractor* Λ, if (f, Λ) has the following properties:

(i) $f : S \to S$ is a $C^{1+\alpha}$ diffeomorphism of a compact surface S with respect to a $C^{1+\alpha}$ structure on S, for some $\alpha > 0$.
(ii) Λ is a hyperbolic invariant subset of S such that $f|\Lambda$ is topologically transitive and Λ has a local product structure.
(iii) There is an open set $O \subset S$ such that $\Lambda = \bigcap_{n \geq 0} f^n O$.

A C^{1+H} diffeomorphism (f, Λ) with codimension 1 hyperbolic attractor has the property that the local stable leaves intersected with Λ are Cantor sets and the local unstable leaves are 1-dimensional manifolds (see Appendix A.1, and also [1] and [30]). Let $HD(\Lambda^s)$ be the Haus- dorff dimension of the stable leaves intersected with the basic set. Furthermore, (f, Λ) has a Markov partition M on Λ with the following *disjointness property*: The unstable leaf bound- aries of any two Markov rectangles do not intersect except, possibly, at their endpoints (see also Appendix A.3).

Let f be a C^{1+H} diffeomorphism with codimension 1 hyperbolic attractor Λ and with a Markov partition M satisfying the disjointness property. In Section 2, we present an explicit construction of a C^{1+H} Cantor exchange system $\Phi_{f,\mathcal{M}}$ induced by $(f, \Lambda, \mathcal{M})$. Let $\mathcal{C}_{f,\mathcal{M}}$ be the topological conjugacy class of $\Phi_{f,\mathcal{M}}$. In Section 3, we present an explicit construction of a renormalization operator $R_{f,\mathcal{M}} : \mathcal{C}_{f,\mathcal{M}} \to \mathcal{C}_{f,\mathcal{M}}$ acting on the topological conjugacy class $\mathcal{C}_{f,\mathcal{M}}$ of the C^{1+H} Cantor exchange system $\Phi_{f,\mathcal{M}}$ induced by (f, Λ, M).

Let F be the set of all C^{1+H} codimension 1 hyperbolic diffeomorphisms topologically conjugate to f (see Appendix A.6).

Theorem 1 *(Teichmüller space). Let (f, Λ, M) be a C^{1+H} diffeomorphism f with a codimension* 1 *hyperbolic attractor Λ and with a Markov partition M for f on Λ satisfying the disjointness property. There is a unique map*

$$\mathcal{T}_{f,\mathcal{M}} : \mathcal{F} = \{[g]_{C^{1+H}} : g \in \mathcal{F}\} \to \mathcal{C} = \{[\Phi]_{C^{1+H}} : \Phi \in \mathcal{C}_{f,\mathcal{M}}\}$$

defined by $T_{f,M}([g]_{C^{1+H}}) = [\Phi_{g,M_g}]_{C^{1+H}}$*, where* M_g *is the pushforward of the Markov partition* M *of* f *by the topological conjugacy between* f *and* g*. The map* $T = T_{f,M} : F \to C$ *has the following properties*:

(a) *If* $[\Phi]_{C^{1+H}} = T[g]_{C^{1+H}}$*, then* $HD(T_\Phi) = HD(\Lambda_g)$*;*
(b) $T(F) = C_R$*, where* $C_R \subset C$ *is the set of all* C^{1+H} *conjugacy classes* $[\Phi]_{C^{1+H}} \in C$ *that are* C^{1+H} *fixed points of renormalization,* $[R\Phi]_{C^{1+H}} = [\Phi]_{C^{1+H}}$*;*
(c) *For every* $[\Phi]_{C^{1+H}} \in C_R$ *there is a unique* C^{1+H} *conjugacy class of* C^{1+H} *hyperbolic diffeomorphisms* $g \in T^{-1}([\Phi]_{C^{1+H}})$ *with a codimension* 1 *hyperbolic invariant set* Λ_g *that admits an invariant measure absolutely continuous with respect to the Hausdorff measure on* Λ_g*;*
(d) *The set* C_R *is characterized by a moduli space consisting of solenoid functions.*

The above solenoid functions are introduced in [19] where they were used to construct a moduli space for the set of all C^{1+H} diffeomorphisms on a surface S with a hyperbolic invariant set Λ contained in S (see Appendices A.10–A.16).

Remark 1. We note that in Theorem 1, if the unstable lamination of the attractor set Λ is ori- entable, then the Cantor exchange systems in $C_{f,M}$ are determined by Cantor exchange maps.

1.5. *Upper smooth bounds for Cantor exchange systems*

Let Φ and Ψ be C^{1+H} Cantor exchange systems such that Ψ is a renormalization of Φ with renormalization sequence set $S = S(\Phi, \Psi)$. Let us suppose that Ψ is topologically conjugate to Φ, i.e., Φ is a C^0 fixed point of renormalization $[R\Phi]_{C^0} = [\Phi]_{C^0}$. In this case Φ is an infinitely renormalizable C^{1+H} Cantor exchange system, i.e., there is an infinite sequence

$$\left(R^m\Phi = \{\phi_i^{(m)} : \tilde{I}_{\phi_i}^{(m)} \to \tilde{J}_{\phi_i}^{(m)};\ i = 1, \dots, n(m)\}\right)_{m \geqslant 1}$$

of Cantor exchange systems inductively determined, for every m � 1, by $R^m\Phi = R(R^{m-1}\Phi)$ with $S(R^m\Phi, R^{m-1}\Phi) = S(\Phi, \Psi)$.

Set

$$L_m^{(1)} = \{\phi_{s_k^i}^{(m)} \circ \dots \circ \phi_{s_1^i}^{(m)}(I_{\phi_i}^{(m+1)}):\ I_{\phi_i}^{(m+1)} \subset I_{\phi_{s_1^i}}^{(m)},\ 0 \leqslant k \leqslant k(\underline{s}_i),\ \underline{s}_i \in \mathcal{S}\}.$$

Set, inductively on j � 1, the sets

$$L_m^{(j)} = \{\phi_{s_k^i}^{(m)} \circ \dots \circ \phi_{s_1^i}^{(m)}(I):\ I \in L_{m+1}^{(j-1)},\ I \subset I_{\phi_{s_1^i}}^{(m)},\ 0 \leqslant k \leqslant k(\underline{s}_i),\ \underline{s}_i \in \mathcal{S}\}.$$

By construction, $L_m^{(j+1)} \subset L_m^{(j)}$ and $\Omega_{R^m\Phi} = \bigcap_{j \geqslant 1} L_m^{(j)}$. We call $L_m^{(j)}$ the *jth level of the partition of* $R^m\Phi$. Let the *j*-gap set $G_m^{(j)}$ of $R^m\Phi$ be the set of all maximal closed intervals I such that $I \subset J$ for some $J \in L_m^{(j-1)}$ and $\operatorname{int} I \cap K = \emptyset$, for every $K \in L_m^{(j)}$. We say that the C^{1+H} Cantor exchange system Φ has bounded geometry, if there are constants $0 < c_1, c_2 < 1$ such that,
for all j � 1 and all intervals $I \in L_0^{(j)} \cup G_0^{(j)}$ contained in a same interval $K \in L_0^{(j-1)}$, we have $c_1 < |\zeta(I)| / |\zeta(K)| < c_2$ atlas B_Φ.

, where the length is measured with respect to any chart ζ in the $C^{1+\alpha}$

Before proceeding, we present the following notion of $C^{1,HD}$ regularity of a function.

Definition 1.2. Let $\varphi : I \to J$ be a homeomorphism between open sets $I \subset \mathbb{R}$ and $J \subset \mathbb{R}$. If $0 < \alpha < 1$, then φ is said to be $C^{1,\alpha}$ if it is differentiable and for all points $x, y \in I$

$$\left|\phi'(y) - \phi'(x)\right| \leqslant \chi_\phi\big(|y - x|\big) \qquad (1)$$

where the positive function $\chi_\varphi(t)$ satisfies $\lim_{t\to 0} \chi_\varphi(t)/t^\alpha = 0$.

In particular, a $C^{1+\beta}$ diffeomorphism is $C^{1,\alpha}$ for all $0 < \alpha < \beta$. Furthermore, a $C^{1,\alpha}$ homeomorphism is $C^{1+\alpha}$. Hence, for all $0 < \alpha < \beta$, we have $C^{1+\beta} \subset C^{1,\alpha} \subset C^{1+\alpha}$.

Theorem 2. *Let $\mathcal{C}_{f,\mathcal{M}}$ be the topological conjugacy class of C^{1+H} Cantor exchange systems determined by a C^{1+H} diffeomorphism f with codimension 1 hyperbolic attractor Λ and with a Markov partition $\mathcal{M}$ satisfying the disjointness property (as in Theorem 1). There is no $C^{1,HD(\Omega_\Phi)}$ Cantor exchange system $\Phi \in \mathcal{C}_{f,\mathcal{M}}$, with bounded geometry, that is a $C^{1,HD(\Omega_\Phi)}$ fixed point of renormalization operator, i.e., $[R_{f,\mathcal{M}}\Phi]_{C^{1,HD(\Omega_\Phi)}} = [\Phi]_{C^{1,HD(\Omega_\Phi)}}$.*

2. Induced Cantor exchange systems

In this section, given a C^{1+H} diffeomorphism f with codimension 1 hyperbolic attractor Λ and with a Markov partition $\mathcal{M}$ satisfying the disjointness property, we present an explicit construction of the induced C^{1+H} Cantor exchange system $\Phi = \Phi_{f,\mathcal{M}}$.

Suppose that M and N are Markov rectangles, and $x \in M$ and $y \in N$. We say that x and y are *stable holonomically related* if (i) there is an unstable leaf segment $\ell^u(x, y)$ such that $\partial\ell^u(x, y) = \{x, y\}$, and (ii) $\ell^u(x, y) \subset \ell^u(x, M) \cup \ell^u(y, N)$. Let $P = P_{\mathcal{M}}$ be the set of all pairs (M, N) such that there are points $x \in M$ and $y \in N$ stable holonomically related.

For every Markov rectangle $M \in \mathcal{M}$, choose a spanning leaf segment ℓ_M in M. Let $\boldsymbol{I} = \{\ell_M : M \in \mathcal{M}\}$. For every pair $(M, N) \in P$, there are maximal leaf segments $\ell^D_{(M,N)} \subset \ell_M$, $\ell^C_{(M,N)} \subset \ell_N$ such that the holonomy $h_{(M,N)} : \ell^D_{(M,N)} \to \ell^C_{(M,N)}$ is well defined (see Appendix A.3). We call such holonomies $h_{(M,N)} : \ell^D_{(M,N)} \to \ell^C_{(M,N)}$ the (stable) primitive holonomies associated to the Markov partition $\mathcal{M}$.

Definition 2.1. The set $H = \{h_{(M,N)} : \ell^D_{(M,N)} \to \ell^C_{(M,N)} ; (M, N) \in P\}$ is a *complete set of stable holonomies*.

In Fig. 1, we consider a derived-Anosov diffeomorphism $g : T \to T$ semi-conjugated, by a map $\pi : T \to T$, to the Anosov automorphism $f : T \to T$ defined by $f(x, y) = (x + y, y)$, where $T = \mathbb{R}^2 \setminus (\mathbf{Z} \times \mathbf{Z})$. We exhibit the complete set of holonomies $H_{f,\mathcal{M}} = \{h_{(A,A)}, h_{(A,B)}, h_{(B,A)}\}$ associated to the Markov partition $\mathcal{M} = \{A, B\}$ of f. The derived-Anosov diffeomorphism g admits a Markov partition $\mathcal{M}_g = \{A_1, A_2, B_1\}$ with the property that $A = \pi(A_1) \cup \pi(A_2)$ and $B = \pi(B_1)$. The complete set of holonomies $H_{g,\mathcal{M}_g}$ is related to $H_{f,\mathcal{M}}$ as follows: $h_{(A,B)} \circ \pi\,|\,\pi(\ell^D_{(A_1,B_1)}) = \pi \circ h_{(A_1,B_1)}$, $h_{(A,A)} \circ \pi\,|\,\pi(\ell^D_{(A_2,A_1)}) = \pi \circ h_{(A_2,A_1)}$, $h_{(B,A)} \circ \pi\,|\,\pi(\ell^D_{(B_1,A_1)}) = \pi \circ h_{(B_1,A_1)}$ and $h_{(B,A)} \circ \pi\,|\,\pi(\ell^D_{(B_1,A_2)}) = \pi \circ h_{(B_1,A_2)}$.

Lemma 1. *The triple $(f, \Lambda, \mathcal{M})$ induces a train track T_f with a set of canonical parametrizations. Furthermore, the atlas $\mathcal{A}^s(f,\rho)$ induces a $C^{1+\alpha}$ atlas $\mathcal{B}^s(f,\rho)$ in T_f.*

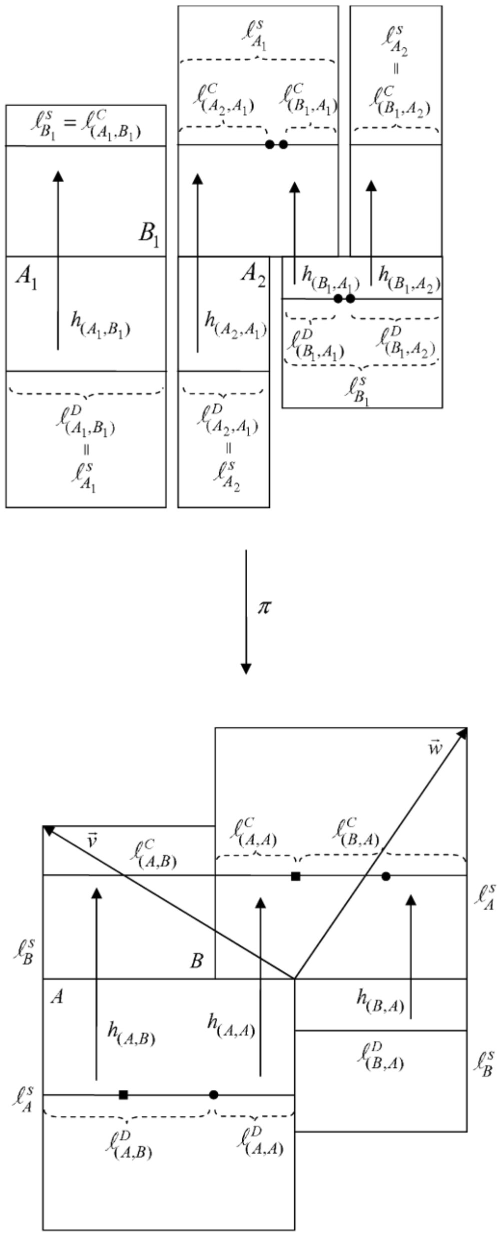

Fig. 1. The complete set of holonomies $H_{g,M_g} = \{h_{(A_1,B_1)}, h_{(A_2,A_1)}, h_{(B_1,A_1)}, h_{(B_1,A_2)}, h^{-1}_{(A_1,B_1)}, h^{-1}_{(A_2,A_1)}, h^{-1}_{(B_1,A_1)}, h^{-1}_{(B_1,A_2)}\}$ for the derived-Anosov diffeomorphism $g : T \to T$ semi-conjugated, by a map $\pi : T \to T$, to the Anosov automorphism $f : T \to T$ defined by $f(x, y) = (x + y, y)$. The complete set of holonomies for the Anosov automorphism $f : T \to T$ associated to the Markov partition $M = \{A, B\}$ is given by $H_{f,M} = \{h_{(A,A)}, h_{(A,B)}, h_{(B,A)}, h^{-1}_{(A,A)}, h^{-1}_{(A,B)}, h^{-1}_{(B,A)}\}$. The complete set of holonomies H_{g,M_g} is related to $H_{f,M}$ as follows: $h_{(A,B)} \circ \pi|\pi(\ell_{(A_1,B_1)}) = \pi \circ h_{(A_1,B_1)}$, $h_{(A,A)} \circ \pi|\pi(\ell_{(A_2,A_1)}) = \pi \circ h_{(A_2,A_1)}$, $h_{(B,A)} \circ \pi|\pi(\ell^D_{(B_1,A_1)}) = \pi \circ h_{(B_1,A_1)}$ and $h_{(B,A)} \circ \pi|\pi(\ell^D_{(B_1,A_2)}) = \pi \circ h_{(B_1,A_2)}$.

Proof. For every leaf segment $\ell_M \in \mathcal{I}$, let $\hat{\ell}_M$ be the smallest full leaf segment containing ℓ_M (see definition in Appendix A). Let $\tilde{\ell}_M$ be a full leaf segment that is a small extension of the leaf $\hat{\ell}_M$, that compactly contains $\hat{\ell}_M$, and does not intersect any other Markov rectangle. By the Stable Manifold Theorem, there are C^{1+H} diffeomorphisms $k_M : \tilde{\ell}_M \to \tilde{K}_M$. We choose the C^{1+H} diffeomorphisms $k_M : \tilde{\ell}_M \to \tilde{K}_M$ with the extra property that their images are pairwise disjoint, i.e., $\tilde{K}_M \cap \tilde{K}_N = \emptyset$ for all $M, N \in \mathcal{M}$ such that $M \neq N$. Set $\tilde{K}_M = k_M(\tilde{\ell}_M)$,

$$\tilde{K}_{\mathcal{M}} = \bigsqcup_{i=1}^{n} \tilde{K}_{M_i}, \qquad \hat{L}_{\mathcal{M}} = \bigsqcup_{i=1}^{n} \hat{\ell}_{M_i}, \qquad \tilde{L}_{\mathcal{M}} = \bigsqcup_{i=1}^{n} \tilde{\ell}_{M_i} \quad \text{and} \quad L_{\mathcal{M}} = \hat{L}_{\mathcal{M}} \cap \Lambda_f. \quad (2)$$

Let $k : \tilde{L}_{\mathcal{M}} \to \tilde{K}_{\mathcal{M}}$ be the map defined by $k|\tilde{\ell}_M = k_M$, for every $M \in \mathcal{M}$. Let

$$\pi : \bigcup_{i=1}^{n} M_i \to L_{\mathcal{M}} \quad (3)$$

be the projection defined by $\pi(x_i) = y_i$, where $y_i \in \ell^u_{M_i}(x_i) \cap L_{\mathcal{M}}$ for every $x_i \in M_i$. The end-points $\tilde{x}_i \in \tilde{\ell}_{M_i}$ and $\tilde{x}_j \in \tilde{\ell}_{M_j}$ are in the same endpoints equivalence class, if there are points $x_i \in \ell_{M_i}$ and $x_j \in \ell_{M_j}$ with the following properties:

(i) $k_{M_i}^{-1}([x_i, \tilde{x}_i]) \cap \Lambda_f = k_{M_i}^{-1}(x_i)$;
(ii) $k_{M_j}^{-1}([x_j, \tilde{x}_j]) \cap \Lambda_f = k_{M_j}^{-1}(x_j)$;
(iii) There is a closed stable leaf segment $\ell^s(y_i, y_j)$ such that

$$\ell^s(y_i, y_j) \cap \Lambda_f = \ell^s(y_i, y_j) \cap \left\{ \operatorname{int} \ell^u_{M_i}\big(k_{M_i}^{-1}(x_i)\big), \operatorname{int} \ell^u_{M_j}\big(k_{M_j}^{-1}(x_j)\big) \right\}.$$

The endpoints equivalence class in $\tilde{L}_{\mathcal{M}}$ is the minimal equivalence class satisfying the above properties.

For every stable leaf segment ℓ^s, consider the smallest full leaf segment $\hat{\ell}^s$ containing ℓ^s and a chart $j : \hat{\ell}^s \to I$ in the atlas $\mathcal{A}^s(f, \rho)$. By [18], for every Markov rectangle M, the holonomy $h : \ell^s \cap M \to \ell_M$ has a $C^{1+\alpha}$ extension with respect to the charts in the atlas $\mathcal{A}^s(f, \rho)$, which implies that the map $k_M \circ h \circ j^{-1}|j(M \cap \ell^s)$ has a $C^{1+\alpha}$ diffeomorphic extension u_M to the train track $\mathbb{T}_f$. Therefore, the map $\pi \circ j^{-1}|(I \cap j(\ell^s))$ has a homeomorphic extension $\tau : I \to \mathbb{T}_f$ such that $k_M^{-1} \circ \tau|j(M \cap \ell^s) = k_M \circ h \circ j^{-1}$ and $k_M^{-1} \circ \tau = u_M$ is a $C^{1+\alpha}$ diffeomorphism, for every Markov rectangle M. The inverse of these parametrizations τ together with the previous charts k_M^{-1}, for every $M \in \mathcal{M}$, form a $C^{1+\alpha}$ atlas $\mathcal{B}^s(f, \rho)$ induced by $\mathcal{A}^s(f, \rho)$. ✷

Lemma 2. *The triple $(f, \Lambda, \mathcal{M})$ induces a C^{1+H} Cantor exchange system*

$$\Phi = \Phi_{f,\mathcal{M}} = \left\{ e_{(M,N)} : k_M\big(\hat{\ell}^D_{(M,N)}\big) \to k_N\big(\hat{\ell}^C_{(M,N)}\big) \;\big|\; (M, N) \in \mathcal{P} \right\},$$

with bounded geometry with respect to the atlas $\mathcal{B}^s(f, \rho)$ on $\mathbb{T}_f = \mathbb{T}_\phi$. Furthermore, for every $(M, N) \in \mathcal{P}$, $e_{(M,N)}|k_M(\ell^D_{(M,N)}) = k_M \circ h_{(M,N)} \circ k_N^{-1}$.

Proof. Define $e_{(M,N)}|k_M(\pounds^D_{(M,N)}) = k_M \circ h_{(M,N)} \circ k_N^{-1}$. By Theorem 2.1 in [18], the map $k_M \circ h_{(M,N)} \circ k_N^{-1} |k_M(\pounds^D_{(M,N)})$ extends (not uniquely) to a $C^{1+\alpha}$ diffeomorphism $e_{(M,N)} : k(\pounds^D_{(M,N)}) \to k(\pounds^C_{(M,N)})$ for some $\alpha > 0$. By Lemma 1, the set $\{e_{(M,N)}: (M,N) \in P\}$ satisfies properties (i), (ii) and (iii) of the definition of a C^{1+H} Cantor exchange system. ✷

3. Renormalization of Cantor exchange systems

In this section, given a C^{1+H} diffeomorphism f with codimension 1 hyperbolic attractor Λ and with a Markov partition $\mathcal{M}$ satisfying the disjointness property, we present an explicit construction of a renormalization operator $R = R_{f,\mathcal{M}}$ acting on the topological conjugacy
class of the C^{1+H} Cantor exchange system $\Phi_{f,\mathcal{M}}$ induced by $(f, \Lambda, \mathcal{M})$. Let the Markov partition $\mathcal{N} = f_* \mathcal{M}$ be the pushforward of the Markov partition $\mathcal{M}$, i.e., for every $M \in \mathcal{M}$, $N = f(M) \in \mathcal{N}$.

Lemma 3. *Let $\Phi_{f,\mathcal{M}}$ and $\Phi_{f,\mathcal{N}}$ be the C^{1+H} Cantor exchange systems induced (as in Lemma 2), respectively, by $(f,\Lambda,\mathcal{M})$ and $(f,\Lambda,\mathcal{N})$. There is a well-defined renormalization operator*

$$R = R_{f,\mathcal{M}} : [\Phi_{f,\mathcal{M}}]_{C^0} \to [\Phi_{f,\mathcal{N}}]_{C^0}.$$

Proof. For simplicity of notation, let us denote k_M by k (see (2)). We choose a map

$$\sigma : \{1,\ldots,n\} \to \{1,\ldots,n\} \tag{4}$$

with the property that $N_i \cap M_{\sigma(i)} \neq \emptyset$, where $N_i \in \mathcal{N}$ and $M_{\sigma(i)} \in \mathcal{M}$. For each $N_i \in \mathcal{N}$, let ℓ_{N_i} be the stable spanning leaf segment $\ell_{M_{\sigma(i)}} \cap \pi(N_i)$, and let $\hat{\ell}_{N_i}$ be the corresponding full stable spanning leaf (i.e., $\hat{\ell}_{N_i} \cap \Lambda = \ell_{N_i}$), where $\pi : \bigsqcup_{i=1}^n M_i \to L_M$ is the natural projection as defined in (2). Set

$$\mathcal{L}_{\mathcal{N}} = \bigcup_{i=1}^n \ell_{N_i} \quad \text{and} \quad \hat{\mathcal{L}}_{\mathcal{N}} = \bigcup_{i=1}^n \hat{\ell}_{N_i}.$$

The set $\hat{L}_N$ determines the train track T_N with atlas $B(f,\rho)$ as constructed in Lemma 1. Let $H_N = \{h_{(N_i,N_j)} : \ell^D_{(N_i,N_j)} \to \ell^C_{(N_i,N_j)} \mid (N_i,N_j) \in P_N\}$ be the (stable) primitive holonomic system associated to the Markov partition N. By construction, for every $(N_i,N_j) \in P_N$ there is a sequence $h_{\alpha_1},\ldots,h_{\alpha_n}$ of holonomies in H_M such that

$$h_{(N_i,N_j)} = h_{\alpha_n} \circ \cdots \circ h_{\alpha_1} | \tilde{\ell}^D_{N_i}.$$

Let

$$e_{(N_i,N_j)} : k_{M_{\sigma(i)}}(\hat{\ell}^D_{(N_i,N_j)}) \to k_{M_{\sigma(j)}}(\hat{\ell}^D_{(N_i,N_j)})$$

be given by $e_{(N_i,N_j)} = e_{\alpha_n} \circ \cdots \circ e_{\alpha_1}$, where $e_{\alpha_i} \in \Phi_{f,\mathcal{M}}$ and $e_{\alpha_i}|k(\ell^D_{(N_i,N_j)}) = k \circ h_{\alpha_i} \circ k^{-1}|k(\ell^D_{(N_i,N_j)})$. Set

$$\Psi = \{e_{(N_i,N_j)} : k(\hat{\ell}^D_{(N_i,N_j)}) \to k(\hat{\ell}^C_{(N_i,N_j)}) \mid (N_i,N_j) \in \mathcal{P}_{\mathcal{N}}\}.$$

By construction, $\Psi = \Phi_{f,N}$ (as constructed in Lemma 2), and so Ψ is a C^{1+H} Cantor exchange system. Since the set $S(\Phi_{f,M}, \Phi_{f,N})$ of all sequences $\alpha_1 \ldots \alpha_n$ such that $e_{(N_i,N_j)} = e_{\alpha_n} \circ \cdots \circ e_{\alpha_1}$, for some $(N_i, N_j) \in P_N$, form a renormalizable sequence set, the C^{1+H} Cantor exchange system $\Phi_{f,N}$ is a renormalization of $\Phi_{f,M}$. Therefore, by Section 1.3, there is a well-defined renormalization operator $R = R_{f,M} : [\Phi_{f,M}]_{C^0} \to [\Phi_{f,N}]_{C^0}$. ✷

Lemma 4. *The C^{1+H} Cantor exchange system $\Phi_{f,M}$ is a C^{1+H} fixed point of renormalization, i.e., $[R\Phi_{f,M}]_{C^0} = [\Phi_{f,M}]_{C^0}$, where $R = R_{f,M} : [\Phi_{f,M}]_{C^0} \to [\Phi_{f,N}]_{C^0}$ is the renormalization operator* (*as constructed in Lemma* 3).

Proof. We construct a $C^{1+\alpha}$ conjugacy $\Theta : K_N \to K_M$ between $\Phi_{f,M}$ and $\Phi_{f,N}$. For every $N \in N$ and $M = f^{-1}(N)$, there is a holonomy ϑ_N between the spanning leaf segments $f^{-1}(\ell_N)$ and ℓ_M. By Theorem 2.1 in [18], the holonomy ϑ_N has a $C^{1+\alpha}$ diffeomorphic extension $\hat{\vartheta}_N : f^{-1}(\hat{\ell}_N) \to \hat{\ell}_M$. Let $\Theta : K_N \to K_M$ be the $C^{1+\alpha}$ diffeomorphism given by

$$\Theta | k(\hat{\ell}_N) = k \circ \hat{\theta}_N \circ f^{-1} \circ k^{-1} \qquad (5)$$

for every $N \in N$. We observe that each pair

$$(N_i, N_j) \in \mathcal{P}_N$$

determines a unique pair $(M_i, M_j) = (f^{-1}(N_i), f^{-1}(N_j)) \in P^D_M$, and vice versa. Hence, it is enough to prove that Θ conjugates $\varphi_{(N_i,N_j)} | \ell_{(N_i,N_j)}$ with $\varphi_{(M_i,M_j)} | \ell_{(M_i,M_j)}$, for every $(N_i, N_j) \in P^D_N$, to show that $\Phi_{f,M}$ is a C^{1+H} fixed point of renormalization.

By construction of the maps ϑ_{N_i} and ϑ_{N_j}, we have that

$$h_{(M_i,M_j)} | \ell^D_{(M_i,M_j)} = \theta_{N_i} \circ f^{-1} \circ h_{(N_i,N_j)} \circ f \circ \theta^{-1}_{N_j},$$

and so

$$\begin{aligned}\Theta \circ e_{(N_i,N_j)} \circ \Theta^{-1} | \ell^D_{(M_i,M_j)} &= k \circ \theta_{N_j} \circ f^{-1} \circ k^{-1} \circ k \circ h_{(N_i,N_j)} \circ k^{-1} \circ k \circ f \circ \theta^{-1}_{N_j} \circ k^{-1} \\ &= k \circ h_{(M_i,M_j)} \circ k^{-1} \\ &= e_{(M_i,M_j)},\end{aligned}$$

which ends the proof. □

4. Markov maps versus renormalization

The map $F : T \to T$ determines a $C^{1+\alpha}$ *Markov map*, with respect to the atlas B and with invariant set Ω, if the following properties are satisfied:

(i) $F : T \to T$ is a local $C^{1+\alpha}$ diffeomorphism with respect to the $C^{1+\alpha}$ atlas B on the train track T.

(ii) There exist $c > 0$ and $\lambda > 1$ such that, for every $x \in \Omega$,

$$\left|d\left(j_n \circ F^n \circ i^{-1}\right)(x)\right| > c\lambda^n, \qquad (6)$$

with respect to charts $i, j_n \in B$.

(iii) The map F admits a Markov partition $\{K_1, \ldots, K_m\}$, i.e., there exists a finite set of arcs $\{\hat{K}_1, \ldots, \hat{K}_m\}$ such that (a) $K_i = \hat{K}_i \cap \Omega$, (b) $\bigcup_{i=1}^{m} \partial \hat{K}_i \subset \Omega$, and (c) $F(\partial \hat{K}_j) \subset \bigcup_{i=1}^{m} \partial \hat{K}_i$, for every $j = 1, \ldots, m$.

Let $\underline{F} : L_M \to L_M$ be the map induced by the action of f on stable leaf segments, i.e., $\underline{F}(x) = \pi \circ f^{-1}(x)$ for every $x \in \underline{L}$ (see (3)). Since f is a local diffeomorphism, the map $\underline{F}$ is a local homeomorphism. Let $\tilde{F} : k_M(L_M) \to k_M(L_M)$ be the map defined by $\tilde{F} = k_M \circ \underline{F} \circ k_M^{-1}$. Since the holonomies have $C^{1+\alpha}$ extensions (see Theorem 2.1 in [18]), and the map f is $C^{1+\alpha}$, for some $\alpha > 0$, the map $\tilde{F}$ has a $C^{1+\alpha}$ extension $F_{f,M} : \mathbb{T}_f \to \mathbb{T}_f$, with respect to the atlas $B^l(f, \rho)$, (not uniquely determined) that is a $C^{1+\alpha}$ Markov map with Markov partition $\{k_M \circ \pi(M_1), \ldots, k_M \circ \pi(M_l)\}$, where $M = \{M_1, \ldots, M_l\}$ is the Markov partition of f (see also A. Pinto and D. Rand [19]). Hence, the map $F_{f,M} : \mathbb{T}_f \to \mathbb{T}_f$ constructed above is a $C^{1+\alpha}$ Markov map. For every $j \geq 1$, let $L_0^{(j)}$ be the jth level of the partition of $\Phi_{f,M}$, as presented in Section 1.5. By construction, the map $F_{f,M}$ sends each interval $I \in L_0^{(j)}$ onto an interval $F_{f,M}(I) \in L_0^{(j-1)}$, for every $j > 0$.

Definition 4.1. Let $h : \Omega_\Phi \to \Omega_\Psi$ be the topological conjugacy between a C^{1+H} Cantor exchange system $\Psi = \{\psi_i : I_{\psi_i} \to J_{\psi_i};\ i = 1, \ldots, m\}$ and $\Phi_{f,M} = \{\varphi_i : I_{\varphi_i} \to J_{\varphi_i};\ i = 1, \ldots, n\}$. We say that Ψ induces a C^{1+H} Markov map

$$F_\Psi : \mathbb{T}_\Psi \to \mathbb{T}_\Psi,$$

if F_Ψ is a $C^{1+\alpha}$ Markov map, for some $\alpha > 0$, and $F_\Psi \circ h(x) = h \circ F_{f,M}(x)$, for every $x \in \Omega_\Psi$.

Lemma 5. *Let $\Phi_{f,M}$ be a C^{1+H} Cantor exchange system induced by (f, Λ, M). A C^{1+H} Cantor exchange system $\Psi \in [\Phi_{f,M}]_{C^0}$, with bounded geometry, determines a C^{1+H} Markov map F_Ψ topologically conjugate to $F_{f,M}$ if, and only if, Ψ is a C^{1+H} fixed point of the renormalization operator $R_{f,M}$.*

Remark 2. Lemma 5 also holds for $C^{1,\alpha}$ regularities.

Proof of Lemma 5. For simplicity of notation, let us denote k_M by k (see (2)). Let $\Theta : K_N \to K_M$ be the $C^{1+\alpha}$ diffeomorphism as constructed in (5). For every $N \in N$, let $M = f^{-1}(N) \in M$. Recall that $\ell_N \subset \ell_{M_{\sigma(i)}} \subset L_M$ (see (4)). By construction of $F = F_{f,M}$ and Θ, the spanning leaf segment $\ell_N \subset L_N$ has the property that $F \circ k(\ell_N) = k(\ell_M)$ and $F \,|\, k(\ell_N) = \Theta$. Therefore,

$$F|K_{\mathcal{N}} = \Theta. \tag{7}$$

Every leaf segment $\ell \subset L_{\mathcal{M}}$ with the property that $F \circ k(\ell) = k(\ell_M)$ is a spanning leaf segment of N. Therefore, there is a sequence $e_{\alpha_1}, \ldots, e_{\alpha_p}$ of Cantor exchange maps in $\Phi = \Phi_{f,\mathcal{M}}$ such that

$$e_{\alpha_p} \circ \cdots \circ e_{\alpha_1}(k(\ell)) = k(\ell_N).$$

Furthermore,

$$F|k(\ell) = \Theta \circ e_{\alpha_p} \circ \cdots \circ e_{\alpha_1}. \tag{8}$$

Let $\xi : \bigcup_{i=1}^{n} I_{\varphi_i} \to \bigcup_{i=1}^{n} I_{\psi_i}$ be a homeomorphic extension of the conjugacy between Ψ and Φ,
and $\Theta_\Psi : \xi(K_N) \to \xi(K_M)$ be the homeomorphic extension of the conjugacy between Ψ and its renormalization $R\Psi$. For every $e \in \Phi$, there is a unique $\underline{e} \in \Psi$ such that $\underline{e} = \xi \circ e \circ \xi^{-1}$. Since F_Ψ is topologically conjugate to F, by (7), we have that

$$F_\Psi | \xi(K_N) = \Theta. \tag{9}$$

Letting $\tilde{\ell}_N$, $\tilde{\ell}$ and $e_{\alpha_1}, \ldots, e_{\alpha_p}$ be as above, by (8), we obtain that

$$F_\Psi | \xi \circ k(\tilde{\ell}) = \Theta_\Psi \circ \underline{e}_{\alpha_p} \circ \cdots \circ \underline{e}_{\alpha_1}. \tag{10}$$

By (9), if F_Ψ is $C^{1+\alpha}$ then Θ_Ψ is $C^{1+\alpha}$. By (10), if Θ_Ψ is $C^{1+\alpha}$ then F_Ψ is $C^{1+\alpha}$.

Let $L_0^{(j)}$ be the jth level of the partition of Ψ (see Section 1.3). By construction, the map F_Ψ sends each interval $I \in L_0^{(j)}$ onto an interval $F_\Psi(I) \in L_0^{(j-1)}$ for every $j > 0$. Hence, if Ψ has bounded geometry we obtain that the length of the sets in $L_0^{(j)}$ converge exponentially fast to 0 when j tends to infinity. Therefore, using the Mean Value Theorem, we obtain that if Ψ has bounded geometry then F_Ψ satisfies property (ii) and, conversely, if F_Ψ satisfies property (ii)
we obtain that Ψ has bounded geometry. So, we conclude that if Ψ is a $C^{1+\alpha}$ Cantor exchange system, with bounded geometry, then F_Ψ is a $C^{1+\alpha}$ Markov map, and vice versa. ✷

5. Proofs of Theorems 1 and 2

Proof of Theorem 1. Let $g \in F$ be topologically conjugated to f by a map $u_g : U_f \to U_g$, where U_f and U_g are open neighborhoods of Λ_f and Λ_g, respectively. The Markov partition $\mathcal{M}$ determines a Markov partition $\mathcal{M}_g$ such that, for every $M \in \mathcal{M}$, we have that $M_g = u_g(M) \in \mathcal{M}_g$. We denote the set $u_g(\hat{L}_{\mathcal{M}})$ by $\hat{L}_g$. Letting $L_g = \hat{L}_g \cap \Lambda_g$, we have that $L_g = u_g(L_{\mathcal{M}})$. By the Stable Manifold Theorem, there is a C^{1+H} diffeomorphism $k_g : \hat{L}_g \to K_g$ with the similar properties as the map $k : \hat{L}_{\mathcal{M}} \to K_{\mathcal{M}}$. The triple $(g, \Lambda_g, \mathcal{M}_g)$ induces a C^{1+H} Cantor exchange system

$$\Phi_{g,\mathcal{M}_g} = \{e_{(M_g,N_g)} : k_g(\hat{\ell}^D_{(M_g,N_g)}) \to k_g(\hat{\ell}^C_{(M_g,N_g)}) \mid (M_g, N_g) \in \mathcal{P}\},$$

with bounded geometry. Let $g_1, g_2 \in F$ be C^{1+H} conjugate by a map $u : U_{g_1} \to U_{g_2}$, where U_{g_1} and U_{g_2} are open neighborhoods of Λ_{g_1} and Λ_{g_2}, respectively. By construction, the map

$\underline{u} = k_{g_2} \circ u \circ k^{-1}_{g_1}$ is a C^{1+H} diffeomorphism, and

$$e_{(M_{g1},N_{g1})}|k_{g1}\left(\ell^D_{(M_{g1},N_{g1})}\right) = \underline{u}^{-1} \circ e_{(M_{g2},N_{g2})} \circ \underline{u}.$$

Therefore, $\underline{u}$ is a C^{1+H} conjugacy between Φ_{g_1}, M_{g_1} and Φ_{g_1}, M_{g_1}, and so the map $T_{f,M}$ is well defined.

Proof of statement (a). Since the holonomies are $C^{1+\alpha}$ (see A. Pinto and D. Rand [18]), the Hausdorff dimension of the stable leaf segments ℓ is the same independently of the stable leaf segment considered, and so equal to $HD(\Lambda_g^s)$. In particular, all leaf segments $\ell_{M_g} \in I_g$ have the same Hausdorff dimension which is equal to the Hausdorff dimension of L_g. Since the Cantor invariant set $T_{\Phi_{g,M_g}}$ is equal to $k(L_g)$, the Hausdorff dimension $HD(T_{\Phi_{g,M_g}})$ is equal to $HD(\Lambda^s_g)$.

Proof of statement (b). By Lemma 4, if $g \in F$, then the C^{1+H} Cantor exchange system Φ_{g, M_g} is a fixed point of the renormalization operator R_{g,M_g} that, by construction, is the same as $R_{f,M}$. Hence, $T(F) \subset C_R$.

The proof that $T(F) \supset C_R$ follows from the proof of the statement (c) below.

Proof of statement (c). Let Φ be a C^{1+H} Cantor exchange system such that $[R\Phi]_{C^{1+H}} = [\Phi]_{C^{1+H}}$. Since $[R\Phi]_{C^{1+H}} = [\Phi]_{C^{1+H}}$, by Lemma 5, the C^{1+H} Cantor exchange system Φ induces a Markov map F_Φ. Therefore, (Φ, F_Φ) is a self-renormalizable structure as defined in [15] and [16]. By Theorem 1.14 in [15] (see also A. Pinto and D. Rand [16]), there is a one-to-one correspondence between C^{1+H} conjugacy classes of (Φ, F_Φ) and C^{1+H} conjugacy classes of C^{1+H} diffeomorphisms $g(\Phi, F_\Phi)$ with hyperbolic invariant set Λ_g, and with an invariant measure absolutely continuous with respect to the Hausdorff measure.

Proof of statement (d). Let Φ be a C^{1+H} Cantor exchange system such that $[R\Phi]_{C^{1+H}} = [\Phi]_{C^{1+H}}$. Since $[R\Phi]_{C^{1+H}} = [\Phi]_{C^{1+H}}$, by Lemma 5, the C^{1+H} Cantor exchange system Φ induces a Markov map F_Φ. Let C_F be the set of all C^{1+H} conjugacy classes of pairs (Φ, F_Φ). Hence, there is a one-to-one map $m_1 : C_R \to C_F$ given by $m_1(\Phi) = (\Phi, F_\Phi)$. By Lemma 4.2 in [15] (see also A. Pinto and D. Rand [16]), there is a well-defined Teichmüller space TS consisting of solenoid functions, and a one-to-one map $m_2 : TS \to C_F$ given by $m_2(s) = (\Phi, F_\Phi)$. Therefore, $m_1^{-1} \circ m_2 : TS \to C_R$ is a one-to-one map. ✷

Lemma 6. *Let $\Phi_{f,M}$ be a C^{1+H} Cantor exchange system induced by (f, Λ, M). The Cantor exchange system Ψ is a $C^{1+\beta}$ fixed point of the renormalization operator $R_{f,M}$, for some $\beta > 0$, if, and only if, Ψ is induced by a $C^{1+\alpha}$ diffeomorphism with a codimension 1 hyperbolic attractor, with α � β, topologically conjugate to f. In particular, $\Phi_{f,M}$ is a $C^{1+\beta}$ fixed point of the renormalization operator $R_{f,M}$, for some $\beta > 0$.*

Proof. By Theorem 1.14 in [15], given a $C^{1+\alpha}$ Cantor exchange system Ψ and a $C^{1+\alpha}$ Markov map F_Ψ, there is a $C^{1+\alpha}$ diffeomorphism with a codimension 1 hyperbolic attractor set with an invariant measure absolutely continuous with respect to the Hausdorff measure, topologically conjugate to f which induces Ψ. Hence, by Lemma 5, Ψ is a $C^{1+\beta}$ fixed point of renormalization, for some $\beta > 0$, which implies Lemma 6. ✷

Proof of Theorem 2. Let us suppose that the Cantor exchange system Ψ is a $C^{1,\alpha}$ fixed point of the renormalization operator $R_{f,M}$ with $\alpha = HD(T_\Psi)$ and with bounded geometry. Hence, by Lemma 5, Ψ induces a $C^{1,\alpha}$ Markov map F_Ψ. Let ξ be the homeomorphic extension of the conjugacy between Φ and Ψ, and set $\eta = \xi \circ k \circ \pi$. Let T_n be the set of all pairs (I, J) such that (i) I is a stable leaf n-cylinder, (ii) J is a stable leaf n-cylinder or a stable n-gap cylinder, and (iii) I and J have a unique common endpoint (see Appendix A.4). Using the Mean Value Theorem and that F_Ψ is a $C^{1,\alpha}$ Markov map, the function $r: \cup_{n \geq 1} T_n \to \mathbb{R}^+$ given by

$$r(I, J) = \lim_{m \to +\infty} \frac{|\eta \circ f^m(J)|}{|\eta \circ f^m(I)|}$$

is well defined, where $|L|$ means the length of the smallest interval containing $L \subset \mathbb{R}$. By bounded geometry of Ψ, we obtain that r is bounded away from zero. Furthermore, using that F_Ψ is a $C^{1,\alpha}$ Markov map, for every pair $(I, J) \in T_n$, we get

$$\frac{|\eta(J)|}{|\eta(I)|}\left(1 - C_n\left(|\eta(I \cup J)|^\alpha\right)\right) \leqslant r(I, J) \leqslant \frac{|\eta(J)|}{|\eta(I)|}\left(1 + C_n\left(|\eta(I \cup J)|^\alpha\right)\right) \qquad (11)$$

where $C_n \in \mathbb{R}^+_0$ converges to zero when n tends to infinity.

Let $h = h_{(M,N)} : \tilde{\mathcal{E}}^D_{(M,N)} \to \tilde{\mathcal{E}}^C_{(M,N)}$ be a ι-primitive holonomy. Since the Cantor exchange system is $C^{1,\alpha}$, for every $(I, J) \in T_n$ such that $I \cup J \subset \tilde{\mathcal{E}}^D_{(M,N)}$, we get

$$1 - C_n|\eta(I \cup J)|^\alpha \leqslant \frac{|\eta(I)|}{|\eta(J)|}\frac{|\eta \circ h(J)|}{|\eta \circ h(I)|} \leqslant 1 + C_n|\eta(I \cup J)|^\alpha \qquad (12)$$

where $C_n \in \mathbb{R}^+_0$ converges to zero when n tends to infinity.

From (11), we obtain that

$$\frac{|\eta(I)|}{|\eta(J)|}\frac{|\eta \circ h(J)|}{|\eta \circ h(I)|}\left(1 - C_n|\eta(I \cup J)|^\alpha\right) \leqslant \frac{r(h(I), h(J))}{r(I, J)} \leqslant \frac{|\eta(I)|}{|\eta(J)|}\frac{|\eta \circ h_\alpha(J)|}{|\eta \circ h_\alpha(I)|}\left(1 + C_n|\eta(I \cup J)|^\alpha\right).$$

Thus, using (12) we get

$$1 - C'_n|\eta(I \cup J)|^\alpha \leqslant \frac{r(h(I), h(J))}{r(I, J)} \leqslant 1 + C'_n|\eta(I \cup J)|^\alpha$$

where $C'_n \in \mathbb{R}^+_0$ converges to zero when n tends to infinity.

Since $\alpha = HD(T_\Psi)$, by the Rigidity Lemma 4.1 in [17], we obtain that r is a stable transversely affine ratio function (see definition in Appendix A.7). However, putting together Theorem 1 and Lemma 1 in [22], there are no stable transversely affine ratio functions with respect to the stable lamination of Λ_f, and so we get a contradiction. ✷

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Appendix A

In this appendix, we present some basic facts for C^{1+H} hyperbolic diffeomorphisms *(f, Λ)*, that we include for clarity of the exposition. We say that *(f, Λ)* is a C^{1+H} *hyperbolic diffeomorphism*, if *(f, Λ)* has the following properties:

(i) $f: S \to S$ is a $C^{1+\alpha}$ diffeomorphism of a compact surface S with respect to a $C^{1+\alpha}$ structure on S, for some $\alpha > 0$.
(ii) Λ is a hyperbolic invariant subset of S such that $f|\Lambda$ is topologically transitive and Λ has a local product structure.

In particular, a C^{1+H} diffeomorphism f with a codimension 1 attractor Λ is a C^{1+H} hyperbolic diffeomorphism.

A.1. *Leaf segments*

Let d be a metric on M, and define the map $f_\iota = f$ if $\iota = u$, or $f_\iota = f^{-1}$ if $\iota = s$. For $\iota \in \{s, u\}$, if $x \in \Lambda$ we denote the local ι-manifolds through x by

$$W^\iota(x, \varepsilon) = \{ y \in M:\ d(f_\iota^{-n}(x), f_\iota^{-n}(y)) \leqslant \varepsilon,\ \text{for all } n \geqslant 0 \}.$$

By the Stable Manifold Theorem (see [9] and [25]), these sets are respectively contained in the stable and unstable immersed manifolds

$$W^\iota(x) = \bigcup_{n \geqslant 0} f_\iota^n\big(W^\iota(f_\iota^{-n}(x), \varepsilon_0)\big)$$

which are the image of a $C^{1+\gamma}$ immersion $\kappa_{\iota,x} : \mathbb{R} \to M$. An *open* (respectively *closed*) *full ι-leaf segment* I is defined as a subset of $W^\iota(x)$ of the form $\kappa_{\iota,x}(I_1)$ where I_1 is an open (respectively closed) subinterval (non-empty) in R. An *ι-open* (respectively *closed*) *leaf segment* is the intersection with Λ of a full open (respectively closed) ι-leaf segment such that the intersection contains at least two distinct points. If the intersection is exactly two points we call this ι-closed leaf segment an *ι-leaf gap*. An *ι-full leaf segment* is either an open or closed ι-full leaf segment. An *ι-leaf segment* is either an open or closed ι-leaf segment. The *endpoints* of a full ι-leaf seg- ment are the points $\kappa_{\iota,x}(u)$ and $\kappa_{\iota,x}(v)$ where u and v are the endpoints of I_1. The *endpoints* of an ι-leaf segment I are the points of the minimal closed full ι-leaf segment containing I. The *interior* of a ι-leaf segment I is the complement of its boundary. In particular, a ι-leaf segment
I has empty interior if, and only if, it is an ι-leaf gap. A map $c: I \to \mathbb{R}$ is an *ι-leaf chart* of an ι-leaf segment I if has an extension $c_E : I_E \to \mathbb{R}$ to a full ι-leaf segment I_E with the following properties: $I \subset I_E$ and c_E is a homeomorphism onto its image. An *ι-full leaf segment* is either an open or close full leaf segment.

A.2. Rectangles

Since Λ is a hyperbolic invariant set of a diffeomorphism $f : M \to M$, for $0 < \varepsilon < \varepsilon_0$ there is $\delta = \delta(\varepsilon) > 0$, such that for all points $w, z \in \Lambda$ with $d(w, z) < \delta$, $W^u(w, \varepsilon)$ and $W^s(z, \varepsilon)$ intersect in an unique point that we denote by $[w, z]$. Since we assume that the hyperbolic set has a *local product structure*, we have that $[w, z] \in \Lambda$. Furthermore, the following properties are satisfied: (i) $[w, z]$ varies continuously with $w, z \in \Lambda$; (ii) the bracket map is continuous on a δ-uniform neighborhood of the diagonal in $\Lambda \times \Lambda$; and (iii) whenever both sides are defined $f([z, w]) = [f(z), f(w)]$. Note that the bracket map does not really depend on δ provided it is sufficiently small.

Let us underline that it is a standing hypothesis that all the hyperbolic sets considered here have such a local product structure.

A *rectangle* R is a subset of Λ which is (i) closed under the bracket, i.e., $x, y \in R \Rightarrow [x, y] \in R$, and (ii) proper, i.e., is the closure of its interior in Λ. This definition imposes that a rectangle has always to be proper which is more restrictive than the usual one which only insists on the closure condition.

If $\hat{\ell}^s$ and $\hat{\ell}^u$ are respectively stable and unstable leaf segments intersecting in a single point then we denote by $[\hat{\ell}^s, \hat{\ell}^u]$ the set consisting of all points of the form $[w, z]$ with $w \in \hat{\ell}^s$ and $z \in \hat{\ell}^u$. We note that if the stable and unstable leaf segments $\hat{\ell}$ and $\hat{\ell}$ are closed then the set $[\hat{\ell}, \hat{\ell}]$ is a rectangle. Conversely in this 2-dimensional situations, any rectangle R has a product structure in the following sense: for each $x \in R$ there are closed stable and unstable leaf segments of Λ, $\hat{\ell}^s(x, R) \subset W^s(x)$ and $\hat{\ell}^u(x, R) \subset W^u(x)$ such that $R = [\hat{\ell}^s(x, R), \hat{\ell}^u(x, R)]$. The leaf segments $\hat{\ell}^s(x, R)$ and $\hat{\ell}^u(x, R)$ are called *stable and unstable spanning leaf segments* for R. For $\iota \in \{s, u\}$, we denote by $\partial \hat{\ell}^\iota(x, R)$ the set consisting of the endpoints of $\hat{\ell}^\iota(x, R)$, and we denote by $\operatorname{int} \hat{\ell}^\iota(x, R)$ the set $\hat{\ell}^\iota(x, R) \setminus \partial \hat{\ell}^\iota(x, R)$. The *interior of R* is given by $\operatorname{int} R = [\operatorname{int} \hat{\ell}^s(x, R), \operatorname{int} \hat{\ell}^u(x, R)]$, and the *boundary of R* is given by $\partial R = [\partial \hat{\ell}^s(x, R), \hat{\ell}^u(x, R)] \cup [\hat{\ell}^s(x, R), \partial \hat{\ell}^u(x, R)]$.

A.3. Markov partitions

By Theorem 3.12 in page 79 of [2] (see also Sinai [26]), a *Markov partition of f* is a collection $R = \{R_1, \ldots, R_k\}$ of such rectangles such that (i) $\Lambda \subset \bigcup_{i=1}^{k} R_i$; (ii) $R_i \cap R_j = \partial R_i \cap \partial R_j$ for all i and j; (iii) if $x \in \operatorname{int} R_i$ and $fx \in \operatorname{int} R_j$ then

(a) $f(\hat{\ell}^s(x, R_i)) \subset \hat{\ell}^s(fx, R_j)$ and $f^{-1}(\hat{\ell}^u(fx, R_j)) \subset \hat{\ell}^u(x, R_i)$;
(b) $f(\hat{\ell}^u(x, R_i)) \cap R_j = \hat{\ell}^u(fx, R_j)$ and $f^{-1}(\hat{\ell}^s(fx, R_j)) \cap R_i = \hat{\ell}^s(x, R_i)$.

The last condition means that $f(R_i)$ goes across R_j just once. In fact, it follows from condition (a) providing the rectangles R_j are chosen sufficiently small (see Mañé [10]). The rectangles which make up the Markov partition are called *Markov rectangles*.

We note that there is a Markov partition R of f with the following *disjointness property* (see R. Bowen [2], S. Newhouse and J. Palis [12], Ya. Sinai [26]):

(i) If $0 < \delta_{f,s} < 1$ and $0 < \delta_{f,u} < 1$ then the stable and unstable leaf boundaries of any two Markov rectangles do not intersect.
(ii) If $0 < \delta_{f,\iota} < 1$ and $0 < \delta_{f,\iota^1} = 1$ then the ι^1-leaf boundaries of any two Markov rectangles do not intersect except, possibly, at their endpoints.

If $\delta_{f,s} = \delta_{f,u} = 1$, the disjointness property does not apply and so we consider that it is trivially satisfied for every Markov partition. For simplicity of our exposition, we consider Markov partitions that satisfy the disjointness property. This result is also used in [5–7,20,21,23] and [24].

A.4. Cylinders and gaps

For $\iota = s$ or u, an *ι-leaf primary cylinder of a Markov rectangle R* is a spanning ι-leaf segment of R. For n � 1, an *ι-leaf n-cylinder of R* is an ι-leaf segment I such that

(i) $f_\iota^n I$ is an ι-leaf primary cylinder of a Markov rectangle M;

(ii) $f_\iota^n(\pounds^{\iota^1}(x, R)) \subset M$ for every $x \in I$.

For n � 2, an *ι-leaf n-gap G of R* is an ι-leaf gap $\{x, y\}$ in a Markov rectangle R such that n is the smallest integer such that both leaves $f_\iota^{n-1}\pounds^{\iota^1}(x, R)$ and $f_\iota^{n-1}\pounds^{\iota^1}(y, R)$ are contained in ι^1-boundaries of Markov rectangles; an *ι-leaf primary gap G* is the image $f_\iota G^1$ by f_ι of an ι-leaf 2-gap G^1.

Let f be a diffeomorphisms with a codimension 1 hyperbolic attractor and π be the projection as constructed in (3). The projection $\pi(I)$ of a stable leaf n-cylinder I is in the n-level $L_0^{(n)}$ of the partition of $\Phi_{f,M}$ (see definition of $L_0^{(n)}$ in Section 1.5).

A.5. Basic holonomies

Suppose that x and z are two points inside any rectangle R of Λ. Let I and J be two stable leaf segments respectively containing x and z and inside R. Then we define $h : I \to J$ by $h(w) = [w, z]$. Such maps are called the *basic stable holonomies*. They generate the pseudo-group of all stable holonomies. Similarly we define the basic unstable holonomies.

A.6. Conjugacies

Let (f, Λ) be a C^{1+H} hyperbolic diffeomorphism. Somewhat unusually we also desire to highlight the C^{1+H} structure on M in which f is a diffeomorphism. By a *C^{1+H} structure on M* we mean a maximal set of charts with open domains in M such that the union of their domains cover M and whenever U is an open subset contained in the domains of any two of these charts i and j then the overlap map $j \circ i^{-1} : i(U) \to j(U)$ is $C^{1+\alpha}$, where $\alpha > 0$ depends on i, j and U. We note that by compactness of M, given such a C^{1+H} structure on M, there is an atlas consisting of a finite set of these charts which cover M and for which the overlap maps are $C^{1+\alpha}$ compatible and uniformly bounded in the $C^{1+\alpha}$ norm, where $\alpha > 0$ just depends upon the atlas.

We denote by $\mathcal{C}_f$ the C^{1+H} structure on M in which f is a diffeomorphism. Usually one is not concerned with this as, given two such structures, there is a homeomorphism of M sending one onto the other and thus, from this point of view, all such structures can be identified. For our discussion it will be important to maintain the identity of the different smooth structures on M.

We say that a map $h : \Lambda_f \to \Lambda_g$ is a *topological conjugacy* between two C^{1+H} hyperbolic diffeomorphisms (f, Λ_f) and (g, Λ_g) if there is a homeomorphism $h : \Lambda_f \to \Lambda_g$ with the following properties:

(i) $g \circ h(x) = h \circ \boldsymbol{f}(x)$ for every $x \in \boldsymbol{\Lambda}_{\boldsymbol{f}}$.

(ii) The pull-back of the ι-leaf segments of g by h are ι-leaf segments of $\boldsymbol{f}$.

Definition 6.1. Let F be the set of all C^{1+H} hyperbolic diffeomorphisms (g,Λ_g) such that (g,Λ_g) and (f,Λ) are topologically conjugate by h.

A.7. *HR-Hölder ratios*

A *HR-structure* associates an affine structure to each stable and unstable leaf segment in such a way that these vary Hölder continuously with the leaf and are invariant under f.

An affine structure on a stable or unstable leaf is equivalent to a *ratio function* $r(I : J)$ which can be thought of as prescribing the ratio of the size of two leaf segments I and J in the same stable or unstable leaf. A *ratio function* $r(I : J)$ is positive (we recall that each leaf segment has at least two distinct points) and continuous in the endpoints of I and J. Moreover,

$$r(I:J)=r(J:I)^{-1} \quad \text{and} \quad r(I_1\cup I_2:K)=r(I_1:K)+r(I_2:K) \tag{13}$$

provided I_1 and I_2 intersect at most in one of their endpoints.

We say that r is a *ι-ratio function* if (i) for all ι-leaf segments K, $r(I:J)$ $(I, J \subset K)$ defines a ratio function on K; (ii) r is invariant under f, i.e., $r(I:J)=r(fI:fJ)$ for all ι-leaf segments; and (iii) for every basic ι-holonomy map $\vartheta : I \to J$ between the leaf segment I and the leaf segment J defined with respect to a rectangle R and for every ι-leaf segment $I_0 \subset I$ and every ι-leaf segment or gap $I_1 \subset I$,

$$\left|\log\frac{r(\theta I_0:\theta I_1)}{r(I_0:I_1)}\right| \leqslant \mathcal{O}\big((d_\Lambda(I,J))^{\epsilon}\big) \tag{14}$$

where $\in (0, 1)$ depends upon r and the constant of proportionality also depends upon R, but not on the segments considered. Since r satisfies the condition (14) and defines an affine structure on each leaf that is f-invariant we say that r is a transversely Hölder ι-ratio function. A *HR-structure*
is a pair (r_s, r_u) consisting of a stable and an unstable ratio function.

Definition 6.2. If an ι-ratio function r is invariant under holonomies h (i.e., $r(I : J) = r(h(I) : h(J))$), then we say that r is a *transversely affine ι-ratio function*.

A.8. *Realized ratio functions*

Let $g \in F$ and $\rho = \rho_g$ be a C^{1+} Riemannian metric on the manifold containing Λ. The *ι-lamination atlas $A^\iota(g,\rho)$ determined by ρ* is the set of all maps $e : I \to \mathbb{R}$ where $I = \Lambda \cap \hat{I}$ with $\hat{I}$ a full ι-leaf segment, such that e extends to an isometry between the induced Riemannian metric on $\hat{I}$ and the Euclidean metric on the reals. We call the maps $e \in A^\iota(g,\rho)$ the *ι-lamination charts*. If I is an ι-leaf segment (or a full ι-leaf segment) then by $|I| = |I|_\rho$ we mean the length in
the Riemannian metric ρ of the minimal full ι-leaf containing I. By hyperbolicity of g in Λ, there are $0 < \nu < 1$ and $C > 0$ such that for all ι-leaf segments I and all m � 0 we get $|g_\iota^m I|$ � $C\nu^m |I|$. Thus, using the mean value theorem and the fact that g_ι is C^r, for all short leaf segments K and all leaf segments I and J contained in it, the ι-realized ratio function $r_{g,\iota}$ given by

$$r_{g,\iota}(I:J)=\lim_{n\to\infty}\frac{|g_{\iota'}^n I|}{|g_{\iota'}^n J|}$$

$$
\begin{aligned}
&= \frac{|g_{\iota'}^m I|}{|g_{\iota'}^m J|} \prod_{n=m}^{\infty} \left(\frac{|g_{\iota'}^{n+1} I|}{|g_{\iota'}^{n+1} J|} \frac{|g_{\iota'}^n J|}{|g_{\iota'}^n I|} \right) \\
&\in \frac{|g_{\iota'}^m I|}{|g_{\iota'}^m J|} \prod_{n=m}^{\infty} \left(1 \pm \mathcal{O}(\nu^n |K|^{\alpha})\right) \\
&\subset \frac{|g_{\iota'}^m I|}{|g_{\iota'}^m J|} \left(1 \pm \mathcal{O}(\nu^n |K|^{\alpha})\right)
\end{aligned}
$$

is well defined, where $\alpha = \min\{1, r-1\}$. This construction gives the HR-structure on Λ determined by g. By [19], we get the following equivalence:

Theorem 3. *The map $g \to (r_{g,s}, r_{g,u})$ determines a one-to-one correspondence between C^{1+H} conjugacy classes in F and HR-structures.*

A.9. Lamination atlas

Given an ι-ratio function r, we define the embeddings $e : I \to \mathbb{R}$ by

$$e(x) = r\big(\ell(\xi, x), \ell(\xi, R)\big) \tag{15}$$

where ξ is an endpoint of the ι-leaf segment I and R is a Markov rectangle containing ξ (see Fig. 2). For this definition it is not necessary that R contains I. We denote the set of all these embeddings e by $A(r)$.

Let $g \in F$ and $A(g, \rho)$ the ι-lamination atlas determined by a Riemannian metric ρ. Putting together Propositions 2.5 and 3.5 of [19], we get that the overlap map $e_1 \circ e_2^{-1}$ between a chart $e_1 \in A(g,\rho)$ and a chart $e_2 \in A(r_{g,\iota})$ has a C^{1+H} diffeomorphic extension to the reals. Therefore, for all short leaf segments K and all leaf segments I and J contained in it, we obtain that

$$r_{g,\iota}(I : J) = \lim_{n\to\infty} \frac{|g_{\iota'}^n I|_{\rho}}{|g_{\iota'}^n J|_{\rho}} = \lim_{n\to\infty} \frac{|g_{\iota'}^n I|_{i_n}}{|g_{\iota'}^n J|_{i_n}}$$

where i_n is any chart in $A(r_{g,\iota})$ containing the segment $g_{\iota'}^n K$ in its domain.

A.10. Realized solenoid functions

For $\iota = s$ and u, let $\mathbf{S}^{\iota}$ denote the set of all ordered pairs (I, J) of ι-leaf segments with the following properties:

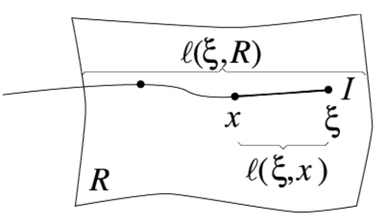

Fig. 2. The embedding $e : I \to \mathbb{R}$.

(i) The intersection of I and J consists of a single endpoint.
(ii) If $\delta_{f,\iota} = 1$ then I and J are primary ι-leaf cylinders.
(iii) If $0 < \delta_{f,\iota} < 1$ then $f_\iota I$ is an ι-leaf 2-cylinder of a Markov rectangle R and $f_\iota J$ is an ι-leaf 2-gap also of the same Markov rectangle R.

See Appendix A.4 for the definitions of leaf cylinders and gaps. Pairs (I, J) where both are primary cylinders are called *leaf-leaf pairs*. Pairs (I, J) where J is a gap are called *leaf-gap pairs* and in this case we refer to J as a *primary gap*. The set $\mathbf{S}^\iota$ has a very nice topological structure. If $\delta_{f,\iota} = 1$ then the set $\mathbf{S}^\iota$ is isomorphic to a finite union of intervals, and if $\delta_{f,\iota} < 1$ then the set $\mathbf{S}^\iota$ is isomorphic to an embedded Cantor set.

We define a pseudo-metric $d_{\mathbf{S}^\iota} : \mathbf{S}^\iota \times \mathbf{S}^\iota \to \mathbb{R}^+$ on the set $\mathbf{S}^\iota$ by

$$d_{\mathbf{S}^\iota}\big((I,J),(I',J')\big) = \max\{d_\Lambda(I,I'), d_\Lambda(J,J')\}.$$

Let $g \in T(f, \Lambda)$. For $\iota = s$ and u, we call the restriction of an ι-ratio function $r_{g,\iota}$ to $\mathbf{S}^i$ a *realized solenoid function* $\sigma_{g,\iota}$. By construction, for $\iota = s$ and u, the restriction of an ι-ratio function to $\mathbf{S}^i$ gives an Hölder continuous function satisfying the matching condition, the boundary condition
and the cylinder-gap condition as we now proceed to describe.

A.11. *Hölder continuity of solenoid functions*

This means that for $t = (I, J)$ and $t^1 = (I^1, J^1)$ in $\mathbf{S}^\iota$, $|\sigma_\iota(t) - \sigma_\iota(t^1)|$ � $O((d_{\mathbf{S}^\iota}(t, t^1))^\alpha)$. The Hölder continuity of $\sigma_{g,\iota}$ and the compactness of its domain imply that $\sigma_{g,\iota}$ is bounded away from zero and infinity.

A.12. *Matching condition*

Let $(I, J) \in \mathbf{S}^\iota$ be a pair of primary cylinders and suppose that we have pairs

$$(I_0, I_1), (I_1, I_2), \ldots, (I_{n-2}, I_{n-1}) \in \mathbf{S}^\iota$$

of primary cylinders such that $f_\iota I = \bigcup_{j=0}^{k-1} I_j$ and $f_\iota J = \bigcup_{j=k}^{n-1} I_j$. Then

$$\frac{|f_\iota I|}{|f_\iota J|} = \frac{\sum_{j=0}^{k-1}|I_j|}{\sum_{j=k}^{n-1}|I_j|} = \frac{1 + \sum_{j=1}^{k-1}\prod_{i=1}^{j}|I_i|/|I_{i-1}|}{\sum_{j=k}^{n-1}\prod_{i=1}^{j}|I_i|/|I_{i-1}|}.$$

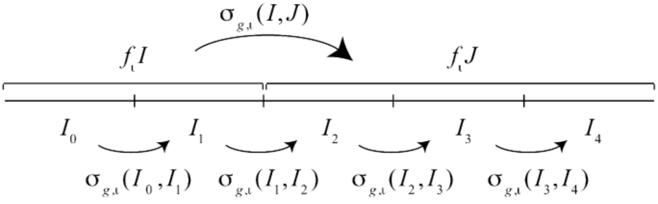

Fig. 3. The f-matching condition for ι-leaf segments.

Hence, noting that $g|\Lambda = f|\Lambda$, the realized solenoid function $\sigma_{g,\iota}$ must satisfy the *matching condition* (see Fig. 3) for all such leaf segments:

$$\sigma_{g,\iota}(I:J) = \frac{1 + \sum_{j=1}^{k-1} \prod_{i=1}^{j} \sigma_{g,\iota}(I_i : I_{i-1})}{\sum_{j=k}^{n-1} \prod_{i=1}^{j} \sigma_{g,\iota}(I_i : I_{i-1})}. \tag{16}$$

A.13. Boundary condition

If the stable and unstable leaf segments have Hausdorff dimension equal to 1, then leaf seg- ments I in the boundaries of Markov rectangles can sometimes be written as the union of primary cylinders in more than one way. This gives rise to the existence of a boundary condition that the realized solenoid functions have to satisfy as we pass to explain.

If J is another leaf segment adjacent to the leaf segment I then the value of $|I|/|J|$ must be the same whichever decomposition we use. If we write $J = I_0 = K_0$ and I as $\bigcup_{i=1}^{m} I_i$ and $\bigcup_{j=1}^{n} K_j$ where the I_i and K_j are primary cylinders with $I_i \neq K_j$ for all i and j, then the above two ratios are

$$\sum_{i=1}^{m} \prod_{j=1}^{i} \frac{|I_j|}{|I_{j-1}|} = \frac{|I|}{|J|} = \sum_{i=1}^{n} \prod_{j=1}^{i} \frac{|K_j|}{|K_{j-1}|}.$$

Thus, noting that $g|\Lambda = f|\Lambda$, a realized solenoid function $\sigma_{g,\iota}$ must satisfy the following *bound- ary condition* (see Fig. 4) for all such leaf segments:

$$\sum_{i=1}^{m} \prod_{j=1}^{i} \sigma_{g,\iota}(I_j : I_{j-1}) = \sum_{i=1}^{n} \prod_{j=1}^{i} \sigma_{g,\iota}(K_j : K_{j-1}). \tag{17}$$

A.14. Scaling function

If the ι-leaf segments have Hausdorff dimension less than one and the ι^1-leaf segments have Hausdorff dimension equal to 1, then a primary cylinder I in the ι-boundary of a Markov rec- tangle can also be written as the union of gaps and cylinders of other Markov rectangles. This gives rise to the existence of a cylinder-gap condition that the ι-realized solenoid functions have to satisfy.

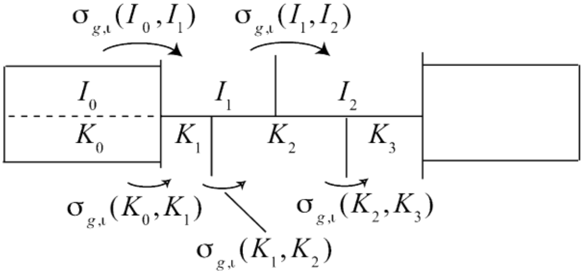

Fig. 4. The boundary condition for ι-leaf segments.

Before defining the cylinder-gap condition, we will introduce the scaling function that will be useful to express the cylinder-gap condition.

Let scl^{ι} be the set of all pairs (K, J) of ι-leaf segments with the following properties:

(i) K is a leaf n_1-cylinder or an n_1-gap segment for some $n_1 > 1$;
(ii) J is a leaf n_2-cylinder or an n_2-gap segment for some $n_2 > 1$;
(iii) $m^{n_1-1}K$ and $m^{n_2-1}J$ are the same primary cylinder.

Lemma 7. *Every function $\sigma_\iota : \mathbf{S}^{\iota} \to \mathrm{R}^+$ has a canonical extension s_ι to scl^{ι}. Furthermore, if σ_ι is the restriction of a ratio function $r_\iota|\mathbf{S}^{\iota}$ to $\mathbf{S}^{\iota}$ then $s_\iota = r_\iota|\mathrm{scl}^{\iota}$.*

See proof of Lemma 7 in [15].

Remark 3. The above map $s_\iota : \mathrm{scl}^{\iota} \to \mathrm{R}^+$ is the *scaling function* determined by the solenoid function $\sigma_\iota : \mathbf{S}^{\iota} \to \mathrm{R}^+$.

A.15. *Cylinder-gap condition*

Let (I, K) be a leaf-gap pair such that the primary cylinder I is the ι-boundary of a Markov rectangle R_1. Then the primary cylinder I intersects another Markov rectangle R_2 giving rise to the existence of a cylinder-gap condition that the realized solenoid functions have to satisfy as we proceed to explain. Take the smallest $l \geq 0$ such that $f^l_{\iota^1} I \cup f^l_{\iota^1} K$ is contained in the intersection of the boundaries of two Markov rectangles M_1 and M_2. Let M_1 be the Markov rectangle with the property that $M_1 \cap f^l_{\iota^1} R_1$ is a rectangle with non-empty interior (and so $M_2 \cap f^l_{\iota^1} R_2$ also has non-empty interior). Then, for some positive n, there are distinct n-cylinder and gap leaf segments $J_1, \ldots, J_m$ contained in a primary cylinder of M_2 such that $f^l_{\iota^1} K = J_m$ and the smallest full ι-leaf segment containing $f^l_{\iota^1} I$ is equal to the union $\bigcup_{i=1}^{m-1} \hat{J}_i$, where $\hat{J}_i$ is the smallest full ι-leaf segment containing J_i. Hence,

$$\frac{|f^l_{\iota'} I|}{|f^l_{\iota'} K|} = \sum_{i=1}^{m-1} \frac{|J_i|}{|J_m|}.$$

Hence, noting that $g|\Lambda = f|\Lambda$, a realized solenoid function $\sigma_{g,\iota}$ must satisfy the *cylinder-gap condition* (see Fig. 5) for all such leaf segments:

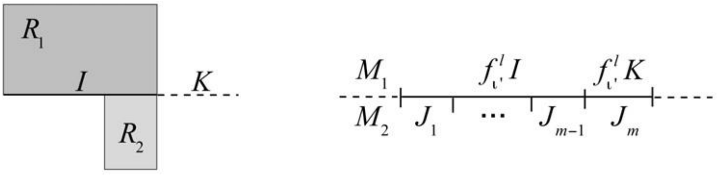

Fig. 5. The cylinder-gap condition for ι-leaf segments.

$$\sigma_{g,\iota}(I,K) = \sum_{i=1}^{m-1} s_{g,\iota}(J_i, J_m)$$

where $s_{g,\iota}$ is the scaling function determined by the solenoid function $\sigma_{g,\iota}$.

A.16. Solenoid functions

Now, we are ready to present the definition of an ι-solenoid function.

Definition 1. An Hölder continuous function $\sigma_\iota : \mathbf{S}^\iota \to \mathbb{R}^+$ is an *ι-solenoid function* if σ_ι satisfies the matching condition, the boundary condition and the cylinder-gap condition.

We denote by $PS(f)$ the set of pairs (σ_s, σ_u) of stable and unstable solenoid functions.

Remark 4. Let $\sigma_\iota : \mathbf{S}^\iota \to \mathbb{R}^+$ be an ι-solenoid function. The matching, the boundary and the cylinder-gap conditions are trivially satisfied except in the following cases:

(i) The matching condition if $\delta_{f,\iota} = 1$.
(ii) The boundary condition if $\delta_{f,s} = \delta_{f,u} = 1$.
(iii) The cylinder-gap condition if $\delta_{f,\iota} < 1$ and $\delta_{f,\iota'} = 1$.

Theorem 4. *The map $r_\iota \to r_\iota | \mathbf{S}^\iota$ gives a one-to-one correspondence between ι-ratio functions and ι-solenoid functions.*

See proof of Theorem 4 in [15].

The set $PS(f)$ of all pairs (σ_s, σ_u) has a natural metric. Combining Theorem 3 with Theorem 4, we obtain that the set $PS(f)$ forms a moduli space for the C^{1+H} conjugacy classes of C^{1+H} hyperbolic diffeomorphisms $g \in T(f, \Lambda)$:

Corollary 1. *The map $g \to (r_{g,s} | \mathbf{S}^s, r_{g,u} | \mathbf{S}^u)$ determines a one-to-one correspondence between C^{1+H} conjugacy classes of $g \in T(f, \Lambda)$ and pairs of solenoid functions in $PS(f)$.*

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