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Comparing Integer and Fractional Models in some Electrical Systems

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Abstract: The Fractional Calculus (FC) is a mathematical tool applied in scientific areas such as electricity, magnetism, fluid dynamics and biology. This paper describes the use of integer and fractional electrical elements, for modelling several diffusion systems. Experimental and theoretical results are analyzed in the frequency domain and the pros and cons of adopting fractional order electrical components for modelling these systems are compared.

Keywords: Fractional calculus, modelling, electrical conduction, electrical circuits.

1. INTRODUCTION

Fractional Calculus (FC) is a generalization of the integration and differentiation to a non-integer order. The fundamental operator is ${}_aD_t^\alpha$, where the order α is a real or, even, complex number, and the subscripts a and t represent the two limits of the operation [Oldham and Spanier (1974a), Oldham and Spanier (1974b), Samko et al. (1993), Miller and Ross (1993) and Kilbas et al. (2006)].

Recent studies brought FC into attention revealing that many physical phenomena can be modelled by fractional differential equations [Debnath (2002), Machado and Jesus (2005), and Jesus et al. (2006b)]. The importance of fractional order models is that they yield a more accurate description and lead to a deeper insight into the physical processes underlying a long range memory behavior.

Capacitors are one of the crucial elements in integrated circuits and are used extensively in many electronic systems [Samavati et al. (1998)]. However, Jonscher [Jonscher (1993)] demonstrated that the ideal capacitor cannot exist in nature, because an impedance of the form $1/[(j\omega C)]$ would violate causality [Bohannan (2002a) and Bohannan (2002b)]. In fact, the dielectric materials exhibit a fractional behavior yielding electrical impedances of the form $1/[(\mathrm{j}\omega C_F)^\alpha]$, with $\alpha \in \Re^+$ [Westerlund and Ekstam (1994)].

Bearing these ideas in mind, this paper analyzes the fractional modelling of several electrical devices and is organized as follows. Section 2 introduces the fundamental concepts of electrical impedances. Sections 3 and 4 describe botanical elements and fractal capacitors, respectively, and presents the experimental results for both the cases. Finally, section 5 draws the main conclusions.

2. ON THE ELECTRICAL IMPEDANCE

In an electrical circuit the voltage $u(t)$ and the current $i(t)$ can be expressed as a function of time t:

$$u(t) = U_0 \cos(\omega t) \tag{1}$$

$$i(t) = I_0 \cos(\omega t + \phi) \tag{2}$$

where U_0 and I_0 are the amplitudes of the signals, ω is the angular frequency and ϕ is the current phase shift. The voltage and current can be expressed in complex form as:

$$u(t) = Re\left\{U_0 e^{j\omega t}\right\} \tag{3}$$

$$i(t) = Re\left\{I_0 e^{j(\omega t+\phi)}\right\} \tag{4}$$

where Re{ } represents the real part and $j = \sqrt{-1}$.

Consequently, in complex form the electrical impedance $Z(j\omega)$ is given by the expression:

$$Z(j\omega) = \frac{U(j\omega)}{I(j\omega)} = Z_0 e^{j\phi} \tag{5}$$

Fractional order elements occur in several fields of engineering.

A brief reference about the Constant Phase Element (CPE) and the Warburg impedance is presented here due to their application in the work. In fact, to model an electrochemical phenomenon it is often used a CPE due to the fact that the surface is not homogeneous [Barsoukov and Macdonald (2005)].

With a CPE we have the expression:

$$Z(j\omega) = \frac{1}{(j\omega C_F)^\alpha} \tag{6}$$

where C_F is a fractional order capacitance and the fractional order $0 < \alpha \leq 1$ is a parameter, occurring the classical ideal capacitor when $\alpha = 1$. It should be noted that, the SI base units of the C_F element are

$[m^{-2/\alpha}kg^{-1/\alpha}s^{(\alpha+3)/\alpha}A^{2/\alpha}]$ [Jesus et al. (2006a) and Jesus et al. (2009)].

Table 1 shows simples series and parallel element associations of integer and fractional order for constructing electrical circuits.

It is well known that, in electrochemical systems with diffusion, the impedance is modelled by the so-called Warburg element [Barsoukov and Macdonald (2005) and Jesus et al. (2009)]. The Warburg element arises from one-dimensional diffusion of an ionic species to the electrode. If the impedance is under an infinite diffusion layer, the Warburg impedance is given by:

$$Z(j\omega) = \frac{R}{(j\omega C_F)^{0.5}} \tag{7}$$

where R is the diffusion resistance. If the diffusion process has finite length, the Warburg element becomes:

$$Z(j\omega) = R\frac{\tanh{(j\omega\tau)}^{0.5}}{(\tau)^{0.5}} \tag{8}$$

with $\tau = \delta^2/D$, where R is the diffusion resistance, τ is the diffusion time constant, δ is the diffusion layer thickness and D is the diffusion coefficient [Jesus et al. (2009) and Jesus and Machado (2009)].

Based on these concepts, and in the previous works developed by the authors [Jesus et al. (2006a), Jesus et al. (2009), Jesus et al. (2008) and Jesus and Machado (2009)], we verify that the $Z(j\omega)$ of the fruits, vegetables and also fractal capacitors, exhibit distinct characteristics according with the frequency range.

This different behavior, for low and for high frequencies difficults the modelling of these systems in all frequency ranges. This fact, motivates us to study both systems, with different type of RC electrical approximation circuits, namely the use of series and parallel two-element associations of integer and fractional order.

In this line of thought, in the following sections the impedances of these systems are analyzed.

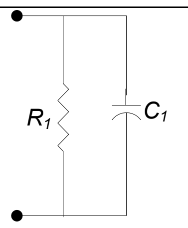
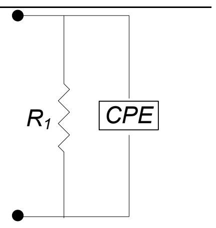
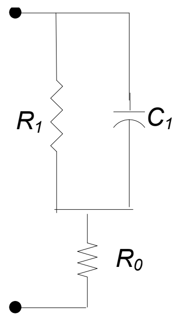
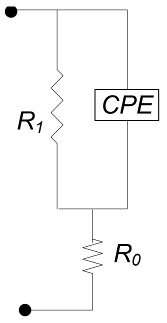
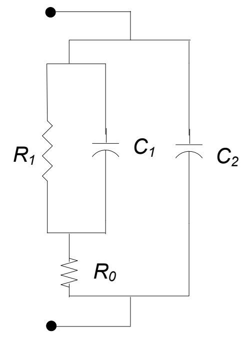
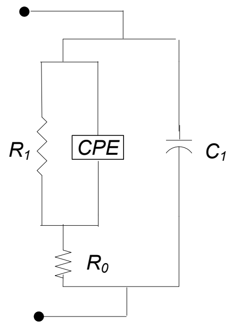
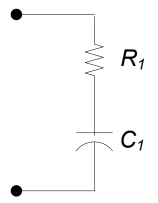
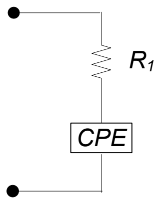
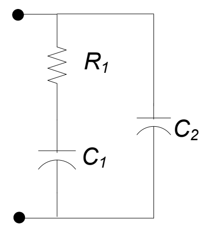
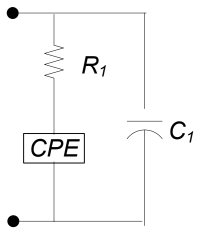
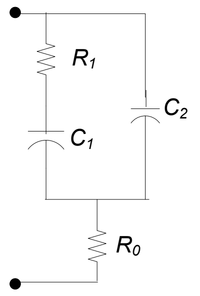
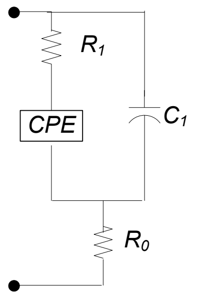
3. BOTANICAL ELEMENTS

The structure of fruits and vegetables have cells that are sensitive to heat, pressure and other stimuli. These systems constitute electrical circuits exhibiting a complex behavior. Bearing these facts in mind, in our work we study the electrical impedance of the *Solanum Tuberosum* (the common potato), under the point of view of FC.

We apply sinusoidal excitation signals $v(t)$, to the potato system, for several distinct frequencies ω (Fig. 1) and the impedance $Z(j\omega)$ is measured based on the resulting voltage $u(t)$ and current $i(t)$.

We analyze the impedance for an amplitude of input signal of $V_0 = 10$ volt, a constant adaptation resistance $R_a = 15$ kΩ, applied to one potato, with an weight $W = 1.24\ 10^{-1}$ kg, environmental temperature $T = 26.5$ degree Celsius, dimension $D = 7.97\ 10^{-2}{\times}5.99\ 10^{-2}$ m, and the electrode length penetration $\Delta = 2.1\ 10^{-2}$ m. Figure 2 presents the polar diagram for the $Z(j\omega)$.

Table 1. Elementar circuits of integer and fractional order.

Case	Circuit	Case	Circuit
1		2	
3		4	
5		6	
7		8	
9		10	
11		12	

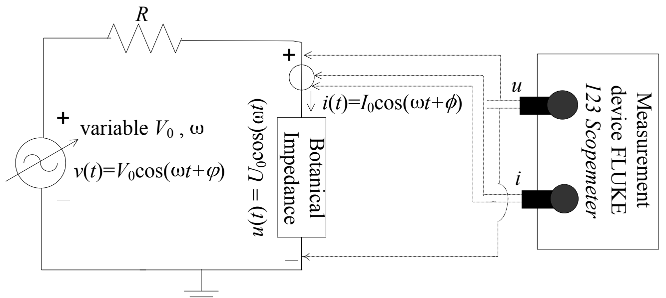


Fig. 1. Electrical circuit for the measurement of the botanical impedance $Z(j\omega)$.

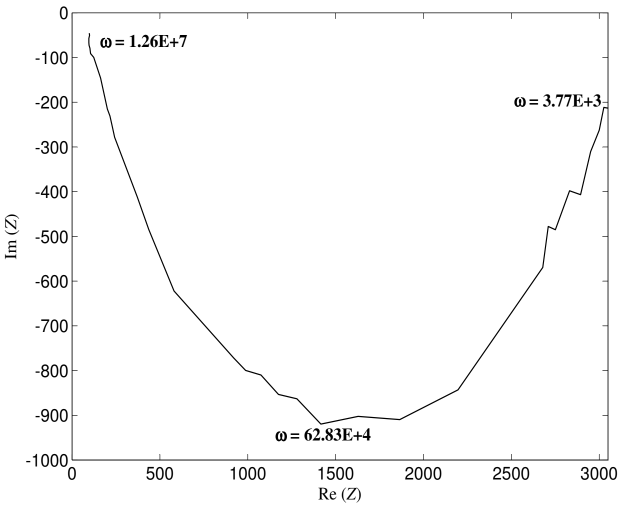


Fig. 2. Polar diagrams of the impedance $Z(j\omega)$ for the *Solanum Tuberosum*.

For the approximate modelling of the results presented in figure 2, we must have in mind the polar plots of the impedance $Z(j\omega)$ for the circuits presented in the Table 1.

In the botanical system are applied the circuits identified by cases 1 up to 6 for modelling $Z(j\omega)$. It is minimized the error (J) between the polar diagram of the experimental data (Z) and the approximation model (Z_{app}), in the perspective of the expression:

$$J = \frac{(Re_Z - Re_{Z_{app}})^2 + (Im_Z - Im_{Z_{app}})^2}{(Re_Z + Re_{Z_{app}})^2 + (Im_Z + Im_{Z_{app}})^2} \quad (9)$$

The resulting numerical values of R_0, R_1, C_1, C_2, C_F and α for the different impedances are shown in Table 2.

Figure 3 presents the approximation error (J) as function as the number of electrical elements (E) and the number of parameters (P) to be adjusted for the cases 1 to 6.

This figure reveals a significant decreasing of the error J with the number of elements and parameters. In figure 4 we present the polar diagrams for $Z(j\omega)$, and the approximations $Z_{app2}(j\omega)$, $Z_{app4}(j\omega)$, $Z_{app5}(j\omega)$ and $Z_{app6}(j\omega)$, for the *Solanum Tuberosum* revealing a very good fit and an error of $J = 1.971$, $J = 0.063$, $J = 0.570$ and $J = 0.060$, respectively.

Table 2. Comparison of circuit parameters for the cases 1 up to 6, for the *Solanum Tuberosum*.

Case	R_0	R_1	C_1	C_2	C_F	α
1		2487.00	1.70E−9			
2		2271.00			1.60E−8	0.84
3	116.00	2309.00	3.70E−9			
4	55.70	3198.00			2.30E−7	0.68
5	115.00	2325.00	3.70E−9	5.00E−14		
6	61.50	3274.00		533800.00	2.30E−7	0.68

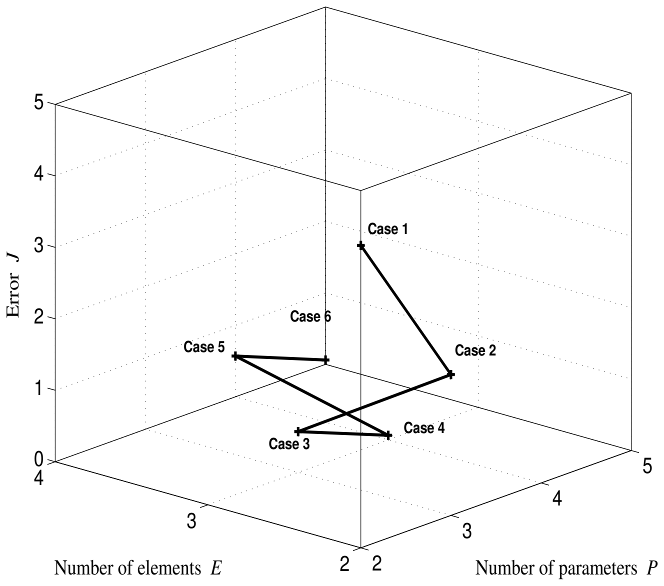


Fig. 3. Approximation error (J) *versus* the number of electrical elements (E) and the number of parameters to be adjusted (P).

4. FRACTAL CAPACITORS

Fractals can be found both in nature and abstract objects. The impact of the fractal structures and geometries, is presently recognized in engineering, physics, chemistry, economy, mathematics, art and medicine.

The concept of fractal is associated with Benoit Mandelbrot, that lead to a new perception of the geometry of the nature [Falconer (1990)]. However, the concept was initially proposed by several well known mathematicians, such as George Cantor (1872), Giuseppe Peano (1890), David Hilbert (1891), Helge von Koch (1904), Waclaw Sierpinski (1916), Gaston Julia (1918) and Felix Hausdorff (1919).

An geometric important index consists in the fractal dimension $(FDim)$ that represents the occupation degree

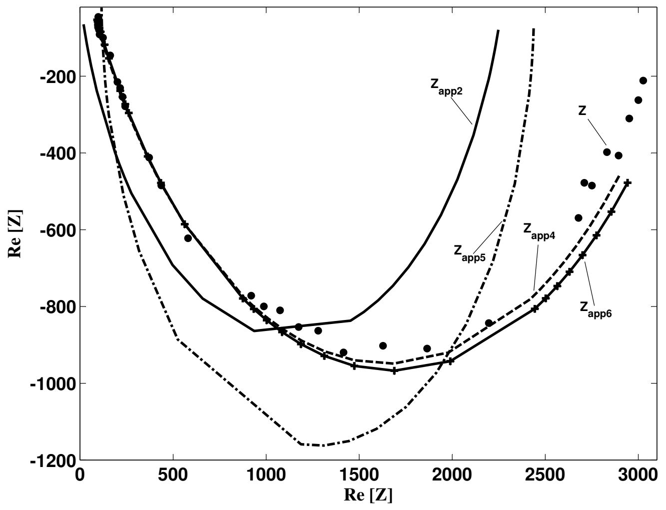


Fig. 4. Polar diagrams of $Z(j\omega)$ and the approximations $Z_{app2}(j\omega)$, $Z_{app4}(j\omega)$, $Z_{app5}(j\omega)$ and $Z_{app6}(j\omega)$ of the electrical impedance of the $Solanum\ Tuberosum$.

in the space and that is related with its irregularity. The $FDim$ is given by:

$$FDim \approx \log(N) / \log(1/\eta) \tag{10}$$

where N represents the number of boxes, with size $\eta(N)$ resulting from the subdivision of the original structure. This is not the only description for the fractal geometry, but it is enough for the identification of groups with similar geometries.

In this work we adopted the classical fractal carpet of Sierpinski with $FDim = 1.893$, depicted in the Figure 5.

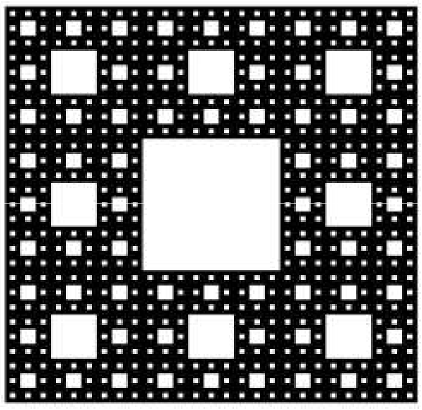


Fig. 5. Carpet of Sierpinski.

The simplest capacitors are constituted by two parallel electrodes separated by a layer of insulating dielectric. There are several factors susceptive of influencing the characteristics of a capacitor. However, three of them have a special importance, namely the surface area of the electrodes, the distance among them and the material that constitutes the dielectric. In this study the capacitors adopts electrodes that are printed with the fractal structure presented in Figure 5.

We apply sinusoidal excitation signals $v(t)$ to the apparatus (Fig. 6), for several distinct frequencies ω, and the impedance $Z(j\omega)$ between the electrodes is measured based on the resulting voltage $u(t)$ and current $i(t)$.

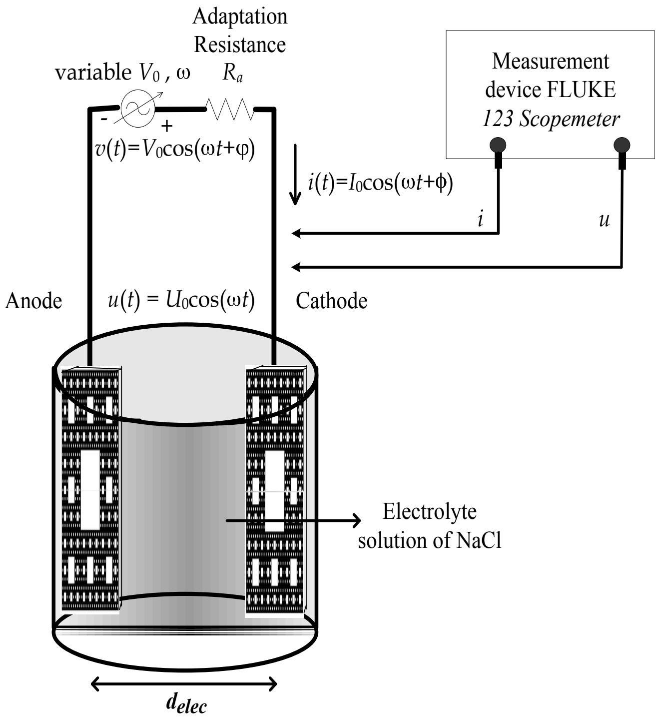


Fig. 6. Electrolyte process.

Are considered two identical single face electrodes. The voltage, the adaptation resistance R_a and the distance between electrodes d_{elec} are respectively, $V_0 = 10\ V$, $R_a = 1.2$ kΩ and d_{elec}= 0.13 m. The electrolyte process consists in an aqueous solution of NaCl with $\Psi = 10$ gl^{-1} and two single face copper electrodes with the carpet of Sierpinski printout and area $S = 0.423$ m^2.

The resulting polar diagram of the electrical impedance $Z(j\omega)$ is depicted in Fig. 7. For this chart we apply the circuits identified by cases 7 up to 12 in Table 2. The resulting numerical values of R_0, R_1, C_1, C_2, C_F and α for the different impedances are shown in Table 3.

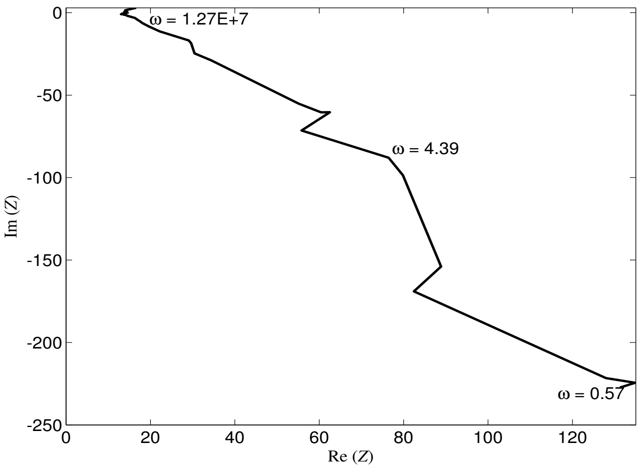


Fig. 7. Polar diagrams of the impedance $Z(j\omega)$ for the Carpet of Sierpinski.

Table 3. Comparison of circuit parameters for the cases 7 up to 12, for the Carpet of Sierpinski.

Case	R_0	R_1	C_1	C_2	C_F	α
7		17.00	2.7E−3			
8		14.20			4.2E−3	0.60
9		17.20	0.0027	1.0E−19		
10		14.30	1.0E−11		0.0041	0.61
11	14.90	41.20	0.0033	8.4E−11		
12	10.20	4.10	4.0E−11		0.0043	0.60

Figure 8 presents the approximation error (J) as function as the number of electrical elements (E) and the number of parameters (P) to be adjusted for the cases 7 to 12.

This figure reveals that the error decreases with the introduction of the fractional order elements. Figure 9 presents the polar diagram of $Z(j\omega)$ and the approximations $Z_{app8}(j\omega)$, $Z_{app10}(j\omega)$, $Z_{app11}(j\omega)$ and $Z_{app12}(j\omega)$, leading to an error $J = 0.199$, $J = 0.203$, $J = 0.585$, $J = 0.387$, respectively.

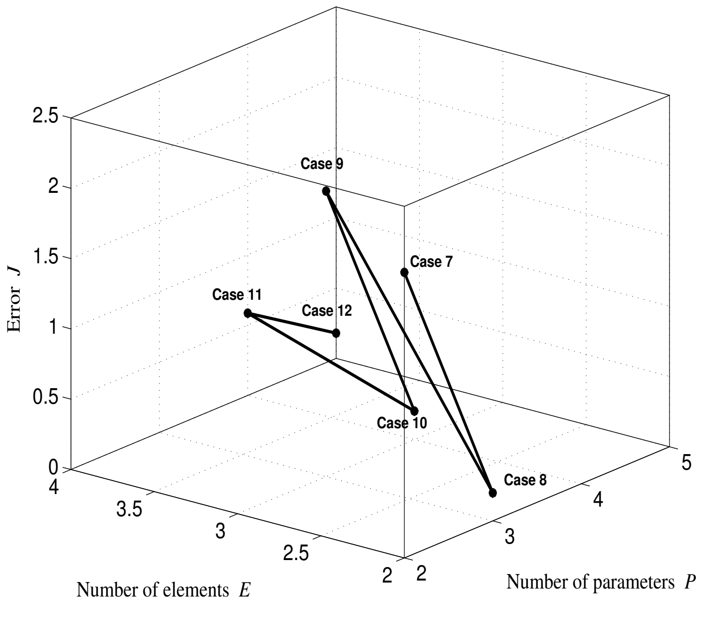


Fig. 8. Approximation error (J) *versus* the number of electrical elements (E) and the number of parameters (P) to be adjusted.

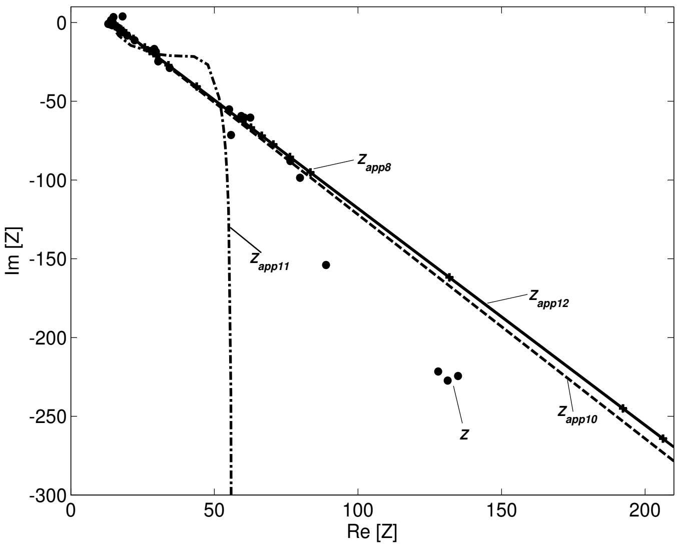


Fig. 9. Polar diagrams of $Z(j\omega)$ and the approximations $Z_{app8}(j\omega)$, $Z_{app10}(j\omega)$, $Z_{app11}(j\omega)$ and $Z_{app12}(j\omega)$ of the electrical impedance of the Carpet of Sierpinski.

5. CONCLUSIONS

During several centuries the FC was developed mainly in a mathematical viewpoint, but presently it addresses a considerable range of applications. In this paper FC concepts were applied to the analysis of electrical fractional impedances, in botanical elements and in electrical capacitors with fractal characteristics. The introduction of the CPE element in the electric circuits, lead us to conclude that for the same number of elements, we have a better approximation model and consequently a decrease in the error value. The different configurations of the polar diagrams of the systems studied, lead us to modelling the systems through electrical circuit with different configurations (series and parallel) and the combination of integer and fractional order elements in the circuits.

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