Fractional Order Nonlinear Control of Heat System

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Abstract: In the last decade, the interest in the area of fractional calculus and its applications has increased significantly. In this paper, we study a heat diffusion system in a fractional calculus perspective. A fractional order nonlinear controller is proposed, and its tuning, in the viewpoint of several performance indices, is analyzed. The simulations demonstrate the good performance of the proposed fractional-order structure.

Keywords: Fractional Calculus, Nonlinear Control, ISE, Heat Diffusion Systems.

1. INTRODUCTION

Fractional calculus (FC) is a generalization of integration and differentiation to a non-integer order $\alpha \in C$, being the fundamental operator ${}_aD_t^\alpha$, where a and t are the limits of the operation (Oldham and Spanier, 1974; Podlubny, 1999).

In the last years, FC has been used increasingly to model the constitutive behavior of materials and physical systems exhibiting hereditary and memory properties. This is the main advantage of fractional derivatives in comparison with classical integer models, where these effects are simply neglected. It is well-known that the fractional-order operator $s^{0.5}$ appears in several types of problems. The transmission lines, heat flow or the diffusion of neutrons in a nuclear reactor are examples where the half-operator is the fundamental element. On the other hand, diffusion is one of the three fundamental partial differential equations of mathematical physics (Courant and Hilbert, 1962).

In this paper we investigate the heat diffusion system in the perspective of applying the FC theory. A nonlinear controller with a fractional order model is presented and compared with other algorithms, namely the fractional-order PID controller. The fractional-order PI$^\alpha$D$^\beta$ controller involves an integrator of order $\alpha \in \Re^+$ and a differentiator of order $\beta \in \Re^+$ (Jesus and Machado, 2008).

Bearing these ideas in mind, the paper is organized as follows. Section 2 gives the fundamentals of fractional-order control systems. Section 3 introduces the heat diffusion system and describes its simulation. Section 4 points out a control strategy for the heat system and discusses the results. Finally, section 5 draws the main conclusions and addresses perspectives towards future developments.

2. FRACTIONAL-ORDER CONTROL SYSTEMS

Fractional controllers are characterized by differential equations that have, in the dynamical system and/or in the control algorithm, an integral and/or a derivative of fractional-order. Due to the fact that these operators are defined by irrational continuous transfer functions, in the Laplace domain, or infinite dimensional discrete transfer functions, in the Z domain, we often encounter evaluation problems in the simulations. Therefore, when analyzing fractional systems, we usually adopt continuous or discrete integer-order approximations of fractional-order operators.

The mathematical definition of a fractional derivative and integral has been the subject of several different approaches (Oldham and Spanier, 1974; Podlubny, 1999). One commonly used definition is given by the Riemann-Liouville expression ($\alpha > 0$ and $n-1 < \alpha < n$):

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau \qquad (1)$$

where $f(t)$ is the applied function and $\Gamma(x)$ is the Gamma function of x. Another widely used definition is given by the Grünwald-Letnikov approach ($\alpha \in \Re$):

$$ {}_aD_t^\alpha f(t) = \lim_{h\to 0} \frac{1}{h^\alpha} \sum_{k=0}^{\left[\frac{t-a}{h}\right]} (-1)^k \binom{\alpha}{k} f(t-kh) \qquad (2a)$$

$$ \binom{\alpha}{k} = \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)} \qquad (2b)$$

where h is the time increment and $[x]$ means the integer part of x.

The "memory" effect of these operators is demonstrated by (1) and (2), where the convolution integral in (1) and the infinite series in (2), reveal the unlimited memory of these operators, ideal for modelling hereditary and memory properties in physical systems and materials.

An alternative definition to (1) and (2), which reveals useful for the analysis of fractional-order control systems, is given by the Laplace transform method. Considering vanishing initial conditions, the fractional differintegration is defined in the Laplace domain, $F(s) = L\{f(t)\}$, as:

$$L\left\{{}_aD_t^{\alpha} f(t)\right\} = s^{\alpha} F(s), \quad \alpha \in \Re \tag{3}$$

An important aspect of fractional-order algorithms can be illustrated through the elemental control system, with open-loop transfer function $G(s) = Ks^{-\alpha}$ $(1 < \alpha < 2)$ in the forward path. The open-loop Bode diagrams of amplitude and phase have correspondingly a slope of -20α dB/dec and a constant phase of $-\alpha\pi/2$ rad over the entire frequency domain. Therefore, the closed-loop system has a constant phase margin of $\text{PM} = \pi(1 - \alpha/2)$ rad, that is independent of the system gain K, and the closed-loop system is robust against gain variations exhibiting step responses with an iso-damping property (Barbosa et al., 2004).

In this paper we adopt discrete integer-order approximations to the fundamental element s^{α} $(\alpha \in \Re)$ of a fractional-order control (FOC) strategy. The usual approach for obtaining discrete equivalents of continuous operators of type s^{α} adopts the Euler, Tustin and Al-Alaoui generating functions (Samko et al., 1987; Miller and Ross, 1993).

It is well known that rational-type approximations frequently converge faster than polynomial-type approximations and have a wider domain of convergence in the complex domain. Thus, by using the Euler operator $\omega(z^{-1}) = (1 - z^{-1})/T_c$, and performing a power series expansion of $[\omega(z^{-1})]^{\alpha} = [(1 - z^{-1})/T_c]^{\alpha}$ gives the discretization formula corresponding to the Grünwald-Letnikov definition (2):

$$D^{\alpha}\left(z^{-1}\right) = \left(\frac{1 - z^{-1}}{T}\right)^{\alpha} = \sum_{k=0}^{\infty} h^{\alpha}(k)\, z^{-k} \tag{4}$$

$$h^{\alpha}(k) = \left(\frac{1}{T}\right)^{\alpha} \binom{k - \alpha - 1}{k} \tag{5}$$

A rational-type approximation can be obtained by applying the Padé approximation method to the impulse response sequence (5) $h^{\alpha}(k)$, yielding the discrete transfer function:

$$H\left(z^{-1}\right) = \frac{b_0 + b_1 z^{-1} + \ldots + b_m z^{-m}}{1 + a_1 z^{-1} + \ldots + a_n z^{-n}} = \sum_{k=0}^{\infty} h(k)\, z^{-k} \tag{6}$$

where $m \leq n$ and the coefficients a_k and b_k are determined by fitting the first $m+n+1$ values of $h^{\alpha}(k)$ into the impulse response $h(k)$ of the desired approximation $H(z^{-1})$. Thus, we obtain an approximation that has a perfect match to the desired impulse response $h^{\alpha}(k)$ for the first $m + n + 1$ values of k. Note that the above Padé approximation is obtained by considering the Euler operator but the determination process will be exactly the same for other types of discretization schemes.

3. HEAT DIFFUSION

The heat diffusion is governed by a linear unidimentional partial differential equation (PDE) of the form:

$$\frac{\partial c}{\partial t} = k \frac{\partial^2 c}{\partial x^2} \tag{7}$$

where k is the diffusivity, t is the time, c is the temperature and x is the space coordinate. The system (7) involves the solution of a PDE of parabolic type for which the standard theory guarantees the existence of a unique solution (Courant and Hilbert, 1962; Crank, 1956).

For the case of a planar perfectly isolated surface we usually apply a constant temperature C_0 at $x = 0$ and analyzes the heat diffusion along the horizontal coordinate x. Under these conditions, the heat diffusion phenomenon is described by a non-integer order model:

$$C(x, s) = \frac{C_0}{s} G(s), \; G(s) = e^{-x\sqrt{\frac{s}{k}}} \tag{8}$$

where x is the space coordinate, C_0 is the boundary condition and $G(s)$ is the system transfer function.

In our study, the simulation of the heat diffusion is performed by adopting the Crank-Nicholson implicit numerical integration based on the discrete approximation to differentiation as (Curtis and Patrick, 1999):

$$\begin{aligned} -rc[j+1, i+1] + (2+r)\, c[j+1, i] - rc[j+1, i-1] = \\ = rc[j, i+1] + (2-r)\, c[j, i] + c[j, i-1] \end{aligned} \tag{9}$$

where $r = k\Delta t(\Delta x^2)^{-1}$, $\{\Delta x, \Delta t\}$ and $\{i, j\}$ are the increments and the integration indices for space and time, respectively (Jesus and Machado, 2008).

4. CONTROL STRATEGY

This section studies a new control strategy for the heat diffusion system. In fact, in previous works developed by the authors (Jesus and Machado, 2007) we analyze the closed-loop system with a conventional PID controller given by the transfer function:

$$G_s(s) = K_p + \frac{K_i}{s} + K_d s \tag{10}$$

Often, the PID parameters $(K_p,\ K_i,\ K_d)$ are tuned by using the so-called Ziegler-Nichols open loop (ZNOL) method (Machado et al., 2006). However, the poor results indicated that the method of tuning might not be the most adequate for the control of the heat system. In fact, the inherent fractional dynamics of the system lead us to consider other configurations. In this perspective, we propose the use of fractional order schemes tuned by the minimization of the index ISE.

In this line of thought, we developed the nonlinear controller (NLC) with a fractional order algorithm, represented in figure 1. The closed-loop system consists in the

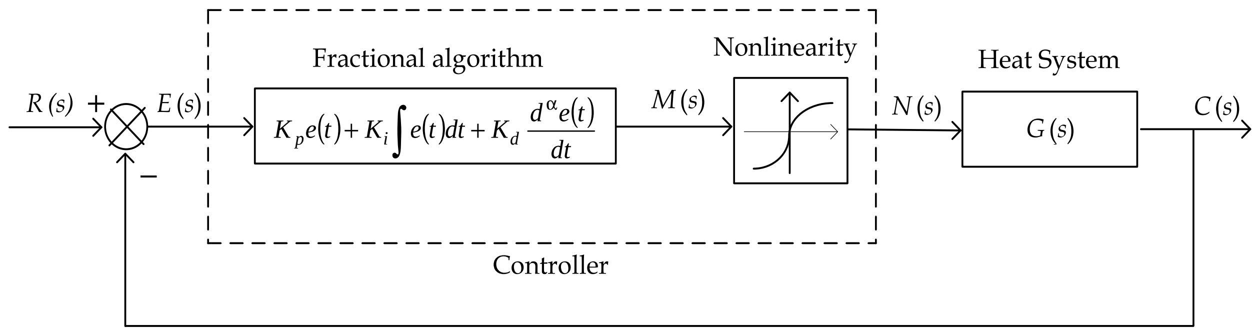

Fig. 1. Nonlinear structure control.

controller given by a fractional order model and nonlinearity described by equations (11) and (12), respectively:

$$m(t) = K_p e(t) + K_i \int e(t)dt + K_d \frac{d^\alpha e(t)}{dt} \tag{11}$$

$$n(t) = Km(t)^\Psi \begin{cases} k\,|m(t)|^\Psi & if \;\; m(t) \geq 0 \\ -k\,|m(t)|^\Psi & if \;\; m(t) < 0 \end{cases} \tag{12}$$

The nonlinearity can be considered as a generalization of the standard variable structure controller (VSC) (Vadim, 1977; Machado, 1996). In fact, in the simplest form, a VSC consists in a saturation-like function which is a special case of (12); therefore, expression (12) gives an extra degree of freedom in the controller design though the tuning of the parameters (K, Ψ). In general nonlinearities must be avoided, but the truth is that VSCs demonstrated good robustness, leading to linear-like responses. In this line of thought, in the sequel we will verify that the quasi-linear response will be a characteristic of the control algorithm (11-12).

In expression (11) the symbol e represents the error, α the order of the fractional derivative term, $0 \leq \alpha \leq 1$, and the constants K_p, K_i and K_d are the proportional, the integral and the derivative gains, respectively.

The fractional derivative term s^β in (11) is implemented through a 4^{th}-order Padé discrete rational transfer function of type (6), with a sampling period of $T = 0.1\ s$. In expression (12) m is the output of fractional algorithm, K is a constant and Ψ is a real number.

The controller is tuned by the minimization of an integral performance index. For that purpose, we analyze the indices that measure the response error, namely the integral square error (ISE) criteria defined as:

$$ISE = \int_0^\infty [r(t) - c(t)]^2 \, dt \tag{13}$$

We can use other performance criteria such as the integral time square error (ITSE), the integral absolute error (IAE) or the integral time absolute error (ITAE); however, in the present case the ISE criterion had produced the best results and is adopted in the study (Jesus and Machado, 2007).

Another possible performance index consists on the energy E_n at the controller output $n(t)$ given by the expression:

$$E_n = \int_0^{T_e} n^2(t)dt \tag{14}$$

where T_e is the time window needed to stabilize the systems output $c(t)$.

A step reference input $R(s) = R_0/s$ is applied at $x = 0.0$ m and the output $c(t)$ is analyzed for $x = 3.0$ m. The heat system is simulated for 3000 seconds and is considered $T_e = 700$ s.

Figure 2 illustrates the variation of the fractional order control parameters $(K_p,\ K_i,\ K_d,\ K,\ \Psi)$ as function of the order's derivative α, when minimizing the ISE criterion.

Figures 3 shows the step responses of the closed-loop system, for the NLC tuned in the ISE perspective, and for $0 \leq \alpha \leq 1$.

The controller parameters $(K_p,\ K_i,\ K_d,\ K,\ \Psi, \alpha)$ corresponding to the minimization of those indices, lead to the values ISE: $(K_p,\ K_i,\ K_d,\ K,\ \Psi,\ \alpha\} \equiv \{0.01,\ 0.28,\ 55.8,\ 0.04,\ 2.40,\ 0.8\}$ for the best case.

The step responses reveal a large diminishing of the overshoot (ov) and the rise time (t_r) when compared with the integer PID $(\{\text{ov}(\%), t_r\} \equiv \{68.56\%, 12.0\})$ (Jesus and Machado, 2007), showing a good transient response and a zero steady-state error.

In order to analyze the system dynamics we evaluate the response of the control system for different input systems amplitudes, namely, $R_0 = \{0.5,\ 1.0,\ 2.0\}$ and $\alpha = 0.8$, for two different cases study. In the first case, all controller parameters correspond to the minimization of the ISE index. Figure 2 depicts all parameters values as function of the parameter α.

The controller parameters $(K_p,\ K_i,\ K_d,\ K,\ \Psi)$ for $R_0 = \{0.5,\ 1.0,\ 2.0\}$ and $\alpha = 0.8$ lead to the values: $(K_p,\ K_i,\ K_d,\ K,\ \Psi) \equiv \{0.01,\ 0.42,\ 143.10,\ 0.07,\ 1.90\}$, $(K_p,\ K_i,\ K_d,\ K,\ \Psi) \equiv \{0.01,\ 0.28,\ 55.80,\ 0.04,\ 2.40\}$, $(K_p,\ K_i,\ K_d,\ K,\ \Psi) \equiv \{0.01,\ 0.44,\ 225.6,\ 0.04,\ 1.60\}$, respectively.

Figure 4 shows, the step response of the closed-loop system, for the fractional order controller tuned in the ISE perspective, for $R_0 = \{0.5,\ 1.0,\ 2.0\}$ and $\alpha = 0.8$. We can verify that this controller reveals good characteristics and that the system output does not change significantly with the variation of input amplitude.

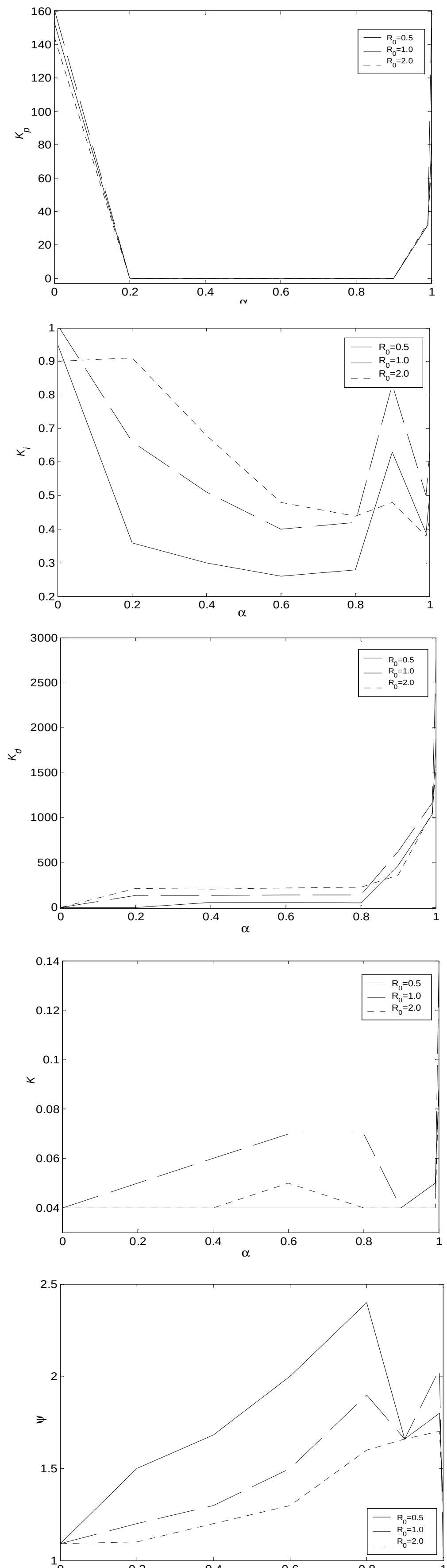

Fig. 2. The fractional order controller parameters (K_p, K_i,K_d, K, Ψ, α) versus α for the ISE criteria and for $R_0 = \{0.5, 1.0, 2.0\}$.

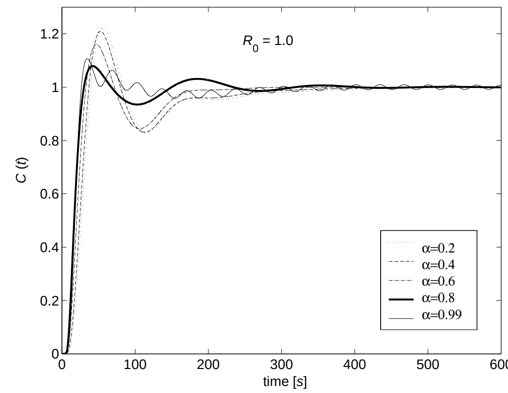

Fig. 3. Step response of the closed-loop system for the ISE indices, with a NLC, $R_0 = 1.0$ and $\alpha = \{0.2, 0.4, 0.6, 0.8, 0.99\}$, for $x = 3.0$ m.

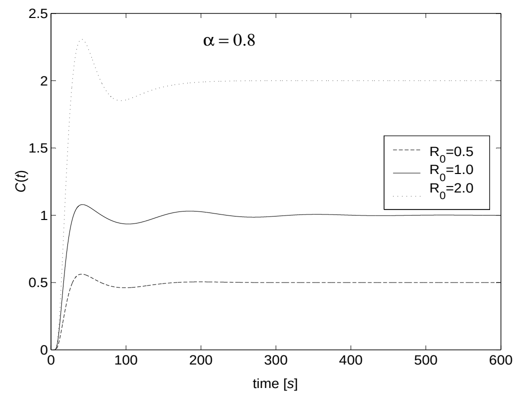

Fig. 4. Step response of the closed-loop system for the ISE, with a fractional order controller, $R_0 = \{0.5, 1.0, 2.0\}$ and $\alpha = 0.8$, for $x = 3.0$ m.

On the other hand, when we compare the controller parameters for these three inputs we verify that they lead almost to similar response time evolutions.

In the second case, we apply the same controller parameters found in the previous study for $U_0 = 1.0$, $(K_p, K_i, K_d, K, \Psi) \equiv \{0.01, 0.28, 55.80, 0.04, 2.40\}$, and analyze the output response for $R_0 = \{0.5, 2.0\}$. Figure 5 depicts the three step responses of the closed-loop system, for the fractional order controller.

The results in this case are clearly worse than the outputs presented previously, when all of the controller parameters are tuned for each one of the input system amplitude. In this line of thought, in all following studies the controller parameters corresponds to the first case study.

Figure 6 depicts the ISE indices for $0.0 \le \alpha \le 1.0$, when $R_0 = \{0.5, 1.0, 2.0\}$ and $x = 3.0$ m. We verify the existence of a minimum for $\beta \approx 0.8$ for the ISE.

The energy E_n (14) at the output $n(t)$ is also analyzed. Figure 7 depicts the energy of the control action E_n as function of the ISE for $0.0 \le \alpha \le 1.0$, $R_0 = \{0.5, 1.0, 2.0\}$ and $x = 3.0$ m. As can be seen, fixing the value of R_0, we verify that the energy increases gradually with α and, for $\alpha > 0.8$ the E_n increases rapidly. On the other hand E_n has minor changes with R_0.

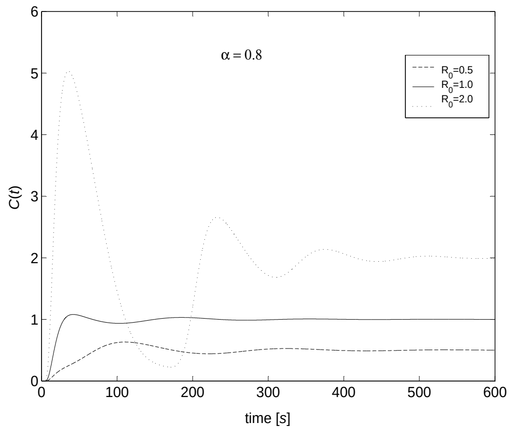

Fig. 5. Step response of the closed-loop system for the ISE, with a fractional order controller, $R_0 = \{0.5, 1.0, 2.0\}$ and $\alpha = 0.8$, for $x = 3.0$ m, tuned through $U_0 = 1.0$.

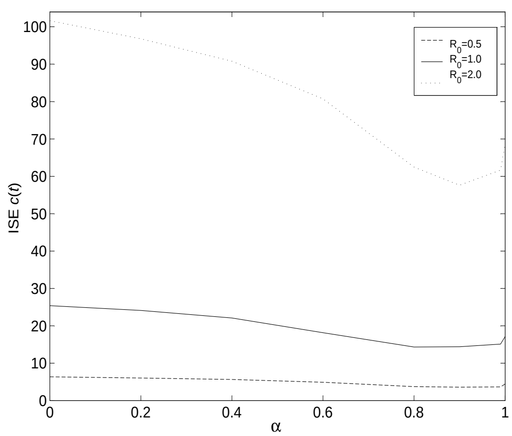

Fig. 6. ISE *versus* $0.0 \leq \alpha \leq 1.0$ for $R_0 = \{0.5, 1.0, 2.0\}$ and for $x = 3.0$ m.

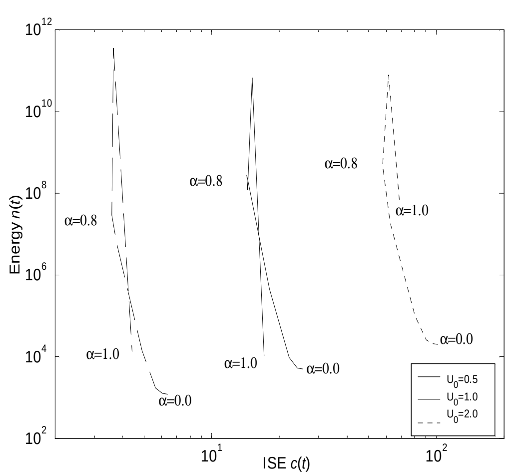

Fig. 7. Energy E_n *versus* the ISE for $0.0 \leq \alpha \leq 1.0$, $R_0 = \{0.5, 1.0, 2.0\}$ and for $x = 3.0$ m.

Figure 8 shows the variation of the settling time t_s, the peak time t_p, the rise time t_r, and the percent overshoot $ov(\%)$, *versus* α, for the closed-loop response tuned through the minimization of the ISE indices. Again, we verify a smooth variation with α with good results for $0.8 \leq \alpha \leq 0.9$ and an abrupt variation for $\alpha = 1.0$.

5. CONCLUSION

This paper presented the fundamental aspects of the FC theory in the control of fractional order systems. We demonstrated that FC is a paradigm allowing a deeper understanding of physical phenomena than traditional methodologies. In this perspective, we studied the heat diffusion system, and its control using a nonlinear controller schemes. The results reveal the superior performance of the NLC based on the fractional order algorithm, namely in the dynamics of systems of non-integer order. Moreover, the fractional order model of the controller is more flexible and gives the possibility of adjusting more carefully the closed-loop system characteristics than the corresponding classical PID controller. These results demonstrate the effectiveness of the fractional order algorithms when used for the control of fractional order systems.

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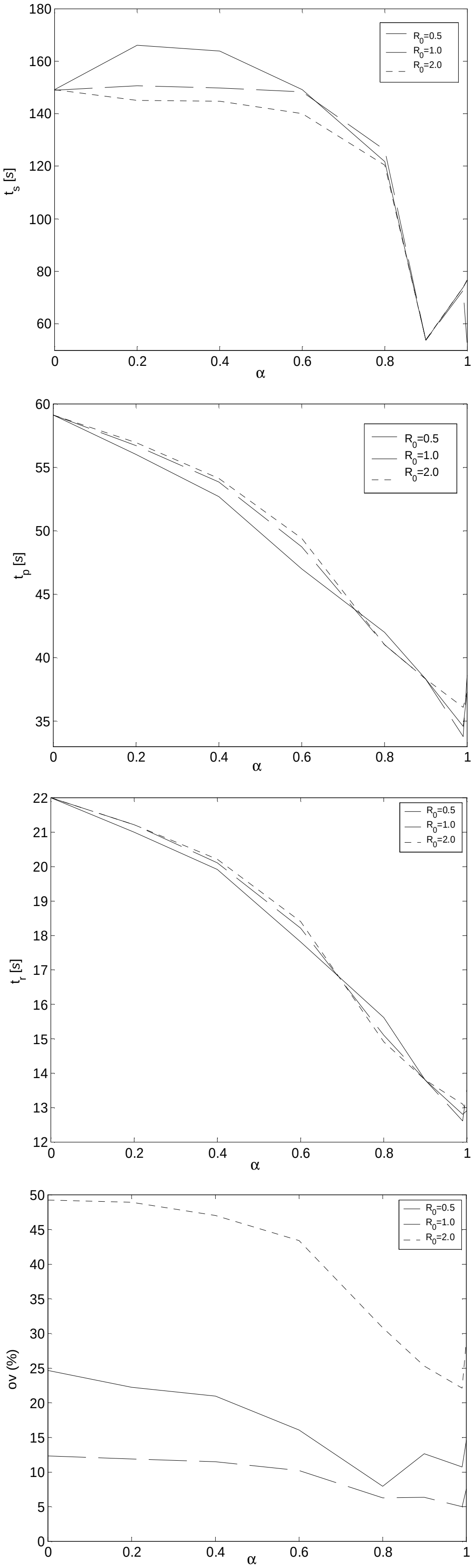

Fig. 8. Parameters t_s, t_p, t_r, $ov(\%)$ *versus* $0.0 \leq \alpha \leq 1.0$ for the step responses of the closed-loop system for the ISE indices, when $R_0 = \{0.5,\ 1.0,\ 2.0\}$ and for $x = 3.0$ m.

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