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Adaptive Nonlinear Vibration Control Based on Causal Time-invariant Green Functions and on a Novel Branch of Soft Computing

József K. Tar[1], Imre J. Rudas[1], János F. Bitó[1], José A. Tenreiro Machado[2]

[1]Institute of Mathematical and Computational Sciences, John von Neumann Faculty of Informatics, Budapest Polytechnic, Népszínház utca 8, H-1081 Budapest, Hungary
E-mail: tar.jozsef@nik.bmf.hu, rudas@bmf.hu, bito.janos@nik.bmf.hu

[2]Department of Electrotechnical Engineering, Institute of Engineering of Porto, Rua Dr. Antonio Bernardino de Almeida4200-072 Porto, Portugal
E-mail: jtm@dee.isep.ipp.pt

Abstract: In the paper a simple nonlinear, adaptive approach inspired by the fractional derivatives based CRONE control is presented for vibration damping. Its key idea is replacement of the fractional derivatives with the mathematically less restricted concept of time-invariant Green functions. Instead of the traditional PID feedback terms it applies positive definite weighted moving average of the square of the error plus a nonlinear term making the error converge to zero. In this way simple kinematic design of the desired damping becomes possible. The adaptive part of the controller guarantees the realization of this kinematic design without making it necessary to the designer to have an accurate and complete dynamic model of the system to be controlled or to design a sophisticated linear controller. The applicability of the approach is illustrated via simulations for a paradigm consisting of a pair of coupled damped linear oscillators under external excitation. One of the oscillators is not modeled by the controller. In similar way, no direct information on the external excitation is available for the controller. The adaptive loop successively maps the observed system behavior to the desired one without exerting any effort to identify the reasons of the differences. The approach was found be useful for solving vibration damping problems with unmodeled and uncontrolled internal degrees of freedom.

Keywords: Soft Computing; Fractional Order Derivatives; Green Functions; Adaptive Control; Vibration Damping.

1 Introduction

Normally vibration is undesired, externally excited phenomenon that occurs in various physical systems therefore its efficient damping is of great practical significance. In a wider context this phenomenon can be modeled as linear or nonlinear interaction between various coupled subsystems having their own internal degrees of freedom. Normally the vibration of certain subsystems has to be reduced only while the other subsystems' vibration is not critical. From the point of control technology the task has the „delicate" nature that the most of the internal degrees of freedom cannot be controlled directly, even the controller cannot be provided with their model or with information on their actual physical state. A novel branch of soft computing has recently been developed on the basis of the simultaneous using of the Modified Renormalization Transformation and simple ancillary methods [1] that is flexible enough to incorporate various algebraic blocks like e.g. special Symplectic Transformations [2] etc. It was shown that in the case of a wide class of physical systems with the aid of this method quite robust adaptive controllers could be developed for the control of very inaccurately and partially modeled physical systems that can have even unmodeled internal degrees of freedom [3]. As an input the method requires the desired trajectory of the generalized coordinates of the subsystem that has to directly be controlled.

This approach should be very efficient for vibration control if the desired trajectory of the generalized coordinates of the subsystems to be controlled could be prescribed with respect to an inertial system of coordinates. However, in the most cases the subsystem the vibration of which needs active damping is a part of a moving object that does not serve as a basis of an inertial frame, as e.g. a car proceeding along a bumpy road crossing hills and valleys. In this case some slow motion along certain average distance between the chassis and the wheel can be prescribed for the controller because it is locally measurable quantity. A feasible kinematic formulation of this goal is the application of some „forgetting integral" that does not allow abrupt changes with respect to the former values of this distance but allows its slow variation. The desired behavior of this distance can practically be prescribed to some extent by the terms used in the traditional linear controllers as frequency filters etc. The most plausible means would be the application of a simple PID type controller to keep a finite error at bay. However, the integrating term of this controller does not „forget" the past, and for an even small but constant error it generates infinite signal for feedback.

As generalization of the concept of the derivative —remaining strictly within the frames of linear physics— the concept of fractional order integrals and derivatives found more and more physical applications to describe the „longer term memory" of various physical systems like in the case of visco-elastic phenomena [4, 5] etc. In general the problem of designing fractional control systems remaining mainly within the frames of linear systems obtained considerable attention recently, e.g.

[6, 7]. The French expression invented by Oustaloup „*CRONE: Commande Robuste d'Ordre Non Entier*" [8] almost became a „trademark" hallmarking a well-elaborated design methodology that obtained application in vibration control, too [9]. Understanding and using this method requires deep engineering knowledge in the realm of linear systems, frequency spectrum analysis, the use of Laplace transforms and complex integrals, various typical diagrams as e.g. the Nichols plot, etc.

The aim of the present paper is to propose an alternative approach not strictly restricted to the traditional way of thinking in the case of linear systems. Tackling the problem from a more general nonlinear basis requires less amount of profound and specific engineering knowledge, the application of which can be evaded by the controller's adaptive nature or learning abilities. For this purpose the „long term memory" or slowly forgetting nature of the fractional order systems is considered in a more general view.

2 Fractional order derivatives and Green functions

In the case of a normal PID-type controller the desired trajectory reproduction can be prescribed in a purely kinematics based manner. For the second time-derivative of the actual coordinate errors the desired relation can be prescribed:

$$\left(\ddot{\mathbf{q}}^{r}-\ddot{\mathbf{q}}^{\mathrm{N}}\right)^{d}=-P\left(\mathbf{q}^{r}-\mathbf{q}^{\mathrm{N}}\right)-D\left(\dot{\mathbf{q}}^{r}-\dot{\mathbf{q}}^{\mathrm{N}}\right)-I\int_{-\infty}^{t}\left(\mathbf{q}^{r}(t')-\mathbf{q}^{\mathrm{N}}(t')\right)dt' \tag{1}$$

which may be apt to result in too big desired joint coordinate acceleration $\ddot{\mathbf{q}}^{d}$. The main idea of this paper is based on the observation that the integrating term in (9) corresponds to a special form of the time-invariant causal Green functions that generally can be expressed as

$$y(t):=\int_{-\infty}^{t}G(t-\tau)f(\tau)d\tau \tag{2}$$

If $G(0)$ is considerable and $G(\xi)\to 0$ as $\xi\to\infty$ (2) describes a long-term memory of slowly forgetting nature. If $G(\xi)=0$ for $\xi\le D>0$, than this function can also model the effect of delay.

It can clearly be observed that the Grünwald-Letnikov form of the fractional derivative of order a [10] also is similar to a finite element approximation of an integral with some Green-function if the $h\to 0$ limit is not exactly executed and some numerical approximation using finite elements for the variable of time is used:

$$\frac{d^a f(t)}{dt^a} := \lim_{h \to 0} \left[(h)^{-a-1} \sum_{j=0}^{\infty} (-1)^j \left(\frac{\Gamma(a+1)}{\Gamma(j+1)\Gamma(a-j+1)} \right) f(t-jh)h \right] \tag{3}$$

Eq. (3) has the property that its result does not depend only from a rigorously restricted „zero distance" vicinity of the variable „*t*": this operation has a more or less slowly forgetting „memory". However, besides this forgetting nature that seems to be the most important fact from „physical point of view" (3) has to satisfy complicated restrictions which stem from the requirement that the fractional order derivatives somehow must be related to the integer order ones in limit cases. In (3) the calculation of the values of the Γ functions means a numerical burden, while a simple function G in (2) may be exempt of such difficulties. These difficulties made the authors to consider a more general, less restricted problem-formulation as follows.

Considering the control problem in purely kinematic terms only, due to the laws of Classical Mechanics just the desired second order time-derivative of the error signal can be prescribed somehow, because it is this quantity that can directly be influenced by the control agent, which in its physical capacity can be torque or force. So let ν, $\mu > 0$, and let the function $h(t)$ be the solution of the initial value problem

$$\ddot{h}(t) = -\nu h(t) + \mu \int_0^t h(\tau) G(\tau - t) d\tau + S(t) \text{ for } t \geq 0, h(0) = h_0, \dot{h}(0) = \dot{h}_0 \tag{4}$$

in which $G(\xi)$ has the following properties:

$$G(\xi) = 0 \text{ if } \xi \geq 0, G(\xi) \geq 0 \text{ if } \xi \leq 0, \int_{-\infty}^{0} G(\xi) d\xi = 1 \tag{5}$$

and let $G(\xi)$ be *continuously differentiable with the exception of certain discrete points of finite number.* It is expedient to introduce a "supplementary" term $S(t)$ that is expected to be necessary for maintaining the decreasing nature of the quadratic error integrated in the "moving window" in (6).

$$V(t) := \int_0^t h^2(\tau) G(\tau - t) d\tau \tag{6}$$

The function $G(\xi)$ in (4, 6) can be interpreted as a weight function of a weighted moving average within a „window" that picks up samples from $h(t)$ and $h^2(t)$ in certain vicinity of t. Since $h(t)$ represents some *error signal* then its convergence to zero is desired as $t \to \infty$. For $\mu = 0$ and $S \equiv 0$ (4) evidently describes exponential error-relaxation. For small positive μ the 2nd term at the right hand side seems to decrease the speed of this relaxation. In comparison with the *common integrating term*, $G(\xi)$ represents some *short-term memory* because from (5) it can evidently be inferred that $G(\xi) \to 0$ as $\xi \to -\infty$, that is the long past's effects are forgotten.

Furthermore, in $G(\tau\text{-}t)$ no any time instant is in „distinguished position". Its behavior is governed exclusively by the *difference of the various time-instants.*

To verify the desired „relaxing" nature of $h(t)$ consider the time-derivative of the „sample" of the square of the error. Because the upper limit of the integral explicitly depends on t, according to the Leibniz rule the derivative of $V(t)$ yields two terms:

$$\dot{V}(t) = h^2(t)G(0) + \int_0^t h^2(\tau)\frac{\partial G(\tau - t)}{\partial t}\,d\tau \tag{7}$$

Utilizing that

$$\frac{\partial G(\tau - t)}{\partial t} = -\frac{\partial G(\tau - t)}{\partial \tau} \tag{8}$$

and that $G(0)=0$ partial integration can be executed in (7) that results in:

$$\dot{V}(t) = -\left[\underbrace{h^2(t)G(0)}_{0} - h^2(0)G(-t)\right] + \int_0^t 2h(\tau)\dot{h}(\tau)G(\tau - t)\,d\tau \tag{9}$$

It is evident that near $t=0$ the first term causes increasing error but its significance decreases with the elapse of time since $G(-t)\to 0$ as $t\to\infty$. To utilize (4) we have to calculate the second derivative of V in its most convenient form in (9) by using again the Leibniz rule and the properties of G as given in (5) and in (8). Due to partial integration certain terms belonging to the upper limit cancel and the 2nd derivative of the error h appear as in the equation as

$$\ddot{V}(t) = h_0^2\dot{G}(-t) + 2\int_0^t \dot{h}^2(\tau)G(\tau - t)\,d\tau + 2\int_0^t \ddot{h}(\tau)h(\tau)G(\tau - t)\,d\tau\,. \tag{10}$$

Now (4) can be substituted into (10) resulting in

$$\begin{aligned}\ddot{V}(t) = &-2\nu V(t) + 2\mu\int_0^t\int_0^\tau h(\tau)G(\tau - t)h(\tau')G(\tau' - t)\,d\tau'\,d\tau + \\ &+ h_0^2\dot{G}(-t) + 2\int_0^t\left[\dot{h}^2(\tau) + S(\tau)h(\tau)\right]G(\tau - t)\,d\tau\end{aligned} \tag{11}$$

In this way a double integral is obtained in which between the variables of integration the following relations hold: $0\le\tau'\le\tau\le t$. Using the rectangular system of coordinates (τ,τ') this domain of integration corresponds to the *lower triangular half of the square shaped area* $[0,t]\times[0,t]$. Introducing the *operator of time ordering T* in the integrand in (11) with the definition

$$T\left[f(\tau)f(\tau')\right] := \begin{cases} f(\tau)f(\tau') \text{ if } \tau \ge \tau' \\ f(\tau')f(\tau) \text{ if } \tau < \tau' \end{cases} \tag{12}$$

the upper limit of integration according to τ' can be extended to the whole $[0,t]$ interval. This exactly corresponds to extending the integration to the *upper triangular half of the square shaped area* $[0,t]\times[0,t]$ with *symmetric integrands.* That is this extension of the upper limit of integration with time-ordering exactly doubles the original integral. Since the integrands are common numbers satisfying a *commutative algebra* the operation of time ordering can simply be omitted. Therefore by transforming the 2nd term in the right hand side of (11) it can be written that

$$\begin{aligned}\ddot{V}(t) = -2\nu\dot{V}(t) + \frac{2\mu}{2}\int_0^t\int_0^t h(\tau)G(\tau-t)h(\tau')G(\tau'-t)d\tau' d\tau + \\ + h_0^2\dot{G}(-t) + 2\int_0^t\left[\dot{h}^2(\tau) + S(\tau)h(\tau)\right]G(\tau-t)d\tau\end{aligned} \tag{13}$$

It is reasonable to define the function $F(t)$ as

$$F(t) := \int_0^t h(\tau)G(\tau-t)d\tau \tag{14}.$$

by the use of which it is obtained that

$$\ddot{V}(t) = -2\nu\dot{V}(t) + \mu F^2(t) + h_0^2\dot{G}(-t) + 2\int_0^t\left[\dot{h}^2(\tau) + S(\tau)h(\tau)\right]G(\tau-t)d\tau \tag{15}$$

To estimate the significance of the function $F(t)$ consider the following *non-negative expression*

$$\begin{aligned}0 \le \int_0^t\left[h(\tau) - F(t)\right]^2 G(\tau-t)d\tau = \\ = \int_0^t\left[h^2(\tau) - 2h(\tau)F(t) + F(t)^2\right]G(\tau-t)d\tau = V(t) + \left[g(t)-2\right]F(t)^2\end{aligned} \tag{16}$$

in which

$$0 \le g(t) := \int_0^t G(\tau-t)d\tau \xrightarrow[t\to\infty]{} 1 \tag{17}$$

is a non-negative monotone increasing function approaching its upper limit 1. Therefore the inequality in (16) can be written as

$$0 \le V(t) + \left[g(t)-2\right]F(t)^2 \le V(t) - F(t)^2 \Rightarrow V(t) \ge F(t)^2 \tag{18}$$

Taking into account (15) it would be a reasonable choice to determine the supplementary term as

$$S(\tau)h(\tau) := -\dot{h}(\tau)^2 - \rho\dot{h}(\tau)h(\tau) \tag{19}$$

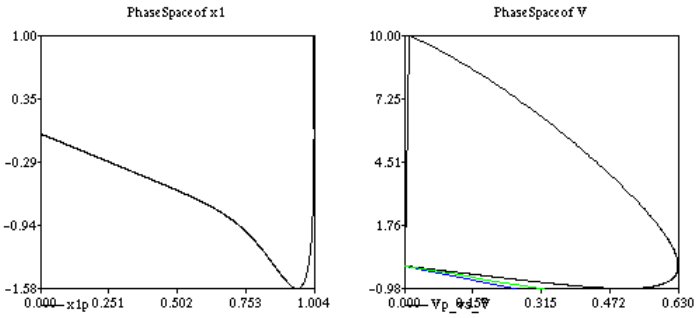

Figure 1: Typical behavior of the solution of (4) using (21) and (22) for h_0=1, $\dot{h}_0 = 1$, *D = 2 ms,* ν*= 100* s^{-2}, μ*= 40* s^{-2}, ρ*=0.5*ν *[*s^{-1}*],* ε*=*10^{-6} *m,* and $exp(\beta \times 10^{-3}\ s) \approx 0.99$

with a constant $\rho>0$ because its results in the appearance of the first time-derivative of $V(t)$ according to (9):

$$\ddot{V}(t) = -2\nu V(t) - \rho \dot{V}(t) + \mu F(t)^2 + h_0^2 \left[\dot{G}(-t) + \rho G(-t)\right] \tag{20}$$

Eq. (20) has a simple and lucid interpretation. Since G can be so constructed that both its value and its derivative converges to zero as $t \to \infty$ the last term in it converges to zero and can be omitted following some transient phase. Regarding the terms remained, if the positive constant μ is small enough, in the phase space determined by v and dV/dt the location of the points in which the 2nd derivative of V can take the value 0 must be within a wedge shaped region in the $V>0$ half plane determined by the origin and the two straight lines of the equations $\dot{V} = -2\nu V/\rho, \dot{V} = (-2\nu + \mu)/\rho$. For a given V outside of the wedge to the dV/dt values bigger than that of the upper limit of the wedge $d^2V/dt^2<0$ belongs, while for the dV/dt values smaller than the lower limit of the wedge $d^2V/dt^2>0$, and the solution cannot enter the $0>V$ half plane. Therefore in the region outside of this wedge it attracts the phase trajectories of V, and asymptotically leads them to the V=0, dV/dt=0 point. That is the moving average of the square of the error $h^2(t)$ approaches zero. So, apart from an initial transient phase, the error decreases in a longer time-interval. Since in the h=0 points (19) may yield infinite S via introducing a small positive value ε instead of its application that of the following approximation seems to be more expedient:

$$S(\tau) := -sign(h(\tau)) \frac{\dot{h}(\tau)^2}{\varepsilon + |h(\tau)|} - \rho \dot{h}(\tau) \tag{21}$$

which for $|h| \rangle\rangle \varepsilon$ well approximates (19) and only for small h differs from it significantly. In choosing $G(\xi)$ we have a great extent of freedom. For instance,

$G(\xi)$ can be the member of the set of the basic functions of fast decrease $\mathcal{D}$ on which the generalized functions (distributions) are defined as linear functionals [12]. These infinitely many times continuously differentiable functions have arbitrary but individually determined finite support guaranteeing exactly zero $G(-t)$ if t is greater than the upper limit of the support of this function. In the sequel (22) will be used with $D>0$, $\beta>0$

$$G(\xi) := \begin{cases} \beta e^{\beta D} e^{\beta \xi} \ if \ \xi \le -D \\ 0 \ otherwise \end{cases} \tag{22}$$

that satisfies the general restrictions imposed in (5). D can be interpreted as the *delay time* of the moderation of the originally prescribed error relaxation. This function takes the value β at $\xi=-D$, it cannot be differentiated only in $\xi=-D$. For illustrative purposes the behavior of the function $x_1(t)$ is described in Figure **1** for an appropriate parameter settings. The figures well illustrate the expectation that over the wedge the 2nd derivative of V is negative while the error is great. The calculations reveal that the small error approximately exponentially approaches zero and that this exponent can be estimated as the slope of the wedge's lines. In the forthcoming part the method is used for active vibration damping. The application of the practically proposed (21) instead of the "theoretically desirable" (19) reveals itself in the "underestimation" of the coefficient of the term quadratic in dh/dt. Therefore in this example the phase curve of V cannot reach the wedge and d^2V/dt^2 becomes positive "over the wedge".

3 Application to vibration control

For this purpose a simple paradigm, a pair of coupled damped linear oscillators under external excitation is considered. A mass point is hanging from the ceiling on a spring. Another spring is attached to the mass point below, the end of which externally is moved along a prescribed height vs. time function (external excitation). In parallel with the mass point hanging from the ceiling a 2nd mass point is inserted between two damped springs. This mass point cannot be controlled and is not modeled by the controller. In parallel with the spring holding the 1st mass point an element exerting active force is inserted. In the simulations sinusoidal external excitation is applied at the frequency that would be equal to the eigenfrequency of the 1st mass point without coupling. For the "active control" there is given an imperfect model of the system containing only the 1st mass point and its spring and neglecting the coupled subsystem and the external excitation. On the basis of that some excitation is calculated to obtain a desired system response $\mathbf{i}^d$ as $\mathbf{e}=\boldsymbol{\varphi}(\mathbf{i}^d)$. The "adaptive active control" refines this model in the following manner. If we apply the above approximate excitation, according to the actual system's inverse dynamics described by the unknown function a

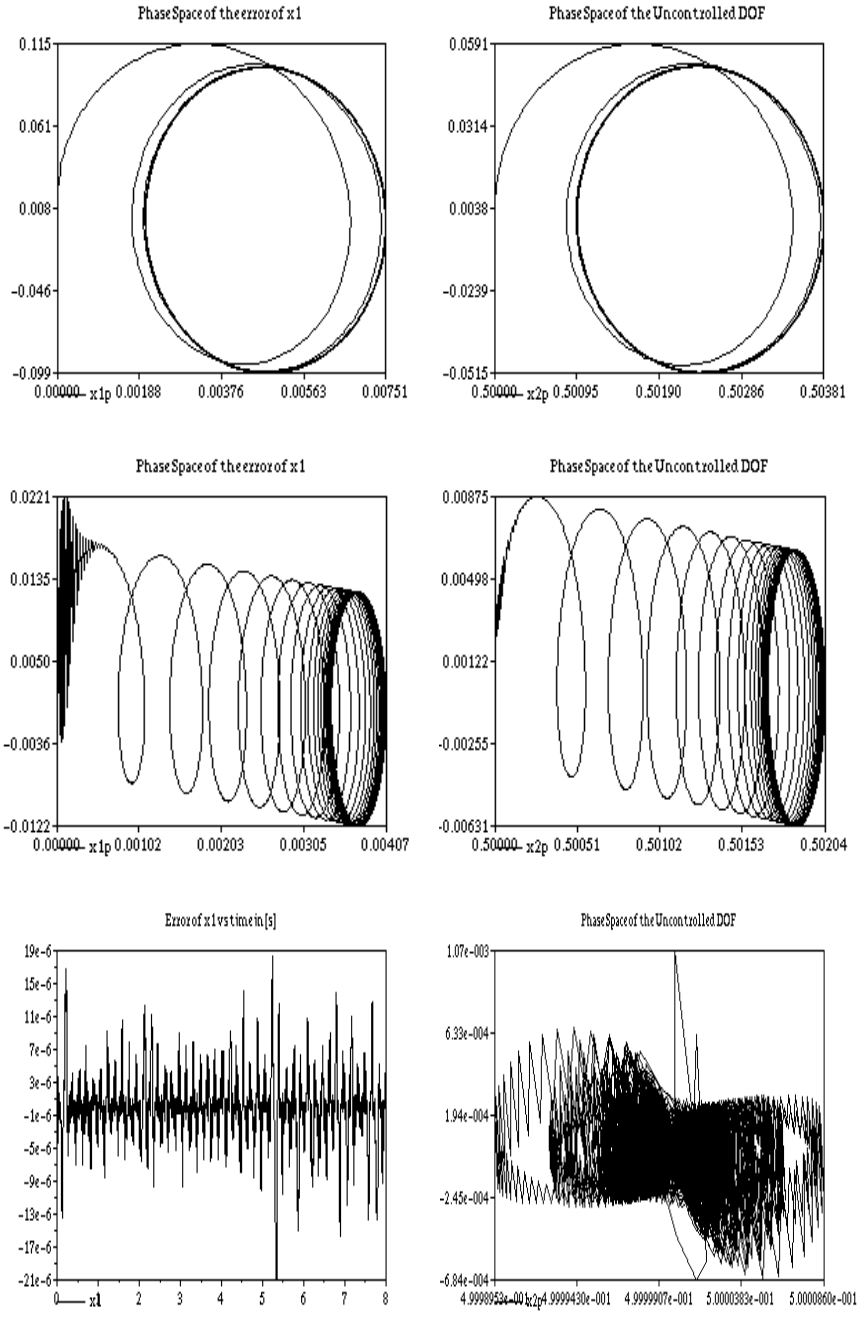

Figure 2: Vibration at the resonance frequency without active damping (upper row), with active non-adaptive damping with the parameters D = 2 ms, ν= 960 s^{-2}, μ= 384 s^{-2}, ρ=1.5ν [s^{-1}], ε=10^{-6} m, and exp(β×10^{-3} s)≈0.99 (middle row), and active damping with adaptivity (lower row).

realized response $\mathbf{i}^r=\psi(\varphi(\mathbf{i}^d))=f(\mathbf{i}^d)$ is obtained instead of the desired one, $\mathbf{i}^d$. Normally one can obtain information via observation only on the function $f()$ considerably varying in time, and no any possibility exists to directly "manipulate" the nature of this function: only $\mathbf{i}^d$ as the input of $f()$ can be “deformed” to $\mathbf{i}^{d*}$ to achieve and maintain the $\mathbf{i}^d=f(\mathbf{i}^{d*})$ state. [Only the *model function* $\varphi()$ can directly be manipulated.] On the basis of the modification of the method of renormalization widely applied in Physics the following "scaling iteration" was suggested for finding the proper deformation:

$$\begin{gathered}\mathbf{i}_0;\mathbf{S}_1\mathbf{f}(\mathbf{i}_0)=\mathbf{i}_0;\mathbf{i}_1=\mathbf{S}_1\mathbf{i}_0;\ldots;\mathbf{S}_n\mathbf{f}(\mathbf{i}_{n-1})=\mathbf{i}_0;\\ \mathbf{i}_{n+1}=\mathbf{S}_{n+1}\mathbf{i}_n;\mathbf{S}_n\xrightarrow[n\to\infty]{}\mathbf{I}\end{gathered} \tag{23}$$

in which the $\mathbf{S}_n$ matrices denote some linear transformations that map the observed response to the desired one, and the construction of each matrix corresponds to a step in the adaptive control. It is evident that if this series converges to the identity operator just the proper deformation is approached, and the controller „learns” the behavior of the observed system by step-by-step amendment and maintenance of the initial model. Since (23) does not unambiguously determine the possible applicable quadratic matrices, we have additional freedom in choosing appropriate ones. At the present application a simple rotation was applied that turns the observed vector to the direction of the desired one in (23). Following that an appropriate shrink/dilatation is applied in only in the direction of the rotated vector to make the two vectors exactly equal to each other. This shrink/dilatation leaves the orthogonal subspace of the rotated vector invariant. All the other algebraic and convergence aspects are detailed in the papers cited in the introduction.

The results of simulation are given in Figure **2**. It can be seen in the 1st column that even the non-adaptive active vibration control designed on simple kinematic prescriptions shrinks the vibration of the controlled DOF from the [1.9 *mm*, 7.5

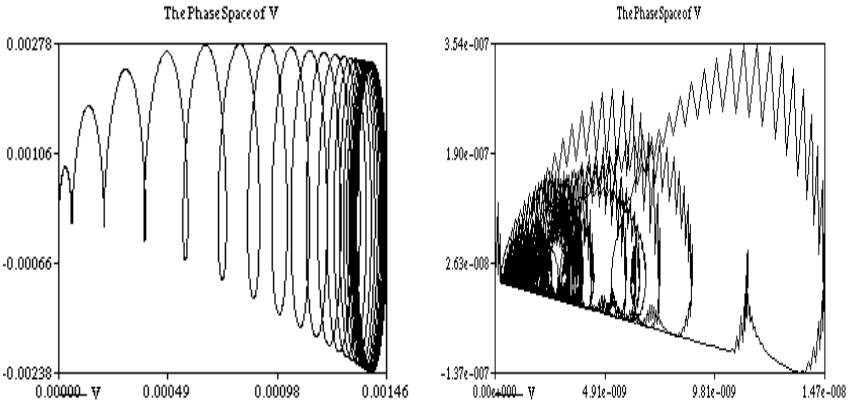

Figure 3: The phase space of *V* for the „non-adaptive” and the „adaptive” case

mm] interval to the [~3 *mm*, ~4 *mm*] region. Due to the external disturbances the active controller designed on purely kinematic basis cannot realize this design. The adaptive law helps the active controller to approach the original "kinematic design" in a far more accurate manner. This shrinks the range of the vibration within the $\pm 2 \times 10^{-5}$ *m* limits, which is a further drastic improvement. As a consequence the vibration of the uncontrolled and unmodeled degree of freedom is also decreased in a drastic manner (the 2nd column in Fig. 2). Figure 3 reveals the phase trajectory of V on which the controller's design was based. It is evidently compatible with the 2nd chart in Fig. 1, especially in the adaptive case when the kinematic design is more strictly approached.

4 Conclusions

In this paper a simple control design was proposed for vibration control purposes. The approach is inspired by the *CRONE* method and tackles the problem from the basis of linear control by replacing the numerical approximation of the fractional derivatives with the more general concept of time-invariant Green functions. Due to the application of a nonlinear feedback term it keeps the task of designing the desired damping within the realm of simple kinematic considerations. The controller's adaptive nature guarantees the realization of this kinematic design. In this manner the burden of creating a very sophisticated linear design can be evaded and the task can be pushed to the adaptive controller. The applicability of the approach was illustrated via simulations based on the behavior of a simple paradigm, on a pair of coupled damped linear oscillators near one of the resonance frequencies. One of the oscillators of the paradigm serves as internal degree of freedom neither modeled nor directly controlled. It was found that this approach is useful for solving even more complicated vibration damping problems in the case of coupled linear or nonlinear systems.

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