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FRACTIONAL ORDER DYNAMICS IN SOME DISTRIBUTED PARAMETER SYSTEMS

J. A. Tenreiro Machado, Isabel S. Jesus
Department of Electrical Engineering, Institute of Engineering of Porto
Rua Dr. António Bernardino de Almeida, 4200-072 Porto
Portugal
{jtm,isj}@dee.isep.ipp.pt

ABSTRACT
Fractional Calculus (*FC*) goes back to the beginning of the theory of differential calculus. Nevertheless, the application of *FC* just emerged in the last two decades, due to the progress in the area of chaos that revealed subtle relationships with the *FC* concepts. In the field of dynamical systems theory some work has been carried out but the proposed models and algorithms are still in a preliminary stage of establishment. Having these ideas in mind, the paper discusses a *FC* perspective in the study of the dynamics and control of some distributed parameter systems.

KEY WORDS
Fractional calculus, modelling, dynamical systems.

1. Introduction

The generalization of the concept of derivative $D^\alpha[f(x)]$ to non-integer values of α goes back to the beginning of the theory of differential calculus. In fact, Leibniz, in his correspondence with Bernoulli, L'Hôpital and Wallis (1695), had several notes about the calculation of $D^{1/2}[f(x)]$. Nevertheless, the development of the theory of Fractional Calculus (*FC*) is due to the contributions of many mathematicians such as Euler, Liouville, Riemann and Letnikov [1-3]. The adoption of the *FC* in control algorithms has been recently studied using the frequency and discrete-time domains [4,5]. Nevertheless, this research is still giving its first steps and further investigation is required.

This article studies the dynamics and control of classical distributed parameter linear systems. In this perspective, the paper is organized as follows. Section 2 presents the main mathematical aspects of the theory of *FC*. Section 3 analyzes the dynamics of partial differential equations, corresponding to electrical transmission lines and to heat diffusion systems, on the perspective of *FC*. Finally, section 4 draws the main conclusions.

2. Theory of Fractional Calculus

2.1. Main Mathematical Aspects

Since the foundation of the differential calculus the generalization of the concept of derivative and integral to a non-integer order α has been the subject of several approaches. Due to this reason there are various definitions of fractional-order integrals (Table I) which are proved to be equivalent.

Table I- Some Definitions of Fractional-Order Integrals

Riemann-Liouville	$\left(I^{\alpha}_{a+}\varphi\right)(x)=\frac{1}{\Gamma(\alpha)}\int_a^x \frac{\varphi(t)}{(x-t)^{1-\alpha}}\,dtt,\ a<x$
	$\left(D^{\alpha}_{a+}\varphi\right)(x)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x \frac{\varphi(t)}{(x-t)^{\alpha}}\,dt,\ a<x$
Grünwald-Letnikov	$\left(I^{\alpha}_{a+}\varphi\right)(x)=\frac{1}{\Gamma(\alpha)}\lim_{h\to+0}\left[h^{\alpha}\sum_{j=0}^{[(x-a)/h]}\frac{\Gamma(\alpha+j)}{\Gamma(j+1)}\varphi(x-jh)\right]$
Laplace	$L\left\{I^{\alpha}_{0+}\varphi\right\}=L\{\varphi\}/s^{\alpha}$, $\mathrm{Re}(\alpha)>0$
	$L\left\{D^{\alpha}_{0+}\varphi\right\}=s^{\alpha}L\{\varphi\}$, $\mathrm{Re}(\alpha)\geq 0$

Table II- Fractional-Order Integrals of Several Functions

$\varphi(x),\ x\in\Re$	$\left(I^{\alpha}_{+}\varphi\right)(x),\ x\in\Re,\ \alpha\in C$
$(x-a)^{\beta-1}$	$\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(x-a)^{\alpha+\beta-1},\ \mathrm{Re}(\beta)>0$
$e^{\lambda x}$	$\lambda^{-\alpha}e^{\lambda x},\ \mathrm{Re}(\lambda)>0$
$\begin{cases}\sin(\lambda x)\\ \cos(\lambda x)\end{cases}$	$\lambda^{-\alpha}\begin{cases}\sin(\lambda x-\alpha\pi/2)\\ \cos(\lambda x-\alpha\pi/2)\end{cases},\ \lambda>0, \mathrm{Re}(\alpha)>1$
$e^{\lambda x}\begin{cases}\sin(\gamma x)\\ \cos(\gamma x)\end{cases}$	$\frac{e^{\lambda x}}{\left(\lambda^2+\gamma^2\right)^{\alpha/2}}\begin{cases}\sin(\gamma x-\alpha\phi)\\ \cos(\gamma x-\alpha\phi)\end{cases},\ \phi=\arctan(\gamma/\lambda),\ \gamma>0, \mathrm{Re}(\lambda)>1$

Based on the proposed definitions it is possible to calculate the fractional-order integrals/derivatives of several functions (Table II). Nevertheless, the problem of devising and implementing fractional-order algorithms is not trivial and will be the matter of the next sections.

2.2. Approximations to Fractional-Order Derivatives

In this section we analyze two methods for implementing fractional-order derivatives, namely the frequency-based and the discrete-time approaches, and its implication in control algorithms.

In order to analyze a frequency-based approach to D^α, $0 < \alpha < 1$, let us consider the recursive circuit represented on Figure 1 such that:

$$I = \sum_{i=0}^{n} I_i \ , \ R_{i+1} = \frac{R_i}{\varepsilon} \ , \ C_{i+1} = \frac{C_i}{\eta} \tag{1}$$

where η and ε are scale factors, I is the current due to an applied voltage V and R_i and C_i are the resistance and capacitance elements of the i^{th} branch of the circuit.

The admittance $Y(j\omega)$ is given by:

$$Y(j\omega) = \frac{I(j\omega)}{V(j\omega)} = \sum_{i=0}^{n} \frac{j\omega C \varepsilon^i}{j\omega CR + (\eta\varepsilon)^i} \tag{2}$$

Figure 2 shows the asymptotic Bode diagram of amplitude of $Y(j\omega)$. The pole and zero frequencies (ω_i and ω'_i) obey the recursive relationships:

$$\frac{\omega'_{i+1}}{\omega'_i} = \frac{\omega_{i+1}}{\omega_i} = \varepsilon\eta \ , \ \frac{\omega_i}{\omega'_i} = \varepsilon \ , \ \frac{\omega'_{i+1}}{\omega_i} = \eta \tag{3}$$

From the Bode diagram of amplitude or of phase, the average slope m' can be calculated as:

$$m' = \frac{\log \varepsilon}{\log \varepsilon + \log \eta} \tag{4}$$

Consequently, the circuit of Figure 1 represents an approach to D^α, $0 < \alpha < 1$, with $m' = \alpha$, based on a recursive pole/zero placement in the frequency domain.

As mentioned in section II, the Laplace definition for a derivative of order $\alpha \in C$ is a 'direct' generalization of the classical integer-order scheme with the multiplication of the signal transform by the s operator. Therefore, in what concerns automatic control theory this means that frequency-based analysis methods have a straightforward adaptation to their fractional-order counterparts. Nevertheless, the implementation based on the Laplace definition (adopting the frequency domain) requires an infinite number of poles and zeros obeying a recursive relationship [4]. In a real approximation the finite number of poles and zeros yields a ripple in the frequency response and a limited bandwidth.

The mathematical definition of a derivative of fractional order has been the subject of several different approaches [1]. For example, Eq. (5) and Eq. (6), represent the Laplace (for zero initial conditions) and the Grünwald-Letnikov definitions of the fractional derivative of order α of the signal $x(t)$:

$$D^\alpha [x(t)] = L^{-1}\{s^\alpha X(s)\} \tag{5}$$

$$D^\alpha x(t) = \lim_{h \to 0} \left[\frac{1}{h^\alpha} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)} x(t-kh) \right] \tag{6}$$

where Γ is the gamma function and h is the time increment. This formulation [5] inspired a discrete-time calculation algorithm, based on the approximation of the time increment h through the sampling period T, yielding the equation in the z domain:

$$\frac{Z\{D^\alpha x(t)\}}{X(z)} \approx \frac{1}{T^\alpha} \sum_{k=0}^{r} \frac{(-1)^k \Gamma(\alpha+1)}{k! \ \Gamma(\alpha-k+1)} z^{-k} = \left(\frac{1-z^{-1}}{T} \right)^\alpha \tag{7}$$

An implementation of (7) corresponds to a r-term truncated series or to a Padé fraction.

An important aspect of fractional-order controllers [4] can be illustrated through the elemental control system represented in Figure 3, with open-loop transfer function $G(s) = Ks^{-\alpha}$ $(1 < \alpha < 2)$ in the forward path. The open-loop Bode diagrams (Figure 4) of amplitude and phase have a slope of -20α *dB/dec* and a constant phase of $-\alpha\pi/2$ *rad*, respectively. Therefore, the closed-loop system has a constant phase margin of $\pi(1 - \alpha/2)$ *rad* that is independent of the system gain K.

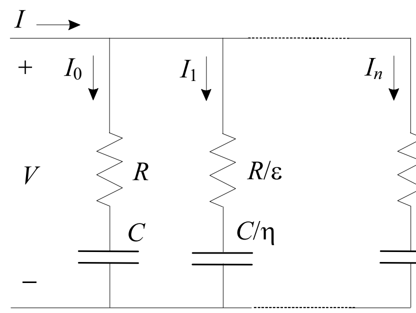

Fig. 1. Electrical circuit with a recursive association of resistance and capacitance elements.

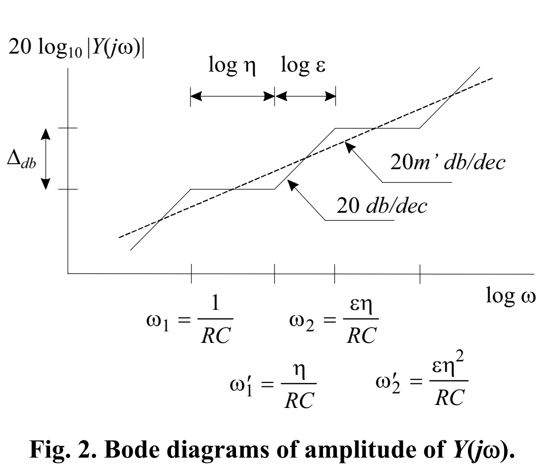

Fig. 2. Bode diagrams of amplitude of $Y(j\omega)$.

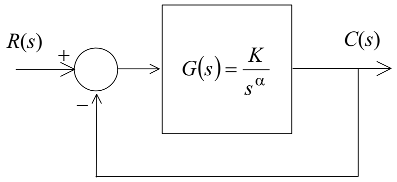

Fig. 3. Block diagram for an elemental feedback control system of fractional order α.

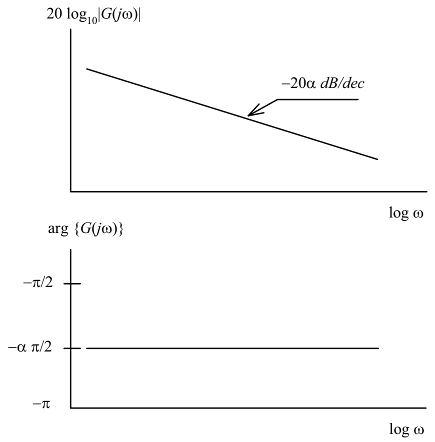

Fig. 4. Open-loop Bode diagrams of amplitude and phase for a system of fractional order $1 < \alpha < 2$.

3. Distributed Parameter Systems

3.1. Electrical Transmission Lines

The main responsible for a complete mathematical analysis of signal propagation on transmissions lines was Olivier Heaviside that published a book, in 1880, based on Maxwell electromagnetic theory [6].

During the twenty century electrical power transmission, telecommunication and microwave engineering, and the subsequent development of innumerable applications, made popular the introduction of transmission line theory in electrical engineering curricula [7,8].

The differential equations for a uniform transmission line are found by considering an infinitesimal length dx located at coordinate x.

This line section has series inductance and resistance Ldx and Rdx and shunt conductance and capacitance Gdx and Cdx as depicted in Fig. 5. The application of the Kirchoff's laws to the circuit leads to the set of *PDE*s:

$$\partial v(x,t)/\partial x = -L\partial i(x,t)/\partial t - Ri(x,t) \quad (8a)$$

$$\partial i(x,t)/\partial x = -C\partial v(x,t)/\partial t - Gv(x,t) \quad (8b)$$

where t represent time, v voltage and i electrical current.

A few simple calculations allow us to eliminate one variable and to explicit the differential equation either to v or to i, yielding:

$$\partial^2 v(x,t)/\partial x^2 = LC\partial^2 v(x,t)/\partial t^2 + (LG+RC)\partial v(x,t)/\partial t + RGv(x,t) \quad (9a)$$

$$\partial^2 i(x,t)/\partial x^2 = LC\partial^2 i(x,t)/\partial t^2 + (LG+RC)\partial i(x,t)/\partial t + RGi(x,t) \quad (9b)$$

It is interesting to note that when $L = 0$ and $G = 0$ equation (9) reduces to the equivalent of the heat diffusion equation (see section 3.2), where v and i are the analogs of the temperature and the heat flux, respectively.

To analyze the transmission lines in the frequency domain it is considered the Fourier transform operator F such that $I(x,j\omega) = F\{i(x,t)\}$ and $V(x,i\omega) = F\{v(x,t)\}$ (with $j = (-1)^{1/2}$) and equations (8) are transformed to:

$$dV(x,j\omega)/dx = -Z(i\omega)I(x,j\omega) \quad (10a)$$

$$dI(x,j\omega)/dx = -Y(i\omega)V(x,j\omega) \quad (10b)$$

where $Z(i\omega) = R + j\omega L$ and $Y(i\omega) = G + j\omega C$. In the same line of thought equations (9) are transformed to:

$$d^2V(x,j\omega)/dx^2 = -Z(i\omega)Y(i\omega)V(x,j\omega) \quad (11)$$

which as a solution in the frequency domain of the type:

$$V(x,j\omega) = A_1 e^{\gamma x} + A_2 e^{-\gamma x} \quad (12a)$$

$$I(x,j\omega) = Z_c^{-1}(A_2 e^{-\gamma x} - A_1 e^{\gamma x}) \quad (12b)$$

where $Z_c(j\omega)=[Z(j\omega)\ Y^{-1}(j\omega)]^{1/2} = [(R+ j\omega L)/(G+j\omega C)]^{1/2}$ (characteristic impedance) and $\gamma(j\omega) = [Z(j\omega)\ Y(j\omega)]^{1/2} = \alpha(\omega) + j\beta(\omega)$.

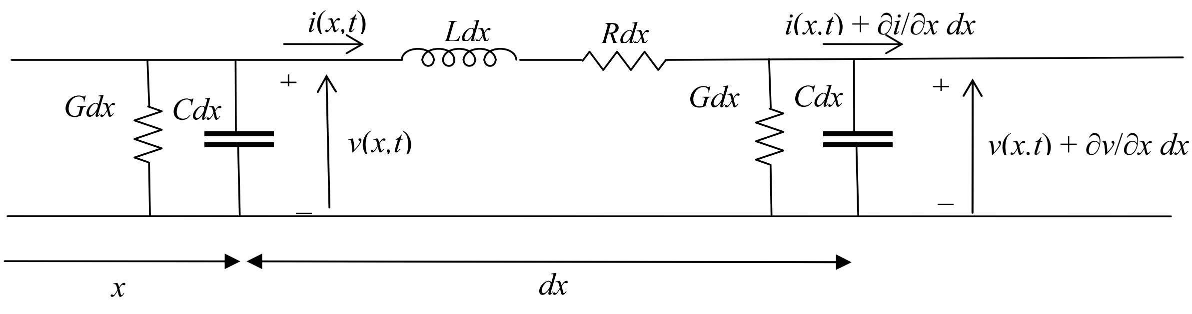

Fig. 5. Electrical circuit of an infinitesimal portion of a uniform transmission line.

These expressions have two terms corresponding to waves traveling in opposite directions: the term proportional to $e^{-\gamma x}$ is due to the signal applied at the line input while the term $e^{\gamma x}$ represents the reflected wave.
For a transmission line of length l it is usual to adopt as variable the distance up to the end given by:

$$y = l - x \tag{13}$$

If V_2 and I_2 represent the voltage and current at the end of the transmission line then the Fourier transforms of equation (8) at coordinate y are given by:

$$V(x, j\omega) = V_2 ch(\gamma y) + I_2 Z_c sh(\gamma y) \tag{14a}$$

$$I(x, j\omega) = V_2 Z_c^{-1} sh(\gamma y) + I_2 ch(\gamma y) \tag{14b}$$

Therefore, for a loading impedance $Z_2(j\omega)$ we have $V_2(j\omega) = Z_2(j\omega)\ I_2(j\omega)$ and the input impedance $Z_i(j\omega)$ of the transmission line results:

$$Z_i(j\omega) = \left[Z_2 ch(\gamma y) + Z_c sh(\gamma y)\right]\left[Z_2 Z_c^{-1} sh(\gamma y) + ch(\gamma y)\right]^{-1} \tag{15}$$

Typically are considered three cases at the end of the line, namely a short circuit, an open circuit and an adapted line, that simplify equation (15) yielding:

$$V_2 = 0, Z_2(j\omega) = 0, Z_i(j\omega) = Z_c(j\omega) th(\gamma l) \tag{16a}$$

$$I_2 = 0, Z_2(j\omega) = \infty, Z_i(j\omega) = Z_c(j\omega) cth(\gamma l) \tag{16b}$$

$$Z_2(j\omega) = Z_c(j\omega), Z_i(j\omega) = Z_c(j\omega) \tag{16c}$$

The classical perspective is to study lossless lines (*i.e.*, $R = 0$ and $G = 0$), reasonable in power systems, and approximations in the frequency domain leading to two-port networks with integer order elements. This is surprising because the transcendental equations (15) and (16) may lead both to integer and fractional-order expressions. For example, in the case of an adapted line (with R, C, L, $G \in \Re^+$), we can have from half-order fractional capacitances up to half-order fractional inductances, that is $-\pi/4 \le \arg\{Z_c(j\omega)\} \le \pi/4$, according with the expressions:

$$L = 0 \text{ and } G = 0 \Rightarrow Z_c(j\omega) = \left[(j\omega)^{-1} RC^{-1}\right]^{1/2} \tag{17a}$$

$$R = kL \text{ and } G = kC\ (k \in \Re^+) \Rightarrow Z_c(j\omega) = (RL^{-1})^{1/2} \tag{17b}$$

$$R = 0 \text{ and } C = 0 \Rightarrow Z_c(j\omega) = (j\omega L G^{-1})^{1/2} \tag{17c}$$

These results, overlooked in the (integer-order point of view) classical textbooks, suggest possible strategies for implementing fractional-order impedances, somehow as standard microstrips and striplines work in microwave circuits. In fact, this hardware strategy of implementing fractional-order derivatives has been recently pointed out in order to avoid computational approximation schemes [9,10]. Therefore, an alternative to exploring fractal geometries [11] and dielectric properties [12] to achieve fractional capacitors we can also turn our attention to the distributed characteristics of this type of system in order to design integrated circuits capable of implementing directly fractional derivatives.

3.2. Heat Diffusion

In many industrial applications it is important that the temperature distribution in the work pieces should be as uniform as possible. It is clearly difficult to determine the temperature distribution in the interior of the material or system, but the measurement of the surface temperature is routine. Therefore, we encounter the problem of the observability and control of the temperature distribution throughout the material from the available surface measurements.
The heat diffusion is represented by a linear partial differential equation (*PDE*) [13-14]:

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{18}$$

where k is the diffusivity, t is the time, u is the temperature and (x,y,z) are the space cartesian coordinates.
This system involves the integration of a *PDE* of parabolic type for which the standard theory of parabolic *PDE*s guarantees the existence of a unique solution.
For the case of a planar perfectly isolated surface we apply a constant temperature U_0 at $x = 0$ and we analyse the heat diffusion along horizontal coordinate x. The heat diffusion, under the previous conditions, is characterized by a model of non-integer order. In fact, the *PDE* solution in the s-*d*omain corresponds to the expression:

$$U(x,s) = \frac{U_0}{s} G(s),\ G(s) = e^{-x\sqrt{\frac{s}{k}}} \tag{19}$$

where x is the space coordinate and U_0 is the boundary condition.
The corresponding solution in the time domain yields:

$$u(x,t) = U_0 erfc\left(\frac{x}{2\sqrt{kt}}\right) = U_0\left(1 - \frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2\sqrt{kt}}} e^{-u^2}\,du\right) \tag{20}$$

In our study we adopt the Crank-Nicolson implicit numerical integration based on the discrete approximation to differentiation, yielding the equation:

$$-r\,u[j+1,i+1] + (2+r)\,u[j+1,i] - r\,u[j+1,i-1] = \\ = r\,u[j,i+1] + (2-r)\,u[j,i] + r\,u[j,i-1] \tag{21}$$

where $r = k\Delta t(\Delta x^2)^{-1}$, $\{\Delta x, \Delta t\}$ and $\{i, j\}$ are the increments and integration indices for space and time, respectively.
We verify that the results obtained through the numerical approach differ from the analytical results for low frequencies. This is illustrated in Figure 6, which depicts the polar diagram of $G(j\omega)$, for $x = 3.0$ m and $k = 0.042$ m^2s^{-1}, both for the theoretical and numerical methods.

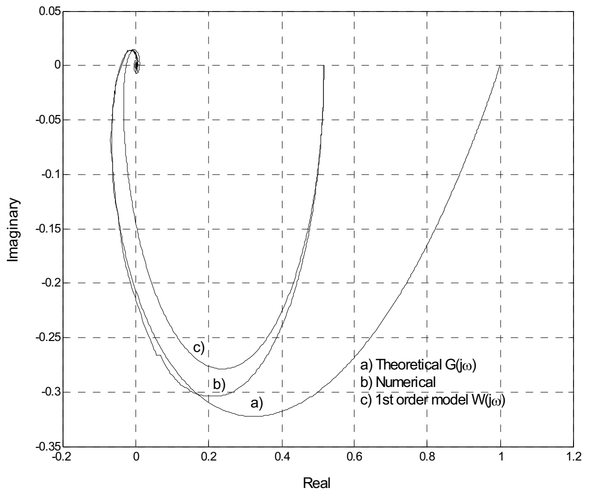

Fig. 6. Polar diagram of $G(j\omega)$ for x=3.0m, k =0.042m^{2}s^{-1}.

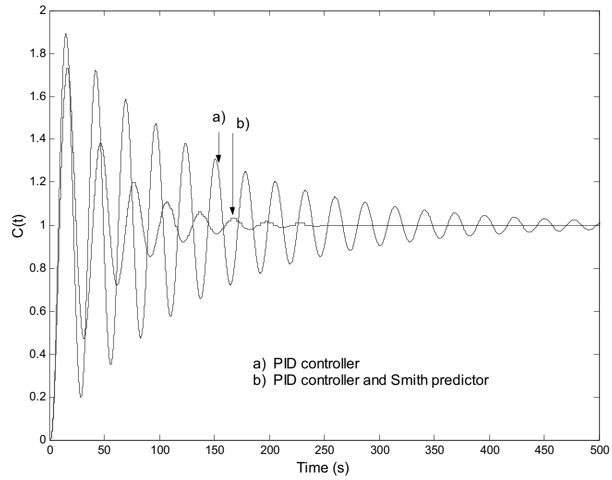

Fig. 8. Step response of closed-loop system for k_p = 0.3484, T_d = 0.0825 s and T_i = 0.33 s

It is clear that the chart has similarities to those of systems with time-delay. In this line of thought we consider the control of the heat system with two types of algorithms. In a first phase (Fig. 7 a) we adopt the simple *PID* controller ($G_c(s) = k_p\ [1+sT_d+(sT_i)^{-1}]$) tuned according with the Ziegler-Nichols open loop method. In this case the tuning heuristics leads to an approximate model $W(s) = k_B e^{-sT}/(s\tau + 1)$ with $k_B = 0.52$, $T = 0.165$, $\tau = 1.235$ and the *PID* parameters $k_p = 0.3484$, $T_d = 0.0825$ *s*, $T_i = 0.33$ *s*. Figure 8 depicts the step response of the closed-loop system for $R(s) = 1/s$ and $x = 3.0$ m.

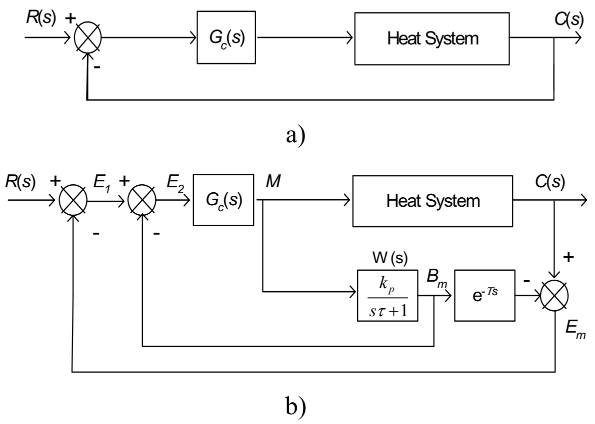

Fig. 7. Block diagram of closed-loop system with a) PID b) PID and Smith predictor.

In a second phase (Fig. 7 b), we adopt the previous *PID* controller but we apply the Smith predictor. This algorithm a well-known dead-time compensation technique that is very effective in improving the control of processes having time delays. Figure 8 shows the corresponding time response for $R(s) = 1/s$.

It is clear that the Smith predictor leads to a superior response, revealing that we can adopt with success classical control algorithms in fractional-order dynamical systems.

4. Conclusions

This paper presented the fundamental aspects of the theory of *FC*, the main approximation methods for the fractional-order derivatives calculation and the implication of the *FC* concepts on the extension of the classical systems theory. Bearing these ideas in mind, two distributed parameter linear systems were described and their dynamics was analyzed in the perspective of fractional calculus. It was shown that fractional-order models capture phenomena and properties that classical integer-order simply neglect. In this line of thought, this article is a step towards the development of systems modeling and control based on the theory of *FC*.

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