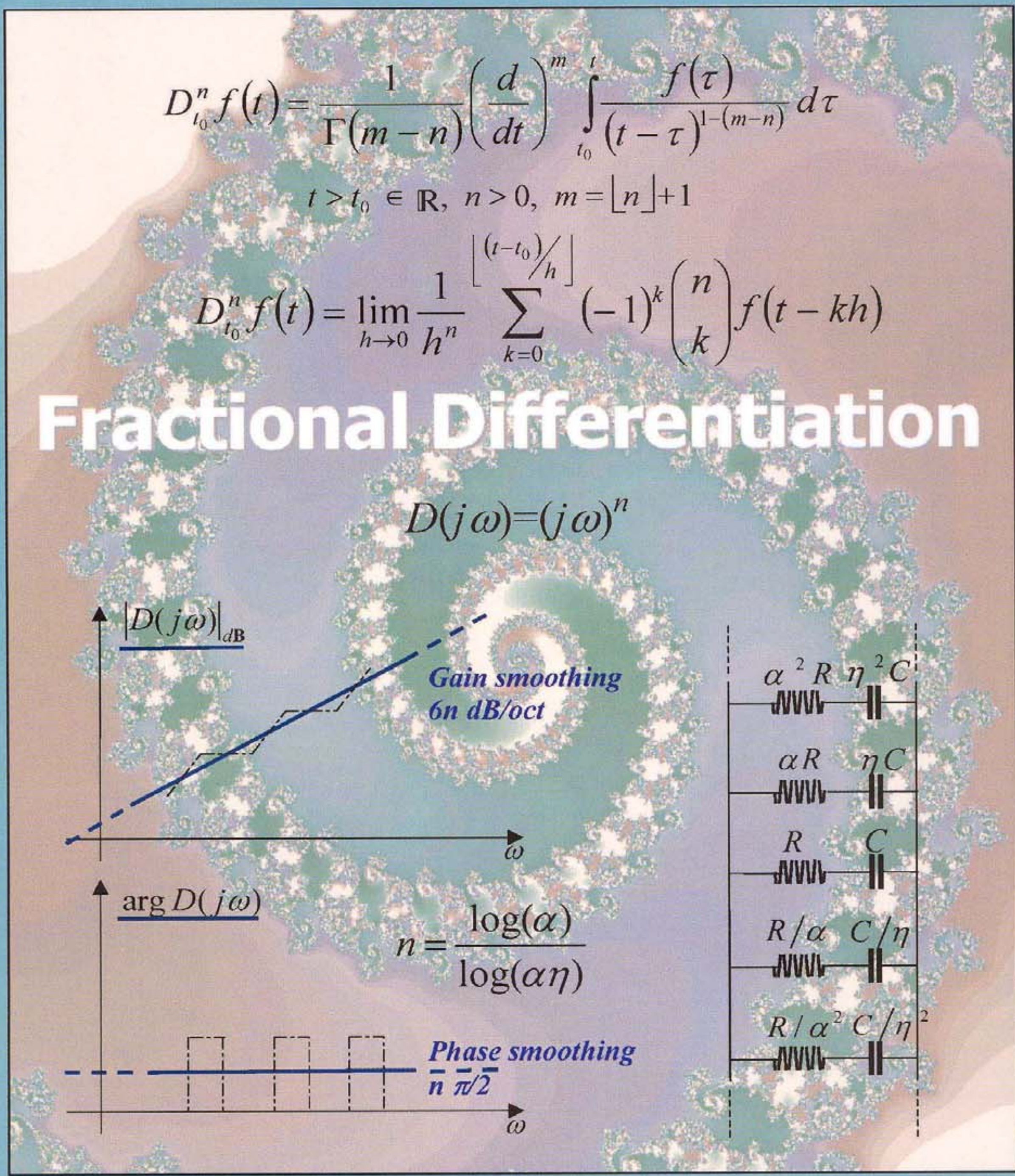
Fractional differentiation and its applications



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Short-Circuit Impedance of Power Transformers: the Fractional Order Calculus Approach

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Abstract –– *This paper reports investigation on the modeling of the short circuit impedance of power transformers, using fractional order calculus, analyzing the influence of the diffusion phenomena in the windings. This aims to better characterize the medium frequency range behavior of leakage inductances of power transformer models, which include terms to represent the magnetic field diffusion in the windings. Calculated and measured values are shown and compared.*

1 Introduction

Electromagnetic field in conductive media is described by a diffusion type equation, in the absence of displacement currents. A non-linear diffusion is considered to occur within the power transformer core, whereas a linear diffusion is supposed to occur in their windings and usually modeled by leakage impedances. The precise estimation of these leakage impedances, prior to transformer building, is very important to tune the de-rating K factor and to optimize several power network characteristics, such as short-circuit currents, network protection sub-systems, and assess power quality.

Conductors, like copper or aluminum, standing diffusion process of magnetic fields at frequency ω ($\omega > 0$), such as skin and proximity effects, are supposed to typically show an impedance Z_{cond} proportional to the square root of ω, $Z_{cond} \propto \omega^{1/2} e^{i\pi/4}$, ($i = (-1)^{1/2}$). This behavior can be obtained solving the magnetic field diffusion equations within the conductor. If measured impedances, including terms from diffusion phenomena, show arguments different from $\pi/4$ and magnitudes not proportional to $\omega^{1/2}$, the diffusion process they represent might be described by a differential equation with non-integer derivatives, usually called a fractional order differential equation.

This paper uses the extension of magnetic field diffusion equations to situations where α, the order of the time partial derivative in the diffusion equation, assumes fractional values [1], leading to conductor impedances taking the form $Z_{cond} \propto \omega^{½} e^{i\,\pi/4}$. The fractional approach, here proposed, will be shown to provide a better description of the transformer short-circuit impedance behavior with frequency.

The magnetic field diffusion at the power transformer windings is studied, with Maxwell equations extended with a fractional order Faraday law. Solving the fractional order differential diffusion equation obtained, the voltage drops in the frequency domain and equivalent leakage impedance components, due to diffusion, are found.

Section 2 proposes a magnetic field fractional order diffusion model. Section 3 applies this fractional order model to power transformers, to calculate the winding fractional dispersion impedance and section 4 give suitable high and low frequency approximations. Section 5 compares experimental and calculated results concerning short-circuit impedances of power transformers.

2 Magnetic Field Fractional Order Diffusion

Motionless magnetic field systems, consisting primarily of magnetizable and conducting materials with conductivity $1/\rho$, permittivity ε, permeability μ and characteristic length l, operated at frequencies $\omega << [l\,(\varepsilon\mu)^{1/2}]^{-1}$ (quasi-steady regime), experience mainly magnetic field diffusion. Assuming all materials to be electrically linear, homogeneous, isotropic, negligible charges [2] and neglecting displacement currents ($1/\rho >> \omega\varepsilon$), when compared to conduction currents, the relevant Maxwell and material equations [3][4], are $\nabla \cdot \vec{B} = 0$, $\nabla \times \vec{H} \approx \vec{J}$, $\vec{B} = \mu\vec{H}$, $\vec{E} = \rho\vec{J}$, where ∇ is the nabla operator, $\vec{B}$ is the magnetic flux density vector, $\vec{H}$ is the magnetic field vector and $\vec{J}$ is the electrical current density vector.

2.1 Fractional Order Faraday's Law

Fractional order Faraday's law [1] is expressed as a differential equation with fractional order α, being an extension of the classical Electrical field $\vec{E}$ Faraday's law:

$$\nabla \times \vec{E} = -{}_m D_t^{\alpha} \vec{B}, \tag{1}$$

where ${}_m D_t^{\alpha}$ represents the time t partial derivative of fractional order α [5][6][7], defined for $t > m$ (here $m = 0$), or:

$${}_m D_t^{\alpha} \vec{B} = \frac{\partial^{\alpha}}{\partial t^{\alpha}} \vec{B} \quad \text{for } 0 < \alpha < 2 \quad \text{and} \quad t > m \tag{2}$$

The Riemann-Liouville partial derivative of fractional order α, applied to function $f(x,y)$, regarding variable x, for $x > m$, is calculated as:

$${}_m D_x^{\alpha} f(x,y) = \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial x^n} \int_m^x \frac{f(\tau,y)}{(x-\tau)^{\alpha-n+1}} d\tau \tag{3}$$

where n is an integer satisfying $n-1 < \alpha < n$; and Γ represents the gamma function [8].

2.2 Fractional Order Diffusion Vector Equation for $\vec{H}$ Field Inside Conductors

Applying the curl operator to $\nabla \times \vec{H} \approx \vec{J}$, $\nabla \times (\nabla \times \vec{H}) = \nabla \times \vec{J}$, substituting $\vec{J}$ from $\vec{E} = \rho\vec{J}$, $\nabla \times (\nabla \times \vec{H}) = \frac{1}{\rho}(\nabla \times \vec{E})$, and $\vec{E}$ from (1), it follows that:

$$\nabla \times (\nabla \times \vec{H}) = -\frac{1}{\rho}\left[{}_0 D_t^{\alpha} \vec{B}\right] \tag{4}$$

As, from $\nabla \cdot \vec{B} = 0$ and $\vec{B} = \mu \vec{H}$, $\vec{B}$ and $\vec{H}$ present zero divergence, using the vector identity $\nabla \times (\nabla \times \vec{H}) = \nabla(\nabla \cdot \vec{H}) - \nabla^2 \vec{H}$ in (4), (5) is obtained.

$$\nabla^2 \vec{H} - \frac{\mu}{\rho}\left[{}_0 D_t^{\alpha} \vec{H}\right] = 0 \tag{5}$$

The above differential equation describes a magnetic field "diffusion phenomenon" [4]. The equation (5) is an ordinary integer order differential equation if $\alpha = 1$, or a fractional order differential equation if $\alpha \neq 1$. Therefore, (5) is the fractional extension of the classical diffusion equation $\nabla^2 \vec{H} - (\mu/\rho)\left[\partial \vec{H}/\partial t\right] = 0$.

In the following section, (5) will be applied to transformer windings with cylindrical symmetry, in order to obtain a better model for the transformer short-circuit impedance behavior with frequency, mainly in the medium to high frequency range (300-6000 Hz) which includes most current harmonics when the transformer supplies non-linear loads.

3. Fractional Order Diffusion Equation Applied to Transformers

3.1 Fractional Order Diffusion Vector Equation for leakage field $\vec{H}$ in one turn with circular cylindrical geometry.

The single-phase transformer has a ferromagnetic core and two coaxial or concentric cylindrical windings (figure 1), with current flowing only in winding 1. The main induced magnetic flux path (shown in dashed line) is assumed to be all inside the core, and links all the turns of all windings (unity magnetic coupling). The leakage flux (shown in solid lines) through the air or insulators, only partially links the windings turns. Since, compared to the core, air or insulating material present much higher reluctance, leakage inductances can be assumed to be linear.

Magnetic flux line A, at the winding 1 head (figure 2), represents flux linking all the turns of winding 1, but only partly the turns of winding 2. Therefore, there is magnetic coupling between windings due to leakage flux (mutual inductance). Flux represented in line B links only all the turns of winding 1, meaning magnetic coupling between all turns of winding 1. Flux in line C means magnetic coupling between some turns of winding 1.

Figure 3 shows the winding q layers, with m turns in each layer, together with a leakage flux path. To calculate the magnetic field $\vec{H}$ in one turn of layer k, assume the turn with internal radius r_k and the magnetic field direction shown in figure 4. To obtain a closed solution, consider the magnetic field with cylindrical geometry, with vector components only in the z-axis direction. Then, in cylindrical coordinates, $\vec{H}$ is $\vec{H} = H(r,t) \cdot \vec{a}_z$. Its Laplacian, in cylindrical coordinates, is $\nabla^2 \vec{H} = \vec{a}_z \, \nabla^2 H(r,t)$, giving:

$$\nabla^2 \vec{H} = \vec{a}_z \left[\frac{\partial^2}{\partial r^2} H(r,t) + \frac{1}{r}\frac{\partial}{\partial r} H(r,t)\right] \tag{6}$$

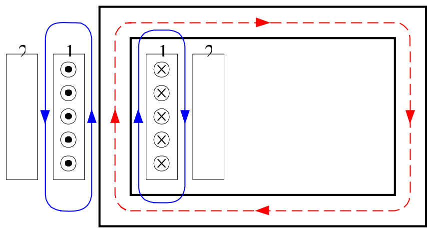

Figure 1: Main (dashed line) and leakage (solid line) fluxes in the transformer.

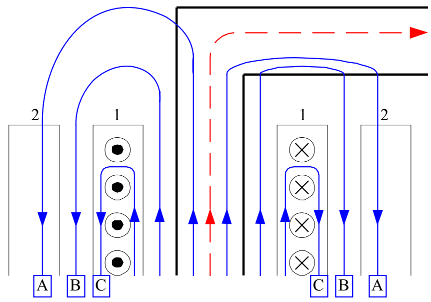

Figure 2: Main (dashed line) and leakage (solid lines) fluxes

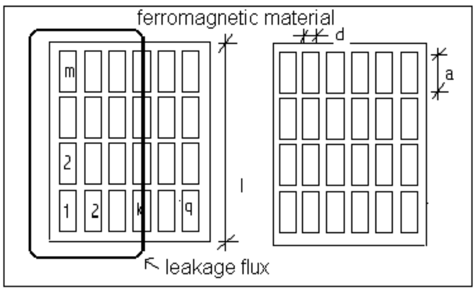

Figure 3: Winding with q layers, and m turns per layer.

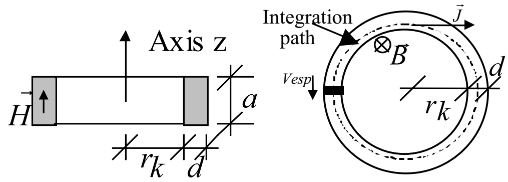

Figure 4: Cross-sections of turn k in layer k of figure 3 winding.

Using (6) in (5), equation (7) is obtained. In cylindrical coordinates, it describes the fractional diffusion phenomenon of the magnetic field strength $H(r,t)$ per winding turn.

$$\frac{\partial^2}{\partial r^2} H(r,t) + \frac{1}{r}\frac{\partial}{\partial r} H(r,t) - \frac{\mu}{\rho}\, {}_0D_t^{\alpha} H(r,t) = 0 \tag{7}$$

3.2 Leakage magnetic field $\vec{H}$

Assuming zero initial conditions and applying Laplace transform (t is the independent variable) to (7), (8) is obtained, where $H(r)$ is the Laplace transform of the magnetic field strength.

$$\frac{d^2}{dr^2} H(r) + \frac{1}{r}\frac{d}{dr} H(r) - \frac{\mu}{\rho} s^{\alpha} H(r) = 0 \tag{8}$$

Multiplying (8) by r^2, and using a new variable x:

$$x = \frac{\mathrm{i}}{\delta} r\,, \tag{9}$$

where $\mathrm{i} = (-1)^{1/2}$, and δ in (10) is the fractional skin depth, (11) is derived (for $n = 0$).

$$\delta = \sqrt{\rho/\left(s^{\alpha}\mu\right)}\,, \tag{10}$$

$$x^2 \frac{d^2}{dx^2} H(x) + x\frac{d}{dx} H(x) + \left(x^2 - n^2\right) H(x) = 0 \tag{11}$$

The previous identity is a classical Bessel differential equation [8] of order n ($n = 0$). For $x_k + d' > x > x_k$, (11) has a family of solutions:

$$H(x) = A\, J_0(x) + B\, Y_0(x) \tag{12}$$

In these solutions $J_0(x)$ and $Y_0(x)$ are respectively the zero order first and second kind Bessel functions. The A and B parameters are calculated using boundary conditions. The domain of the x variable, in terms of r_k and d (figure 4) is:

$$x_k = \frac{\mathrm{i}}{\delta} r_k, \qquad x_k + d' = \frac{\mathrm{i}}{\delta}\left(r_k + d\right) \tag{13}$$

Boundary conditions can be obtained using the integral form of Ampère's law [2] and considering figure 3 and 4. The magnetic field, at the surface inside the turn with radius $r = r_k$, with the current i_s (in the Laplace domain), and considering l as the length of the leakage flux path out of the core (figure 3), is:

$$\oint \vec{H} \cdot d\vec{l} = \int_S \vec{J} \cdot d\vec{s} = F_{mm} = m\ k\ i_s \Rightarrow H(r_k) = \frac{F_{mm}(r_k)}{l} \tag{14}$$

where F_{mm} is the "magnetomotive force" due to the currents crossing the surface limited by the integration contour. Similarly, the magnetic field in the outside surface of the turn, $H(r_k + d)$, is:

$$H(r_k + d) = \frac{F_{mm}(r_k + d)}{l} = \frac{m\,(k-1)\, i_s}{l} \tag{15}$$

Using (10), (12), (13), (14) and (15), the values A and B of (12) can be determined and substituted in (12), to obtain the leakage magnetic field strength:

$$H(x) = \frac{1}{l}\left[\frac{F_{mm}(r_k + d)Y_0(x_k) - F_{mm}(r_k)Y_0(x_k + d')}{J_0(x_k + d')Y_0(x_k) - J_0(x_k)Y_0(x_k + d')}\right]J_0(x) + \frac{1}{l}\left[\frac{F_{mm}(r_k)J_0(x_k + d') - F_{mm}(r_r + d)J_0(x_k)}{J_0(x_k + d')Y_0(x_k) - J_0(x_k)Y_0(x_k + d')}\right]Y_0(x) \tag{16}$$

This equation is difficult to use, since Bessel functions $J_0(x)$ and $Y_0(x)$ are given by infinite series. A possible simplification, suitable for high-frequency modeling of leakage inductances, is to use an asymptotic approximation $J_n(x)$ and $Y_n(x)$ [8] of Bessel functions for high values of x, since, from (9) and (10), the argument x of the Bessel functions increases with increasing frequency.

$$J_n(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) \quad \text{for } n = 0, 1, 2, 3, \ldots \tag{17}$$

$$Y_n(x) = \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) \quad \text{for } n = 0, 1, 2, 3, \ldots \tag{18}$$

Using (9), (13), (16), (17), (18) and simplifying, the fractional equation of the leakage magnetic field (in the Laplace domain) $\vec{H} = H(r) \cdot \vec{a}_z$ is obtained for $r_k < r < r_k + d$, being $H(r)$ given by (19), where sinh is the hyperbolic sinus:

$$H(r) = \frac{\sqrt{r_k (r_k + d)}}{l\ \sinh(d/\delta)}\left[\frac{F_{mm}(r_k + d)}{\sqrt{r_k\, r}} \sinh\left(\frac{r - r_k}{\delta}\right) - \frac{F_{mm}(r_k)}{\sqrt{r (r_k + d)}} \sinh\left(\frac{r - r_k - d}{\delta}\right)\right] \tag{19}$$

After calculating the leakage magnetic field, next the current density vector $\vec{J}$ in the winding turns is determined. Both are needed to estimate leakage inductances.

3.3 Current Density Vector $\vec{J}$ in the Winding Turns

The current density vector $\vec{J}$ is given from $\nabla \times \vec{H} \approx \vec{J}$ and (12), giving $\nabla \times \vec{H} = \nabla \times [A\, J_0(x) + B\, Y_0(x)] \cdot \vec{a}_z \approx \vec{J}$, which can be solved, in cylindrical coordinates:

$$\vec{J} = \left[-\frac{d}{dx}\left[A\, J_0(x) + B\, Y_0(x)\right] \cdot \frac{d}{dr} x \right] \cdot \vec{a}_\phi \tag{20}$$

Differentiating the Bessel equations [8], $\vec{J} = \frac{\sqrt{-1}}{\delta}[A\, J_1(x) + B\, Y_1(x)] \cdot \vec{a}_\phi$, and using (9), (10), (13), the values A, B of (12), (17), (18), and simplifying, in the Laplace domain, $\vec{J} = J(r) \cdot \vec{a}_\phi$ is obtained for $r_k < r < r_k + d$, being $J(r)$ given by (21), where cosh is the hyperbolic co-sinus.

$$J(r) = \frac{\sqrt{r_k\,(r_k + d)}}{l\,\delta \sinh(d/\delta)} \left[\frac{F_{mm}(r_k)}{\sqrt{r\,(r_k + d)}} \cosh\left(\frac{r - r_k - d}{\delta}\right) - \frac{F_{mm}(r_k + d)}{\sqrt{r\, r_k}} \cosh\left(\frac{r - r_k}{\delta}\right) \right] \tag{21}$$

Equations (21) and (19) are useful to calculate the voltage in each winding turn.

3.4 Voltage per Winding Turn, Considering Resistivity and Leakage Flux

The transformer is first considered to have a two turn winding (figure 5). Vector Φ_j represents the leakage flux linked by the conductor of turn j, Φ_k is the leakage flux linked by the conductor of turn k, and Φ_a is the leakage flux across the insulating layers.

Voltage at k turn due to i) the conductor resistance; ii) the self leakage flux Φ_k*.*
The winding voltage of turn k depends on *i*) the conductor resistance; *ii*) the self-leakage flux Φ_k, *iii*) the leakage fluxes Φ_j and Φ_a. The voltage of turn j depends also on *i*) the conductor resistance; *ii*) the self-leakage flux Φ_j, but leakage fluxes Φ_k and Φ_a do not induce a voltage since they do not link the turn j.

The magnetic flux density $\vec{B}$ has zero divergence, being defined as the curl of an auxiliary vector, a potential vector $\vec{A}$, $\vec{B} = \nabla \times \vec{A}$, or considering (1):

$$\nabla \times \vec{E} = -{}_0 D_t^{\alpha} \left(\nabla \times \vec{A} \right) \tag{22}$$

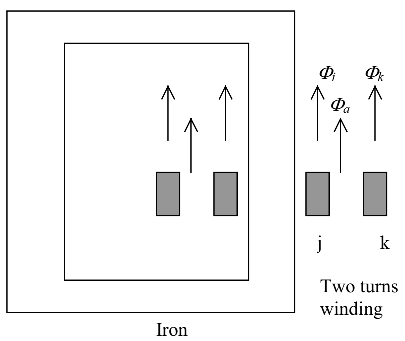

Figure 5: Transformer with a two-turn winding

Rewriting the previous equation $\nabla\times\left[\vec{E}+{}_0 D_t^{\alpha}\vec{A}\right]=0$, it is shown that the curl of the sum of the two vectors is zero. Therefore, ($\vec{E}+{}_0 D_t^{\alpha}\vec{A}$) is a conservative field and can be defined as the gradient of the scalar potential function V, $\vec{E}+{}_0 D_t^{\alpha}\vec{A}=-\nabla V$.

Integrating the potential along a closed path, $\oint_l \vec{E}\cdot d\vec{l}+\oint_l {}_0 D_t^{\alpha}\vec{A}\cdot d\vec{l}=-\oint_l \nabla V\cdot d\vec{l}=0$, applying the Stokes theorem to vector $\vec{A}$ in the last identity, $\oint_l \vec{E}\cdot d\vec{l}+\int_S {}_0 D_t^{\alpha}\left(\nabla\times\vec{A}\right)\cdot d\vec{s}=-\oint_l \nabla V\cdot d\vec{l}=0$, and using $\vec{B}=\nabla\times\vec{A}$, it follows that:

$$\oint_l \vec{E}\cdot d\vec{l}+{}_0 D_t^{\alpha}\int_S \vec{B}\cdot d\vec{s}=-\oint_l \nabla V\cdot d\vec{l}=0 \qquad (23)$$

From the electromagnetism viewpoint, the previous equation is the fractional Kirchhoff voltage law along a closed path. It will be used to calculate the winding voltage V_{kk} of turn k in layer k (figure 4) of the winding (figure 3), due to the conductor resistance and to the self-flux Φ_k (figure 5).

The calculation uses (23) and Ohm's law $\vec{E}=\rho\,\vec{J}$. Considering the integration path l embracing the surface S, shown in figure 4, the voltage V_{kk} at turn k, in Laplace domain, is $V_{k,k}=\int_l \rho\ \vec{J}\cdot d\vec{l}+s^{\alpha}\int_S \mu\ \vec{H}\cdot d\vec{s}$. Solving V_{kk} for $r_k<r<r_k+d$:

$$V_{k,k}=\int_0^{2\pi}\rho\ r\ \vec{J}\cdot\vec{a}_{\phi}\ d\phi+s^{\alpha}\int_{r_k}^{r}\int_0^{2\pi}\mu\ r\ \vec{H}\cdot\vec{a}_z\ d\phi dr \qquad (24)$$

Substituting $\vec{J}$ from (21), $\vec{H}$ from (19), and the F_{mm} value from (14) and (15), (25) is obtained, where coth is the hyperbolic co-tangent, and csch is the hyperbolic co-secant. Equation (25) defines the voltage per turn due to resistance and self-flux.

$$V_{k,k}=\frac{2\ \pi\ \rho}{a\,\delta}\left[r_k k\coth\left(\frac{d}{\delta}\right)-\left[(k-1)\sqrt{r_k\left(r_k+d\right)}\right]\operatorname{csch}\left(\frac{d}{\delta}\right)\right]i_s \qquad (25)$$

Next, the voltage per turn due to the leakage flux of the remaining turns is determined.

Voltage at turn k due to iii) the leakage fluxes Φ_j. To calculate the voltage $V_{k\,j}$ of turn k (figure 5) due to the flux Φ_j across turn j, start with (23) without the term for the resistive voltage drop, since it is already included in (25), to write $V_{k,j}=s^{\alpha}\int_{S_j}\mu\ \vec{H}\cdot d\vec{s}$ for $k\neq j$.

The magnetic flux is evaluated at the surface S_j of turn j to give $V_{k,j}=s^{\alpha}\int_{r_j}^{r_j+d}\int_0^{2\pi}\mu\ r\ \vec{H}\cdot\vec{a}_z d\phi\ dr$ for $k\neq j$. Using $\vec{H}=H(r)\cdot\vec{a}_z$ in the previous equation, and the F_{mm} value from (14) and (15), for $k\neq j$, $V_{k\,j}$ is:

$$V_{k,j}=\frac{2\pi\ \rho}{a\ \delta}\left[\left[r_j(2j-1)+d(j-1)\right]\coth\left(\frac{d}{\delta}\right)-\left[(2j-1)\ \sqrt{r_j\left(r_j+d\right)}\right]\operatorname{csch}\left(\frac{d}{\delta}\right)\right]i_s \quad \text{for } k\neq j \qquad (26)$$

3.5 Winding Fractional Dispersion Impedance

The total voltage V_b at the winding of figure 3, is obtained adding the voltages $V_{k\,k}$ and $V_{k\,j}$ of all the winding turns, $V_b = m\sum_{k=1}^{q}\left[V_{k,k} + \sum_{j=k+1}^{q} V_{k,j}\right]$, giving, from (25) and (26), V_b as:

$$V_b = \frac{2\pi\rho m}{a\,\delta}\sum_{k=1}^{q}\left[r_k\ k\ \coth\left(\frac{d}{\delta}\right) - \left[(k-1)\sqrt{r_k\ (r_k+d)}\right]\cdot \operatorname{csch}\left(\frac{d}{\delta}\right) + \sum_{j=k+1}^{q}\left[r_j(2j-1)+d(j-1)\right]\coth\left(\frac{d}{\delta}\right) - \sum_{j=k+1}^{q}\left[(2j-1)\sqrt{r_j(r_j+d)}\right]\operatorname{csch}\left(\frac{d}{\delta}\right)\right] i_s \tag{27}$$

Since (25) and (26) do not include the induced voltage due to the main flux (core flux, figure 1), then, the ratio V_b/i_s is the winding fractional dispersion impedance Z_σ, in the Laplace domain:

$$Z_\sigma = \frac{2\pi\rho m}{a\,\delta}\sum_{k=1}^{q}\left[r_k\ k\ \coth\left[\frac{d}{\delta}\right] - \left[(k-1)\sqrt{r_k\ (r_k+d)}\right]\operatorname{csch}\left(\frac{d}{\delta}\right) + \sum_{j=k+1}^{q}\left[r_j(2j-1)+d(j-1)\right]\coth\left(\frac{d}{\delta}\right) - \sum_{j=k+1}^{q}\left[(2j-1)\sqrt{r_j(r_j+d)}\right]\operatorname{csch}\left(\frac{d}{\delta}\right)\right] \tag{28}$$

Considering δ given by (10), and P_1, P_2 and P_4 as real terms, given in (29) and (30), only dependent on the turn dimensions and conductivity, Z_σ in (28) is simplified as (31).

$$P_1 = \frac{2\pi\ \rho\ m}{a\ d}, \quad P_2 = \sum_{k=1}^{q}\left[(k-1)^2(r_k+d) + k^2 r_k\right] \tag{29}$$

$$P_4 = \sum_{k=1}^{q}\left[2\ \ k(1-k)\sqrt{r_k(r_k+d)}\right] \tag{30}$$

$$Z_\sigma = P_1\,\frac{d}{\delta}\left[P_2\ \coth\left(\frac{d}{\delta}\right) + P_4\ \operatorname{csch}\left(\frac{d}{\delta}\right)\right] \tag{31}$$

4 Winding Fractional Dispersion Impedance Behavior

4.1 High Frequency Asymptotic Behavior

Considering the fractional skin depth δ of (10) in the frequency domain, $\delta = \sqrt{\rho/\left[(i\omega)^\alpha\,\mu\right]}$, and frequencies ω high enough to satisfy $d >> |\delta|$, $\coth(d/\delta) \approx 1$ for $d >> |\delta|$ and $\cosh(d/\delta) \approx 0$ for $d >> |\delta|$, the use of (29) in (31), gives the high-frequency asymptotic behavior (valid for $d >> |\delta|$) of the fractional dispersion impedance Z_σ:

$$Z_\sigma = \sqrt{d^2\mu/\rho}\left[\omega^{\alpha/2}\ \ e^{i\pi\alpha/4}\right]P_1\ P_2 \tag{32}$$

Impedance Z_σ is proportional to $\omega^{\alpha/2}\ e^{i\pi\alpha/4}$, or, in the Laplace domain, to $s^{\alpha/2}$. Therefore, even in the classical diffusion phenomenon ($\alpha = 1$) the high frequency dispersion impedance Z_σ shows a fractional derivative behavior of order 1/2 [2], being $Z_\sigma \propto \omega^{1/2}\ e^{i\pi/4}$. Moreover, if one impedance, related to diffusion phenomena, departs from the behavior expressed in $Z_\sigma \propto s^{\alpha/2}$, it might obey a fractional order differential diffusion equation (5), in which $\alpha \neq 1$.

However, this approximation is only valid for high frequencies. For dc and low frequency, (32) is no longer valid, as can be seen in the next section.

4.2 Winding dc Resistance

To obtain the dc resistance, $R_k = 1/G_k$, first consider the differential conductance of the dashed path shown in figure 4 to be $dG = \frac{a\,dr}{\rho\,2\pi r}$ and integrate to obtain the conductance G_k of one turn, $G_k = \int_{r_k}^{r_k+d} \frac{a\ dr}{\rho\ 2\ \pi\ r} = \frac{a}{\rho\ 2\pi} ln\left(\frac{r_k + d}{r_k}\right)$, where ln is the natural logarithm. Therefore:

$$R_k = \frac{\rho\ 2\pi}{a\,d}\, d \Big/ \ln\left(\frac{r_k + d}{r_k}\right) \tag{33}$$

The winding resistance R_{dc} is obtaining adding the resistances of all the turns:

$$R_{dc} = \frac{\rho\ 2\pi\ m}{a\ d} \sum_{k=1}^{q} \left[d \Big/ \ln\left(\frac{r_k + d}{r_k}\right)\right] = P_1\ P_3 \tag{34}$$

where P_1 is given in (29) and P_3 is:

$$P_3 = \sum_{k=1}^{q} \left[d \Big/ \ln\left(\frac{r_k + d}{r_k}\right)\right] \tag{35}$$

Equation (34) is useful to show that Bessel asymptotic approximations (17) and (18) lead to low frequency errors in (31), since R_{dc} should be the limit of (31) as $\omega \to 0$, or $R_{dc} = \lim_{\omega\to 0} Z_\sigma$, giving:

$$R_{dc} = \lim_{\omega\to 0}\left[P_1\left[\frac{d}{\delta}\right]\left[P_2 \coth\left[\frac{d}{\delta}\right] + P_4 \operatorname{csch}\left[\frac{d}{\delta}\right]\right]\right] = P_1 P_2 + P_1 P_4 \tag{36}$$

This result does not equal (34), since the originating equation (31) is not valid for low frequencies, due to the high frequency Bessel asymptotic approximations. However, from (31), the P_2 coth component defines the high frequency behavior, as the P_2 csch term is negligible for those frequencies, being only meaningful for low frequencies. Therefore, we propose an approximation for (31), which tries to enhance the behavior at low frequencies, without disturbing the validity for high frequencies [8]:

$$Z_\sigma = P_1\left[\frac{d}{\delta}\right]\left[P_2 \coth\left[\frac{d}{\delta}\right] + (P_3 - P_2) \operatorname{csch}\left[\frac{d}{\delta}\right]\right] \tag{37}$$

where the values of P_1, P_2 and P_3, are given by (29) and (35).

4.3 Fractional Transfer Function High and Low Frequency Approximation for the Winding Fractional Dispersion Impedance

The fractional dispersion impedance Z_σ (31), (32), is valid only for high frequencies [9]. To obtain an approximation to (31), able to describe both the high frequency and low frequency behavior, the contribution [10] of the self (38) and mutual inductance of the windings must be considered.

$$L_{dc} = \frac{m^2 \pi \mu}{2l} \sum_{j=1}^{q} \left[\frac{1}{\ln\left(\frac{r_j + d}{r_j}\right)} \cdot \left\{ \left[1 + 2(j-1) \cdot \ln\left(\frac{r_j + d}{r_j}\right) \right] (r_j + d)^2 - \left[1 + 2j \cdot \ln\left(\frac{r_j + d}{r_j}\right) \right] r_j^2 \right\} \right] +$$

$$+ \frac{m^2 \pi \mu}{2l} \sum_{k=1}^{q} \left[\frac{k-1}{\left[\ln\left(\frac{r_k + d}{r_k}\right)\right]^2} \cdot \left[\left[1 + (k-1) \cdot \ln\left(\frac{r_k + d}{r_k}\right) \right] \cdot (r_k + d)^2 + \left\{ 1 + \left[2k \cdot \ln\left(\frac{r_k + d}{r_k}\right) + (k+1) \right] \cdot \ln\left(\frac{r_k + d}{r_k}\right) \right\} \cdot r_k^2 \right] \right] \quad (38)$$

The approximation must give the low frequency term $R_{dc} = P_1 P_3$ when $\omega \to 0$, and the high frequency factor $\omega^{\alpha/2} e^{i\pi\alpha/4}$ of (32) when $\omega \to \infty$. A useful candidate transfer function, contains one zero and one pole, both fractional:

$$Z_\sigma = R_{dc} \frac{(1 + s\tau_1)^\alpha}{(1 + s\tau_2)^{\alpha/2}} = P_1 P_3 \frac{(1 + s\tau_1)^\alpha}{(1 + s\tau_2)^{\alpha/2}} \quad (39)$$

where τ_1 is:

$$\tau_1 = (L_{dc} / R_{dc})^{1/\alpha} \quad (40)$$

The constant τ_2 is calculated for high frequency, considering $i\omega\tau_1 >> 1$ and $i\omega\tau_2 >> 1$ in (39):

$$Z_\sigma = P_1 P_3 \frac{(1 + i\omega\tau_1)^\alpha}{(1 + i\omega\tau_2)^{\alpha/2}} \approx P_1 P_3 \frac{(i\omega\ \tau_1)^\alpha}{(i\omega\ \tau_2)^{\alpha/2}} \quad (41)$$

This equation must equal (32). Thus:

$$Z_\sigma \approx P_1 P_3 \frac{(i\omega\ \tau_1)^\alpha}{(i\omega\ \tau_2)^{\alpha/2}} = \sqrt{\frac{d^2 \mu}{\rho}}\ (i\omega)^{\alpha/2} P_1\ P_2 \quad (42)$$

$$\tau_2 = \left[P_3\ \tau_1{}^\alpha \Big/ \left(P_2 \sqrt{\frac{d^2 \mu}{\rho}} \right) \right]^{\frac{2}{\alpha}} \quad (43)$$

Using (40) and (43), the fractional model (39) tries to reproduce the low frequency behavior, without disturbing the high frequency validity. The fractional order α can be estimated using a non-linear regression.

5 Results: Evaluation of Short-Circuit Impedance of Power Transformers

Data from a single-phase toroidal power transformer of 25 kVA, 7200 V in the high voltage side and 240 V/120 V in the low voltage secondary was used [11]. The proposed model for the transformer short-circuit impedance (figure 6) includes an equivalent capacitor C, associated with the high frequency displacement currents, not considered in the previous analysis, L, the frequency independent inductance, and Z_σ, the fractional dispersion impedance associated with (39).

The values of $L = 45$ mH and $C = 890$ pF were calculated in a previous work [9][11]. From the dimensions and short-circuit experimental data of the transformer [10][12],

R_{dc}, τ_1 and τ_2 were calculated and a non-linear regression was used to obtain the fractional order α that characterizes the Z_σ impedance in (39). Table 1 shows the obtained results for fractional α (fractional order diffusion) and for α = 1 (integer order diffusion). The best fit for short-circuit experimental data was obtained with α = 0.949.

The magnitude, angle and real part values of the fractional impedance Z_σ, obtained using (39) and table 1 values, are shown respectively in figures 7, 8 and 9.

Parameter	α = 0.949	α = 1
R_{dc}	20.02 [Ω]	20.02[Ω]
τ_1	1.59 [ms]	1.44 [ms]
τ_2	5.27 [μs]	9.73 [μs]

Table 1: Parameters of Z_σ (α obtained by non-linear regression)

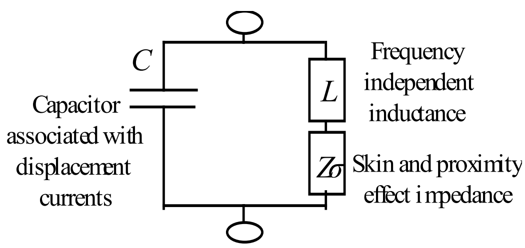

Figure 6: Frequency dependent short-circuit impedance model for the power transformer.

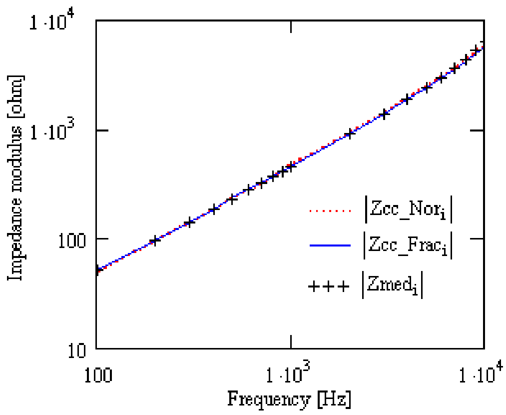

Figure. 7: Measured (Zmed) and calculated magnitude of the dispersion impedance Z_σ versus frequency, showing integer (Zcc_Nor) and fractional (Zcc_Frac) approaches.

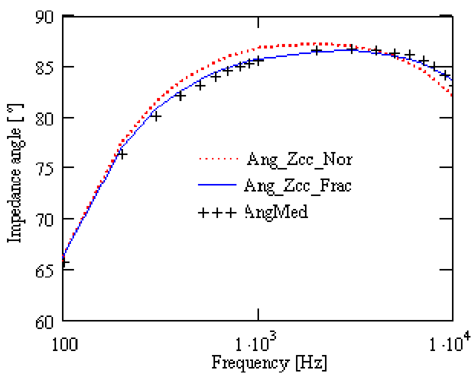

Figure 8: Measured (AngMed) and calculated angle of the fractional dispersion impedance Z_σ versus frequency, showing integer (Ang_Zcc_Nor) and fractional (Ang_Zcc_Frac) approaches.

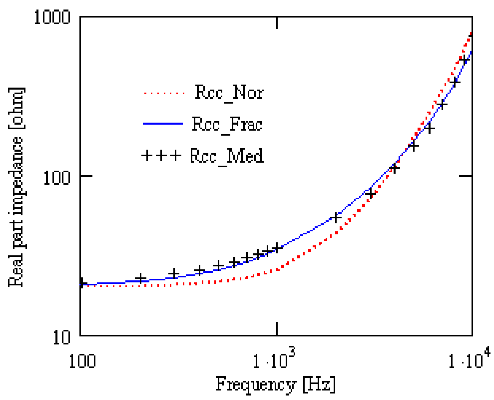

Figure 9: Measured (Rcc_Med) and calculated real part value of the fractional dispersion impedance Z_σ versus frequency, showing integer (Rcc_Nor) and fractional (Rcc_Frac) approaches.

Figure 7 shows that the Z_σ magnitude has not significant variations for the two values of α (α = 1 and α = 0.949), both integer and fractional approximations giving good results.
The Z_σ angle is slightly better approximated with α = 0.949 (figure 8, solid curve), mainly in the medium frequency range (300 Hz to 6000 Hz), the classical approach (α = 1) giving nearly 2% errors in that region.
The classical approach (α = 1) for the Z_σ real part gives nearly 15% errors in the frequency range 300 Hz to 6000 Hz. The Z_σ real part is clearly better approximated taking α = 0.949 (figure 9, solid curve), also in the medium frequency range (300 Hz to 6000 Hz), a range of interest for power quality studies.

6 Conclusion

The assumption that electromagnetic fields diffusion phenomena in conducting media obeys differential equations of fractional order, can be worked out with the same mathematics used to solve the problem using classical integer derivatives. The analytical results obtained extend the existing studies, offer an extra degree of freedom, and enable better diffusion modeling, when compared to models with integer differential equations.
In the measured power transformer, the obtained fractional order is very close to unity (α = 0.949), suggesting that the classical integer approximation is good enough for most purposes. However, the fractional zero-pole model, here parameterized, enables a better approximation, suggesting that this approach can be used to optimize the design and estimation of short-circuit transformer resistance, specially in the medium frequency range (300 Hz to 6000 Hz) where harmonics, transformer heating and power quality related problems can be significant.

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