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Research Article

Application of Integer and Fractional Models in Electrochemical Systems

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This paper describes the use of integer and fractional electrical elements, for modelling two electrochemical systems. A first type of system consists of botanical elements and a second type is implemented by electrolyte processes with fractal electrodes. Experimental results are analyzed in the frequency domain, and the pros and cons of adopting fractional-order electrical components for modelling these systems are compared.

1. Introduction

Fractional calculus (FC) is a generalization of the integration and differentiation to a noninteger order. The fundamental operator is ${}_aD_t^{\alpha}$, where the order α is a real or even, a complex number and the subscripts a and t represent the two limits of the operation [1–8].

Recent studies brought FC into attention revealing that many physical phenomena can be modelled by fractional differential equations [9–17]. The importance of fractional-order models is that they yield a more accurate description and lead to a deeper insight into the physical processes underlying a long-range memory behavior.

Capacitors are one of the crucial elements in integrated circuits and are used extensively in many electronic systems [18]. However, Jonscher [19] demonstrated that the ideal capacitor cannot exist in nature, because an impedance of the form $1/[(j\omega)C]$ would violate causality [20, 21]. In fact, the dielectric materials exhibit a fractional behavior yielding electrical impedances of the form $1/[(j\omega)^{\alpha}\ C_F]$, with $\alpha \in \Re^+$ [22, 23].

Bearing these ideas in mind, this paper analyzes the fractional modelling of several electrical devices and is organized as follows. Section 2 introduces the fundamental concepts of electrical impedances. Sections 3 and 4 describe botanical elements and fractal capacitors, respectively, and present the experimental results for both cases. Finally, Section 5 draws the main conclusions.

2. On the Electrical Impedance

In an electrical circuit, the voltage $u(t)$ and the current $i(t)$ can be expressed as a function of time t:

$$\begin{aligned} u(t) &= U_0 \cos(\omega t), \\ i(t) &= I_0 \cos(\omega t + \phi), \end{aligned} \tag{2.1}$$

where U_0 and I_0 are the amplitudes of the signals, ω is the angular frequency, and ϕ is the current phase shift. The voltage and current can be expressed in complex form as

$$\begin{aligned} u(t) &= \mathrm{Re}\left\{U_0 e^{j\omega t}\right\}, \\ i(t) &= \mathrm{Re}\left\{I_0 e^{j(\omega t+\phi)}\right\}, \end{aligned} \tag{2.2}$$

where Re{ } represents the real part and $j = \sqrt{-1}$.

Consequently, in complex form, the electrical impedance $Z(j\omega)$ is given by the expression:

$$Z(j\omega) = \frac{U(j\omega)}{I(j\omega)} = Z_0 e^{j\phi}. \tag{2.3}$$

Fractional-order elements occur in several fields of engineering [24]. A brief reference about the constant phase element (CPE) and the Warburg impedance is presented here due to their application in the work. In fact, to model an electrochemical phenomenon, it often used a CPE due to the fact that the surface is not homogeneous [24].

In the case of a CPE, we have the model:

$$Z(j\omega) = \frac{1}{(j\omega)^{\alpha} C_F}, \tag{2.4}$$

where C_F is a "capacitance" of fractional order $0 < \alpha \leq 1$, occurring at the classical ideal capacitor when $\alpha = 1$ [25, 26].

It is well known that, in electrochemical systems with diffusion, the impedance is modelled by the so-called Warburg element [24, 26]. The Warburg element arises from one-dimensional diffusion of an ionic species to the electrode. If the impedance is under an infinite diffusion layer, the Warburg impedance is given by

$$Z(j\omega) = \frac{R}{(j\omega)^{0.5} C_F}, \tag{2.5}$$

where R is the diffusion resistance. If the diffusion process has finite length, the Warburg element becomes

$$Z(j\omega) = R\frac{\tanh{(j\omega\tau)}^{0.5}}{(\tau)^{0.5}}, \tag{2.6}$$

with $\tau = \delta^2/D$, where R is the diffusion resistance, τ is the diffusion time constant, δ is the diffusion layer thickness, and D is the diffusion coefficient [26, 27].

Based on these concepts, and in the previous works developed by the authors [12, 26–29], we verify that the $Z(j\omega)$ of the fruits, vegetables, and also fractal capacitors exhibit distinct characteristics according with the frequency range.

This different behavior, for low and for high frequencies, makes difficult the modelling of these systems in all frequency range. This fact motivated the study of both systems with different type of RC electrical approximation circuits, namely, the use of series and parallel two-element associations of integer and fractional order.

Table 1 shows simple series and parallel element associations of integer and fractional order for constructing electrical circuits, that are adopted in this work.

In this line of thought, in the following two sections, the impedances of botanical and fractal electrolyte systems are analyzed. In both cases, a large number of measurements were performed in order to understand and minimize the effect of nonlinearities, initial conditions, and experimental and instrumentation limitations.

3. Botanical Elements

The structures of fruits and vegetables have cells that are sensitive to heat, pressure, and other stimuli. These systems constitute electrical circuits exhibiting a complex behavior. Bearing these facts in mind, in our work, we study the electrical impedance of the *Solanum tuberosum* (the common potato) and the *Actinidia deliciosa* (the common Kiwi), under the point of view of FC.

We apply sinusoidal excitation signals $v(t)$ to the botanical system for several distinct frequencies ω and the impedance $Z(j\omega)$ is measured based on the resulting voltage $u(t)$ and current $i(t)$.

We start by analyzing the impedance for an amplitude of input signal of $V_0 = 10$ volt, a constant adaptation resistance $R_a = 15\,\mathrm{k\Omega}$, applied to one potato, with a weight $W = 1.24 \times 10^{-1}$ kg, environmental temperature $T = 26.5°\mathrm{C}$, dimension $D = (7.97 \times 10^{-2}) \times (5.99 \times 10^{-2})$ m, and electrode length penetration $\Delta = 2.1 \times 10^{-2}$ m. Figure 1 presents the corresponding polar and Nichols diagrams for the $Z(j\omega)$.

For the approximate modelling of the results presented in Figure 1, we must have in mind the polar plots of the impedance $Z(j\omega)$ for the circuits presented in Table 1.

Table 1: Elementar circuits of integer and fractional order.

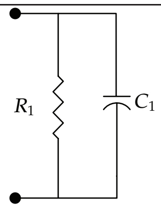
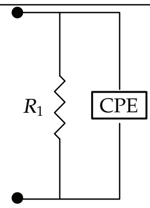
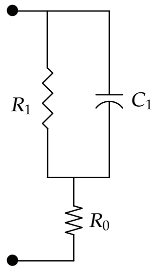
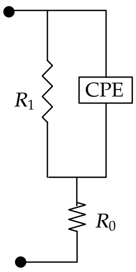
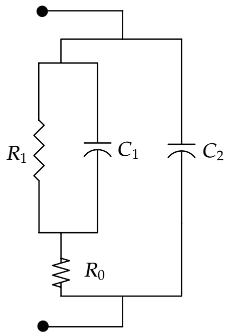
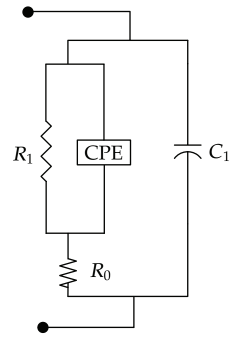
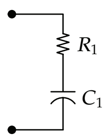
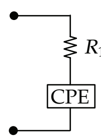
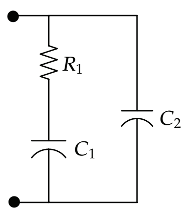
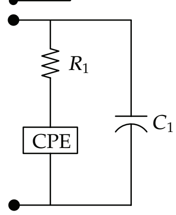
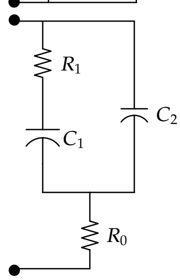
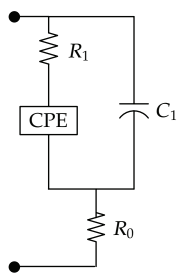
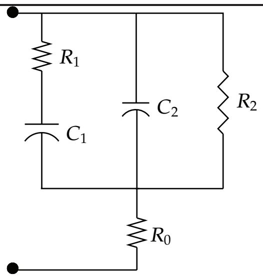
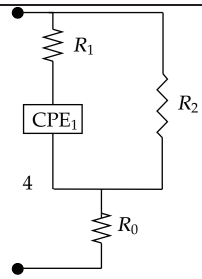
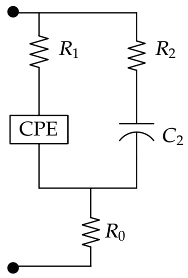
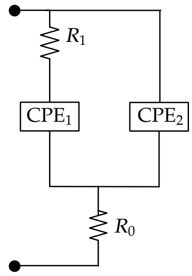
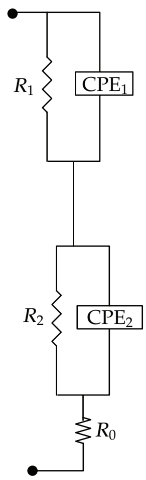
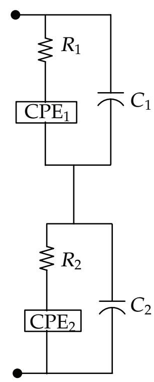
Circuit	Circuit
 $i = 1$	 $i = 2$
 $i = 3$	 $i = 4$
 $i = 5$	 $i = 6$
 $i = 7$	 $i = 8$
 $i = 9$	 $i = 10$
 $i = 11$	 $i = 12$

Table 1: Continued.

Circuit	Circuit
 $i = 13$	 $i = 14$
 $i = 15$	 $i = 16$
 $i = 17$	 $i = 18$

In the botanical system are applied the circuits $i = \{1, \ldots, 6\}$ and $i = \{13, \ldots, 18\}$, for modelling $Z(j\omega)$. It minimized the errors J_{iA} for the Polar diagram and J_{iB} for the Nichols diagram, between the experimental data (Z), and the approximation model (Z_{app}), in the perspective of the expressions:

$$J_{iA} = \sum_{w=w_1}^{w_2} \frac{\left(\mathrm{Re}_Z - \mathrm{Re}_{Z\text{app}}\right)^2 + \left(\mathrm{Im}_Z - \mathrm{Im}_{Z\text{app}}\right)^2}{\left(\mathrm{Re}_Z + \mathrm{Re}_{Z\text{app}}\right)^2 + \left(\mathrm{Im}_Z + \mathrm{Im}_{Z\text{app}}\right)^2},$$

$$J_{iB} = \sum_{w=w_1}^{w_2} \left[\left(\frac{20\log(M_2/M_1)}{K_1} \right)^2 + \left(\frac{F_2 - F_1}{K_2} \right)^2 \right], \tag{3.1}$$

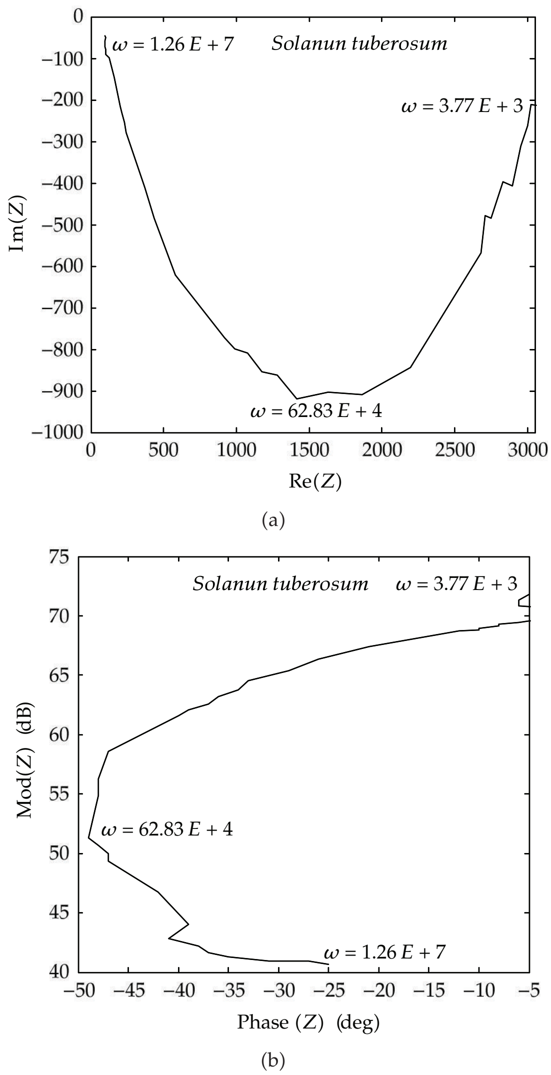


Figure 1: Polar and Nichols diagrams of the impedance $Z(j\omega)$ for the *Solanum tuberosum*.

where $M_1 = \text{Max}\,|Z|$, $M_2 = \text{Max}\,|Z_{\text{app}}|$, $F_1 = \text{Max(phase } Z)$, $F_2 = \text{Max(phase } Z_{\text{app}})$, $K_1 = 20\log(\text{Max}(|Z|)/\min(|Z|))$, and $K_2 = \text{Max(phase } Z) - \min(\text{phase } Z)$, and where $w_1 \le w \le w_2$ is the frequency range.

The resulting numerical values of $\{R_0, R_1, R_2, C_1, C_2, C_{F1}, \alpha_1, C_{F2}, \alpha_2, J\}$ for the different impedances are depicted in Tables 2 and 3 for J_{iA} and J_{iB}, respectively. It is possible to analyze the approximation errors J_{iA} and J_{iB} as function as the number of electrical elements E and the number of parameters P to be adjusted for the circuits $i = \{1, \ldots, 6\}$ and $\{13, \ldots, 18\}$. We verify a significant decreasing of the errors J_{iA} and J_{iB}, respectively, for the polar and Nichols diagrams, with the number of elements and parameters. In Figure 2, we present the polar and Nichols diagrams for $Z(j\omega)$, and the approximations $Z_{\text{app}}(j\omega)$, $i = \{2, 4, 5, 6, 13, 14, 15, 16, 17, 18\}$, and for J_{iA} and J_{iB}, for the *Solanum tuberosum*. The results for circuits $\{1, 3\}$ are not presented because they lead to severe errors.

Table 2: Comparison of circuit parameters for the circuits $\{1,\ldots,6\}$ and $\{13,\ldots,18\}$, with J_{iA} and for the *Solanum tuberosum*.

Circuit i	R_0	R_1	R_2	C_1	C_2	C_{F1}	α_1	C_{F2}	α_2	J_{iA}
1		2487.00		$1.70E-9$						4.23
2		2271.00				$2.83E-7$	0.84			1.97
3	116.00	2309.00		$3.70E-9$						0.57
4	55.70	3198.00				$3.06E-5$	0.68			0.06
5	115.00	2325.00		$3.70E-9$	$5.00E-14$					0.57
6	61.50	3274.00			$5.34E+5$	$3.06E-5$	0.68			0.06
13	137.90	0.10	2384.80	$4.00E-9$	$1.00E-10$					0.63
14	18.50	0.0002	4998.00			$1.09E-3$	0.52			0.46
15	18.80	0.0002	4968.90	$2.70E-1$		$1.09E-3$	0.52			0.46
16	17.50	0.0002				$1.09E-3$	0.52	$9.99E-1$	0.001	0.46
17	62.60	2976.90	759.00			$1.53E-4$	0.62	$2.25E-4$	0.64	0.18
18		662.40	79.30	$1.20E-9$	$5.56E+2$	$4.00E-2$	0.31	$1.98E-4$	0.65	2.67

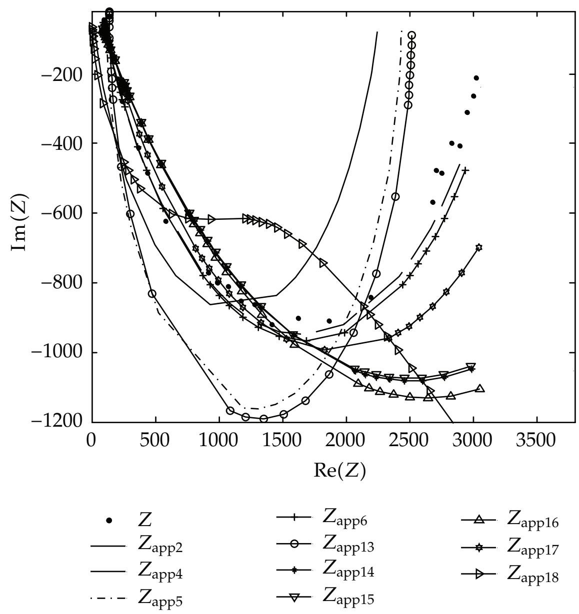
Table 3: Comparison of circuit parameters for the circuits $\{1,\ldots,6\}$ and $\{13,\ldots,18\}$, with J_{iB} and for the *Solanum tuberosum*.

Circuit i	R_0	R_1	R_2	C_1	C_2	C_{F1}	α_1	C_{F2}	α_2	J_{iB}
1		1472.10		$1.00E-9$						15.70
2		2889.60				$2.13E-4$	0.59			1.58
3	104.50	2309.10		$3.10E-9$						1.91
4	67.50	3198.00				$2.97E-5$	0.68			0.19
5	103.60	2325.00		$3.10E-9$	$4.00E-14$					1.91
6	64.30	2691.90			$5.34E+5$	$2.55E-5$	0.69			0.14
13	104.90	0.10	2384.80	$3.00E-9$	$4.00E-11$					1.91
14	35.90	0.0002	3019.00			$5.79E-4$	0.55			0.83
15	34.60	0.0002	4968.90	$2.70E-1$		$7.90E-4$	0.53			1.28
16	43.40	0.0002				$3.12E-4$	0.58	1.00	$2.00E-7$	0.62
17	59.10	2189.70	630.90			$1.47E-4$	0.62	$1.22E-6$	0.85	0.16
18		0.10	79.30	$9.00E-11$	$5.56E+2$	$2.27E-2$	0.35	$1.73E-4$	0.66	3.78

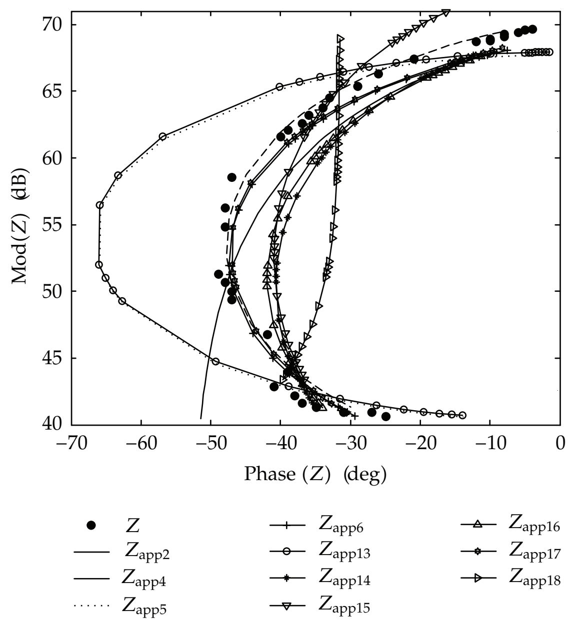
In a second experiment, we organized similar studies for a Kiwi. In this case, the constant adaptation resistance is $R_a = 750\,\Omega$, with a weight $W = 8.95 \times 10^{-2}$ kg, dimension $D = (6.52 \times 10^{-2}) \times (5.50 \times 10^{-2})$ m. Figure 3 presents the polar and the Nichols diagrams for $Z(j\omega)$.

In this case, for modelling $Z(j\omega)$, we apply again the circuits adopted for the potato, namely, the circuits $i = \{1,\ldots,6\}$ and $i = \{13,\ldots,18\}$, and the same expressions for the error measures (J). The resulting numerical values of $\{R_0, R_1, R_2, C_1, C_2, C_{F1}, \alpha_1, C_{F2}, \alpha_2, J\}$ for the different impedances are depicted in Tables 4 and 5, for J_{iA} and J_{iB}, respectively. We can analyze the approximation errors J_{iA} and J_{iB} as function as the number of electrical elements E and the number of parameters P to be adjusted for the circuits $i = \{1,\ldots,6\}$ and $i = \{13,\ldots,18\}$, for the *Actinidia deliciosa*.

We verify that a significant decreasing of the error J_{iA} and J_{iB} with the number of elements and parameters occurs. In Figure 4, we present the polar and Nichols diagrams for $Z(j\omega)$, and the approximations $Z_{\mathrm{app}\,iB}(j\omega)$, $i = \{2,4,5,6,13,14,15,16,17,18\}$, for the *Actinidia deliciosa* revealing a very good fit.



(a)



(b)

Figure 2: Polar and Nichols diagrams of $Z(j\omega)$ and the approximations $Z_{\text{app}}(j\omega)$ for $i = \{2, 4, 5, 6, 13, 14, 15, 16, 17, 18\}$ of the electrical impedance of the *Solanum tuberosum* for J_{iA} and J_{iB}, respectively.

4. Fractal Capacitors

Fractals can be found both in nature and abstract objects. The impact of the fractal structures and geometries, is presently recognized in engineering, physics, chemistry, economy, mathematics, art, and medicine [9, 30].

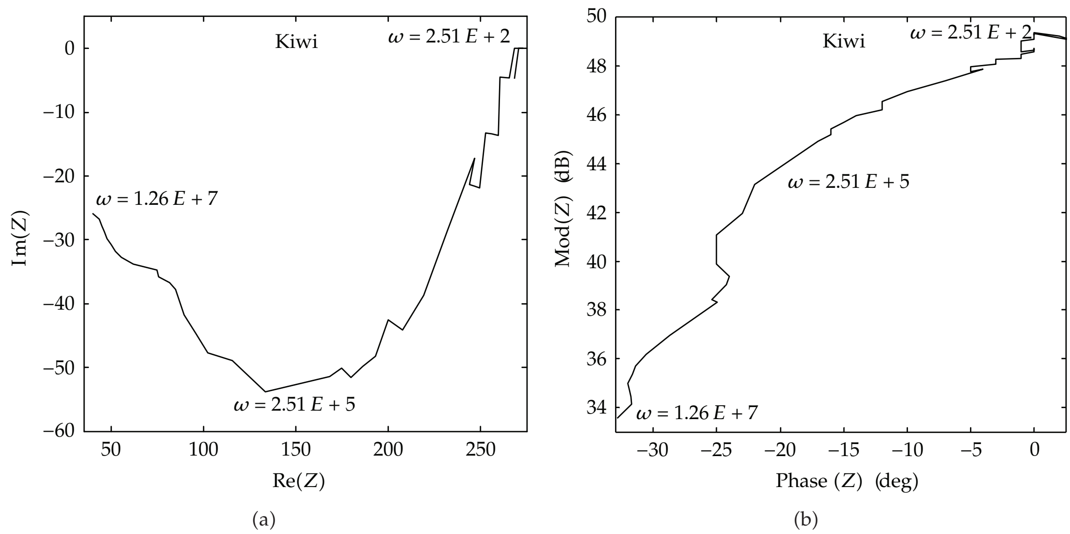


Figure 3: Polar and Nichols diagrams of the impedance $Z(j\omega)$ for the *Actinidia deliciosa*.

Table 4: Comparison of circuit parameters for $i = \{1, \ldots, 6\}$ and $i = \{13, \ldots, 18\}$, with J_{iA} and for the *Actinidia deliciosa*.

Circuit i	R_0	R_1	R_2	C_1	C_2	C_{F1}	α_1	C_{F2}	α_2	J_{iA}
1		178.00		$1.80E-9$						2.48
2		222.00				$3.30E-5$	0.69			0.35
3	56.00	180.40		$1.90E-8$						0.60
4	27.30	250.70				$1.49E-3$	0.54			0.09
5	56.30	180.50		$1.90E-8$	$4.00E-12$					0.61
6	34.60	257.10			2887	$2.40E-4$	0.63			0.13
13	60.20	192.9	$2.42E+3$	$3.00E-3$	$2.00E-8$					0.62
14	19.90	0.009	284.10			$7.68E-3$	0.45			0.11
15	20.10	0.0004	283.60	11.30		$7.68E-3$	0.45			0.12
16	30.20	0.0013				$6.89E-3$	0.46	0.994	0.001	0.19
17	28.10	249.30	755.10			$1.24E-3$	0.55	0.86	1.64	0.09
18		16.80	30.20	$1.10E-9$	86.50	$4.33E-1$	0.13	0.0069	0.49	0.68

The concept of fractal is associated with Benoit Mandelbrot, that led to a new perception of the geometry of the nature [31]. However, the concept was initially proposed by several well-known mathematicians, such as George Cantor (1872), Giuseppe Peano (1890), David Hilbert (1891), Helge von Koch (1904), Waclaw Sierpinski (1916), Gaston Julia (1918), and Felix Hausdorff (1919).

A geometric important index consists in the fractal dimension (FDim) that represents the occupation degree in the space and that is related with its irregularity. The FDim is given by

$$\text{FDim} \approx \frac{\log(N)}{\log(1/\eta)}, \tag{4.1}$$

Table 5: Comparison of circuit parameters for $i = \{1, \ldots, 6\}$ and $i = \{13, \ldots, 18\}$, with J_{iB} and for the *Actinidia deliciosa*.

Circuit i	R_0	R_1	R_2	C_1	C_2	C_{F1}	α_1	C_{F2}	α_2	J_{iB}
1		142.80		$8.00E-10$						9.63
2		217.80				$3.37E-4$	0.59			1.48
3	39.70	158.40		$5.00E-9$						3.13
4	14.20	241.30				$1.25E-3$	0.54			0.21
5	39.60	158.20		$5.00E-9$	$7.00E-14$					3.13
6	21.80	242.00			2887	$1.86E-4$	0.63			0.44
13	39.70	169.9	$2.42E+3$	$8.00E-4$	$5.00E-9$					3.13
14	4.30	0.009	279.00			$9.54E-3$	0.43			0.32
15	3.50	0.0004	280.90	11.30		$9.54E-3$	0.43			0.33
16	8.50	0.0013				$3.18E-3$	0.49	1.00	$6.00E-6$	0.19
17	16.60	239.30	755.10			$1.07E-3$	0.55	0.86	1.64	0.25
18		0.10	30.20	$7.00E-10$	86.50	$4.29E-1$	0.13	0.0069	0.49	2.85

where N represents the number of boxes, with size $\eta(N)$ resulting from the subdivision of the original structure. This is not the only description for the fractal geometry, but it is enough for the identification of groups with similar geometries.

In this work, we adopted the classical fractal Carpet of Sierpinski and the Triangle of Sierpinski with FDim = 1.893 and FDim = 1.585, respectively.

The simplest capacitors are constituted by two parallel electrodes separated by a layer of insulating dielectric. There are several factors susceptive of influencing the characteristics of a capacitor [32]. However, three of them have a special importance, namely, the surface area of the electrodes, the distance among them, and the material that constitutes the dielectric. In this study, the capacitors adopt electrodes that are construted with the fractal structures of Carpet of Sierpinski and Triangle of Sierpinski. The size of the fractals was adjusted so that their copper surface yields identical values, namely, $S = 0.423\,\mathrm{m}^2$.

We apply sinusoidal excitation signals $v(t)$ to the apparatus, for several distinct frequencies ω, and the impedance $Z(j\omega)$ between the electrodes is measured based on the resulting voltage $u(t)$ and current $i(t)$.

For the first experiment with fractal structures, we consider two identical single-face electrodes. The voltage, the adaptation resistance R_a, and the distance between electrodes d_{elec} are, respectively, $V_0 = 10\,\mathrm{V}$, $R_a = 1.2\,\mathrm{k\Omega}$, and $d_{\text{elec}} = 0.13\,\mathrm{m}$. The electrolyte process consists in an aqueous solution of NaCl with $\Psi = 10\ \mathrm{gl}^{-1}$ and two single-face copper electrodes with the Carpet of Sierpinski printout.

The resulting polar and Nichols diagrams of the electrical impedance $Z(j\omega)$ are depicted in Figure 5. For this chart, we apply the circuits $i = \{7, \ldots, 18\}$ in Table 2. The resulting numerical values of $\{R_0, R_1, R_2, C_1, C_2, C_{F1}, \alpha_1, C_{F2}, \alpha_2, J\}$ for the different impedances are shown in Tables 6 and 7, for J_{iA} and J_{iB}, respectively.

We can analyze the approximation errors J_{iA} and J_{iB} as function as the number of electrical elements E and the number of parameters P to be adjusted for the circuits $i = \{7, \ldots, 18\}$. The tables reveal that the error decreases with the introduction of the fractional-order elements. Figure 6 presents the polar and Nichols diagrams of $Z(j\omega)$ and the approximations $Z_{\text{app}\,iB}(j\omega)$, for $i = \{8, 10, 11, 12, 13, 14, 15, 16, 17, 18\}$.

In order to study the influence of the fractal printed in the surface of the electrode, we adopted another fractal, namely, the Triangle of Sierpinski.

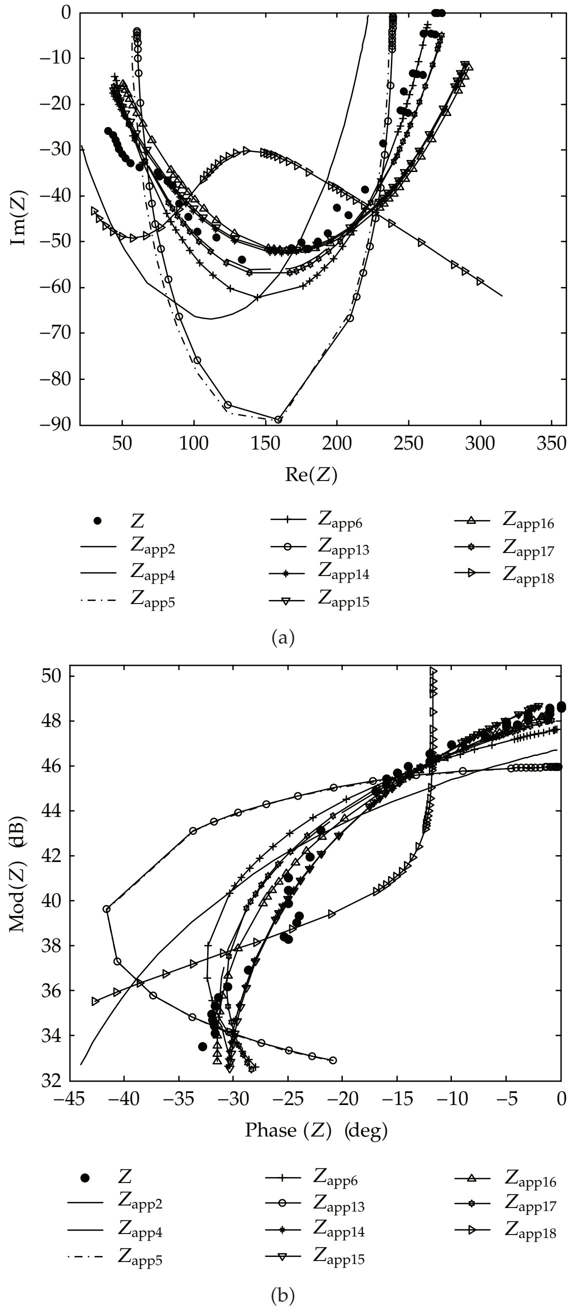
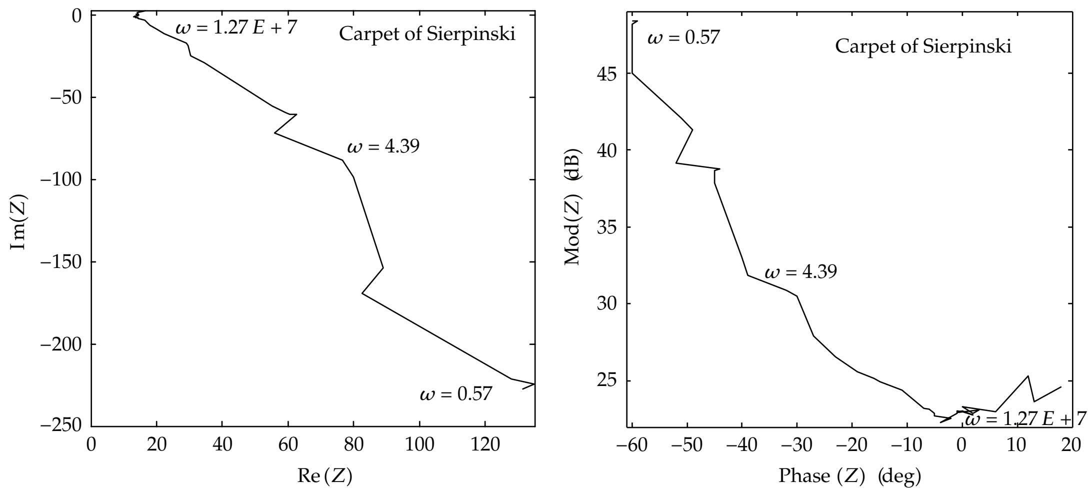


Figure 4: Polar and Nichols diagrams of $Z(j\omega)$ and the approximations $Z_{\text{app}\,iB}(j\omega)$, $i = \{2, 4, 5, 6, 13, 14, 15, 16, 17, 18\}$ of the electrical impedance of the *Actinidia deliciosa* for J_{iA} and J_{iB}, respectively.

In this case the voltage, R_a, d_{elec}, the solution and the area remain identical to the previous example.

The resulting polar and Nichols diagrams of the electrical impedance $Z(j\omega)$ is depicted in Figure 7. For this chart, we apply the circuits $i = \{7, \ldots, 18\}$ in Table 1. The resulting numerical values of $\{R_0,\ R_1,\ R_2, C_1, C_2, C_{F1}, \alpha_1, C_{F2}, \alpha_2, J\}$ for the different impedances are



(a) (b)

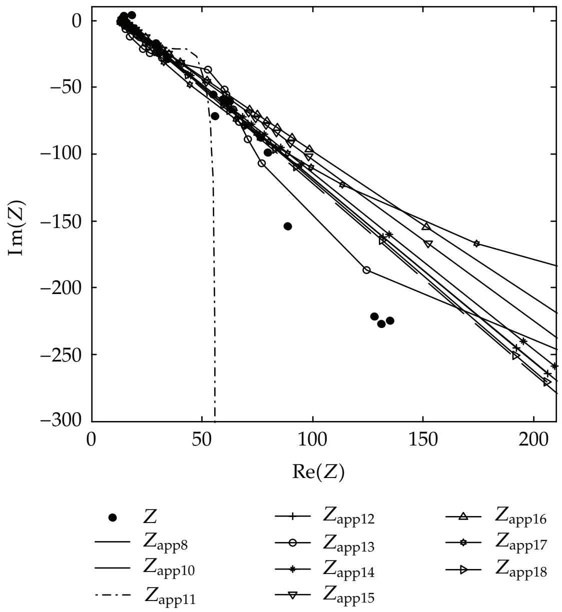
Figure 5: Polar and Nichols diagrams of the impedance $Z(j\omega)$ for the Carpet of Sierpinski.

Table 6: Comparison of circuit parameters for $i = \{7, \ldots, 18\}$, with J_{iA} and for the Carpet of Sierpinski.

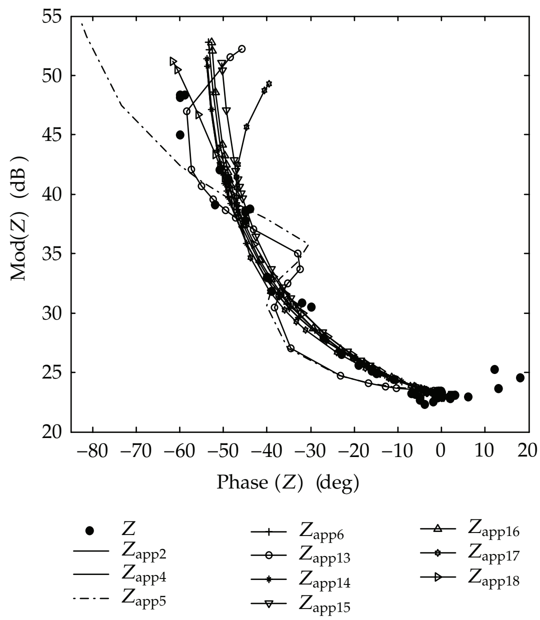
Circuit i	R_0	R_1	R_2	C_1	C_2	C_{F1}	α_1	C_{F2}	α_2	J_{iA}
7		17.00		0.0027						2.03
8		14.20				0.037	0.60			0.20
9		17.20		0.0027	$1.00E-19$					2.03
10		14.30			$1.00E-11$	0.035	0.61			0.20
11	14.90	41.20		0.0033	$8.40E-11$					0.58
12	10.20	4.10			$4.00E-11$	0.038	0.60			0.18
13	15.00	70.6	$6.30E+2$	0.0026	0.001					0.35
14	14.10	0.0001	$1.01E+5$			0.040	0.59			0.20
15	13.90	$3.00E-5$	20.90	$2.00E-8$		0.045	0.56			0.27
16	14.10	3.50				0.057	0.52	$9.34E-4$	0.86	0.31
17	9.80	782.40	4.30			0.019	0.68	$7.03E-6$	0.79	0.26
18		11.90	2.30	$2.00E-11$	$1.00E-11$	0.042	0.59	$1.59E-2$	0.76	0.20

Table 7: Comparison of circuit parameters for $i = \{7, \ldots, 18\}$, with J_{iB} and for the Carpet of Sierpinski.

Circuit i	R_0	R_1	R_2	C_1	C_2	C_{F1}	α_1	C_{F2}	α_2	J_{iB}
7		18.30		0.004						3.36
8		14.30				0.032	0.62			0.23
9		17.60		0.0032	$1.00E-19$					2.42
10		14.30			$1.00E-12$	0.032	0.62			0.23
11	15.00	54.00		0.0034	$2.00E-10$					0.55
12	10.20	4.10			$1.00E-11$	0.031	0.61			0.28
13	14.90	52.50	$6.3E+2$	0.0027	0.001					0.37
14	14.30	0.0001	$9.76E+4$			0.032	0.62			0.23
15	14.00	$3.00E-5$	20.90	$1.00E-8$		0.041	0.58			0.28
16	14.20	0.001				0.031	0.60	0.006	0.63	0.28
17	10.50	782.50	3.90			0.026	0.66	$4.22E-6$	0.60	0.43
18		11.80	2.30	$1.50E-12$	$9.00E-12$	0.043	0.58	0.009	1.30	0.20



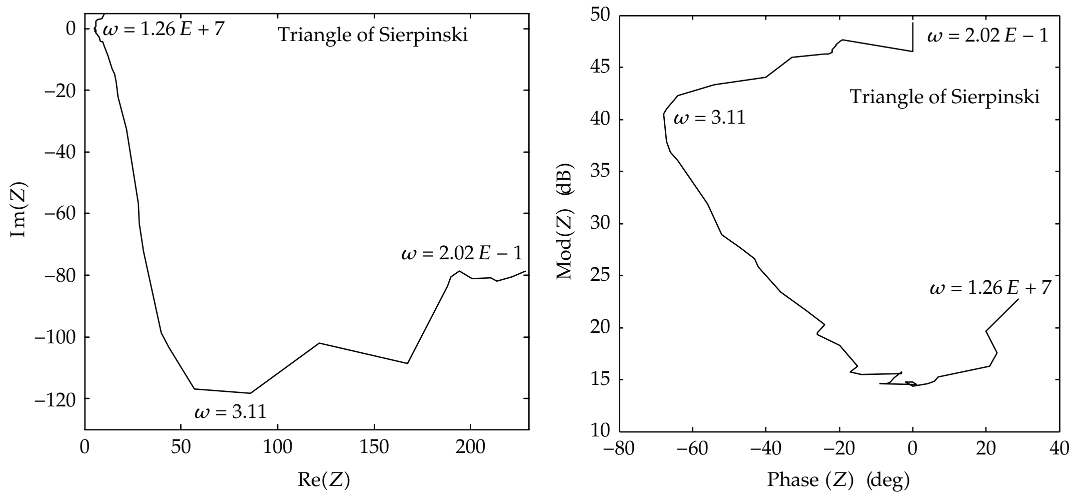
(a)



(b)

Figure 6: Polar and Nichols diagrams of $Z(j\omega)$ and the approximations $Z_{\text{app}\, iB}(j\omega)$, $i = \{8, 10, 11, 12, 13, 14, 15, 16, 17, 18\}$ of the electrical impedance of the Carpet of Sierpinski for J_{iA} and J_{iB}, respectively.

shown in Tables 8 and 9, for J_{iA} and J_{iB}, respectively. We analyze the approximation errors J_{iA} and J_{iB} as function as the number of electrical elements E and the number of parameters P to be adjusted for $i = \{7, \ldots, 18\}$. The results lead to the same conclusions, revealing that the error decreases with the introduction of the fractional-order elements. Figure 8 presents



(a) (b)

Figure 7: Polar and Nichols diagrams of the impedance $Z(j\omega)$ for the Triangle of Sierpinski.

Table 8: Comparison of circuit parameters for $i = \{7, \ldots, 18\}$, with J_{iA} and for the Triangle of Sierpinski.

Circuit i	R_0	R_1	R_2	C_1	C_2	C_{F1}	α_1	C_{F2}	α_2	J_{iA}
7		7.40		$3.50E-3$						7.81
8		5.40				0.072	0.52			1.48
9		7.20		$3.50E-3$	$1.00E-19$					7.81
10		5.40			$3.00E-11$	0.072	0.52			1.48
11	7.00	267.70		$1.10E-2$	$1.20E-6$					2.07
12	4.20	1.50			$5.20E-10$	0.069	0.53			1.48
13	6.80	$4.00E-3$	$2.23E+2$	$1.50E-3$	$4.00E-4$					1.07
14	5.70	0.0002	987.00			0.039	0.61			0.60
15	5.60	$7.00E-5$	29.3	$2.00E-8$		0.063	0.60			0.56
16	5.80	0.0002				0.051	0.56	0.0009	0.87	0.64
17	5.60	337.30	30.90			0.031	0.64	0.0001	1.70	0.13
18		3.30	2.30	$3.00E-11$	$1.00E-11$	0.046	0.59	0.257	0.59	0.66

the polar and Nichols diagrams of $Z(j\omega)$ and the approximations $Z_{\text{app}\,iB}(j\omega)$ for $i = \{8, 10, 11, 12, 13, 14, 15, 16, 17, 18\}$.

In conclusion, we verify that, in general, the adoption of fractional electrical elements leads to modelling circuits well adapted to the experimental data and that this direction of research should be further explored in other complex systems.

5. Conclusions

FC is a mathematical tool applied in scientific areas such as electricity, magnetism, fluid dynamics, and biology. In this paper, FC concepts were applied to the analysis of electrical fractional impedances, in botanical elements and in electrical capacitors with fractal characteristics. The introduction of the CPE element in the electric circuits led us to conclude that,

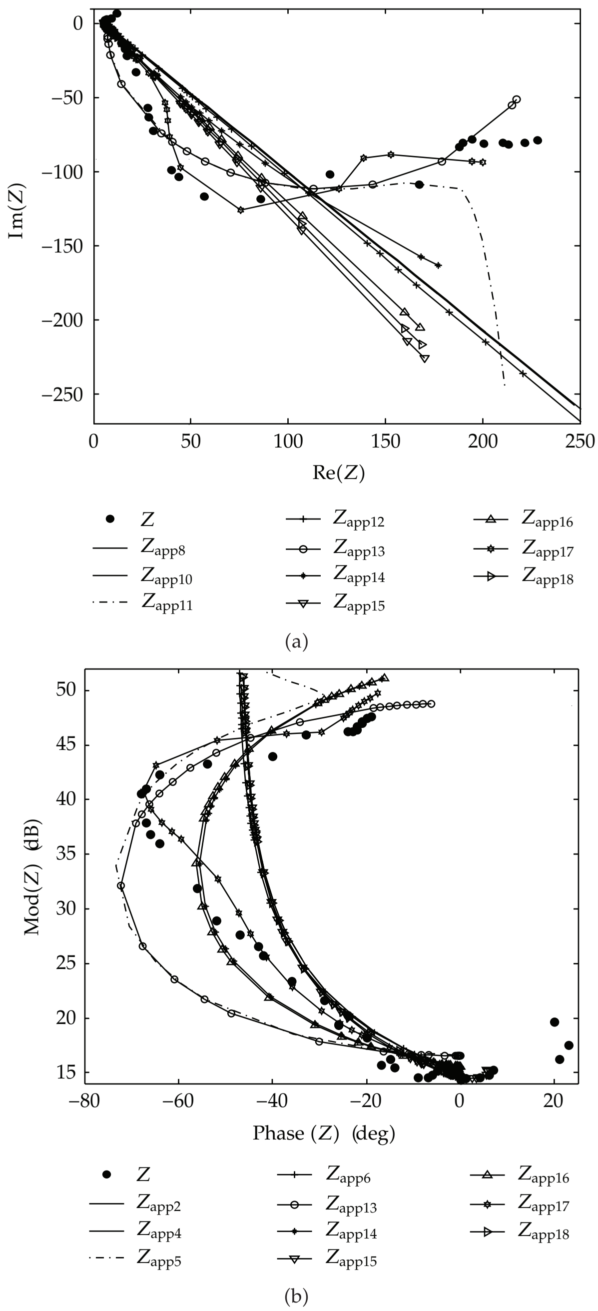


Figure 8: Polar and Nichols diagrams of $Z(j\omega)$ and the approximations $Z_{\text{app}\,iB}(j\omega)$, $i = \{8, 10, 11, 12, 13, 14, 15, 16, 17, 18\}$ of the electrical impedance of the Triangle of Sierpinski for J_{iA} and J_{iB}, respectively.

for the same number of elements, we have a better approximation model and consequently a decrease in the error value. The different configurations of the polar and Nichols diagrams of the systems studied led us to modelling the systems through electrical circuit with different configurations (series and parallel) and the combination of integer and fractional-order elements in the circuits.

Table 9: Comparison of circuit parameters for $i = \{7, \ldots, 18\}$, with J_{iB} and for the Triangle of Sierpinski.

Circuit i	R_0	R_1	R_2	C_1	C_2	C_{F1}	α_1	C_{F2}	α_2	J_{iB}
7		7.20		$3.20E-3$						6.35
8		5.70				0.072	0.52			1.45
9		7.20		$3.20E-3$	$1.00E-19$					6.35
10		5.70			$2.00E-12$	0.072	0.52			1.45
11	6.80	309.70		$2.00E-2$	$1.00E-5$					1.18
12	4.20	1.50			$5.10E-10$	0.068	0.53			1.45
13	6.70	$4.00E-3$	$2.71E+2$	$1.80E-3$	$2.40E-4$					1.05
14	6.00	0.0002	420.40			0.013	0.75			0.59
15	5.30	$7.00E-5$	29.3	$2.00E-8$		0.072	0.52			1.25
16	6.00	0.0002				0.997	0.0005	0.012	0.76	0.59
17	5.80	363.90	53.20			0.035	0.63	0.0001	1.60	0.34
18		3.40	2.00	$9.00E-12$	$1.00E-11$	0.072	0.52	1.476	0.57	1.38

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