Analysis of financial indices by means of the windowed Fourier transform

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Abstract The goal of this study is to analyze the dynamical properties of financial data series from nineteen worldwide stock market indices (SMI) during the period 1995–2009. SMI reveal a complex behavior that can be explored since it is available a considerable volume of data. In this paper is applied the window Fourier transform and methods of frac- tional calculus. The results reveal classification patterns typ- ical of fractional order systems.

Keywords Fourier transform · Windowed Fourier transform · Power law · Dynamics

1 Introduction

The financial time series are inherently non-stationary, noisy, and deterministic chaotic. The noisy characteristic refers to the unavailability of complete information from the past behavior to capture the dependency between future and past values. The non-stationary characteristic implies that the dis- tribution of series is changing over time. The literature that deals with long memory dependency in financial markets is important. Most of these papers are based on several types of statistics. For example, [7,25] show important differences among developed and emerging markets. Recent literature [6,8,9,11] shows that the degree of memory in a given mar- ket or stock is contingent to some of their characteristics.

The methods and algorithms that have been explored for description of physical phenomena become an effective background and inspiration for new methods to be used in the analysis of economical data [26–28,30]. Nevertheless, many factors interact in finance including political events, general economic conditions, and traders' expectations.

In this paper, we study several national indices at a daily time horizon at the closing and the continuously compounded return, in the perspective of fractional calculus [14,15].

The study of fractional order systems has received consid- erable attention, due to the fact that many physical systems are well characterized by fractional models. The importance of fractional order mathematical models is that it can be used to make a more accurate description and it can even give a deeper insight into the processes underlying long-range memory behavior [18,20] than classical approaches. The methods and algorithms that have been explored for descrip- tion of physical phenomena become an inspiration for very productive methods used in the analysis of economical data [1,10,21,24,31].

In this paper, the particular interest is in application of classical concepts of signal analysis, Fourier transform (FT), windowed Fourier transform (WFT), and methods of Frac- tional Calculus to analyze the stock signal indices proprieties.

Bearing these ideas in mind, the outline of our paper is as follows. In Sect. 2 are introduced the fundamentals concepts namely FT and WFT analysis. Those tools are then applied to daily data on nineteen stock market indices (SMI), including major European, American, and Asian/Pacific stock markets.

Finally, in Sect. 3 we conclude the paper with some final remarks and potential topics for further research.

Table 1 Nineteen financial indices

k	Stock market index	Abbreviation	Country	Region
1	Dutch Euronext Amsterdam	aex	Netherlands	European
2	All Ordinaries Index	aord	Australia	Asian/Pacific
3	Index of the Vienna Bourse	atx	Austria	European
4	EURONEXT BEL-20	bfx	Belgium	European
5	S.Paulo Stock	bvsp	Brazil	Americas
6	Cotation Assistée en Continu	cac	France	European
7	Deutscher Aktien Index	dax	Germany	European
8	Dow Jones Industrial	dji	USA	Americas
9	Footsie	ftse	United Kingdom	European
10	Stock market index in Hong Kong	hsi	Hong Kong	Asian/Pacific
11	Iberia Index	ibex	Spain	European
12	Malaysia Bourse	klse	Malaysia	Asian/Pacific
13	Bolsa Mexicana de Valores	mxx	Mexico	Americas
14	NASDAQ	ndx	USA	Americas
15	Tokyo Stock Exchange	nikkei	Japan	Asian/Pacific
16	New York Stock Exchange	nya	USA	Americas
17	Standard & Poor's	sp500	USA	Americas
18	Swiss Market Index	ssmi	Switzerland	European
19	Straits Times Index	sti	Singapore	Asian/Pacific

2 Dynamics of financial indices

In this section, we study nineteen SMI, namely six American, eight European, and five Asian/Pacific indices.

Our data consist of the n daily close values of $S = 19$ SMI, listed in Table 1, from January 1995 up to December 2009, to be denoted as $x_k(t)$, $1 \leq t \leq n$, $k = 1, \ldots, S$. The data are obtained from data provided by Yahoo Finance web site [32], and measure indices in local currencies.

In order to compare the indices, reducing the effects differ- ent operating levels, magnitudes, and currencies, we proceed to a 'normalization' of each index. With this process, we get a new value $x_k^*(t) \leftarrow \frac{x_k(t) - x_{av}(t)}{x_{rms}(t)}$, where $x_{av}(t)$ is the average value and $x_{rms}(t)$ is the root mean squared value of the observed values in the period under study.

Figure 1 depicts, for example, the time evolution of the normalized values for the Dax signal index, from January 1995 to December 2009, revealing the well-known chaotic characteristics.

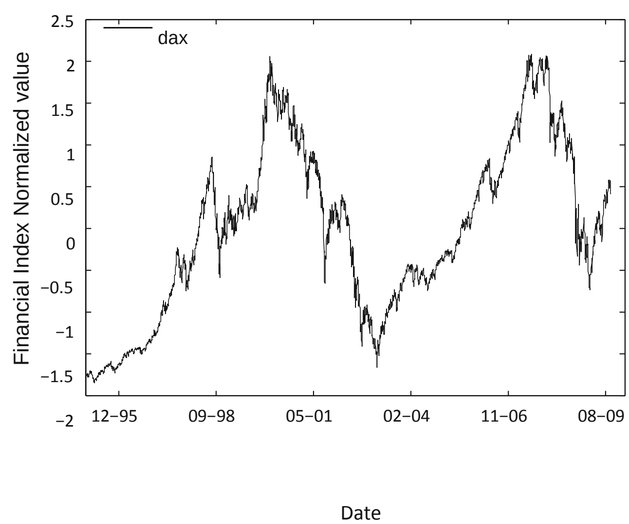


Fig. 1 The temporal evolution of the daily normalized value ($x_j^*(t)$) of the Dax signal index, from January 1995 to December 2009

2.1 Classical Fourier analysis

With these new values the corresponding FT is calculated, according to:

$$\mathcal{F}\{x_k^*(t)\} = \int_{-\infty}^{+\infty} x_k^*(t) e^{-j\omega t} dt \quad (1)$$

where F is the Fourier operator, $x_k^*(t)$ is the k-th normalized SMI, t is time, $j = \sqrt{-1}$ and ω is the angular frequency.

The amplitude $|F\{x_k^*(t)\}|$ of the FT versus the angular frequency ω can be approximated by the power law (PL):

$$|\mathcal{F}\{x_k^*(t)\}| \approx p_k \omega^{q_k}, \; p_k \in \mathbb{R}^+, \; q_k \in \mathbb{R}, k = \{1, \ldots, 19\} \quad (2)$$

with the parameters to be determined by a least square fit procedure, where p_k is a positive constant that depends on the FT amplitude and q_k is the trendline slope [12].

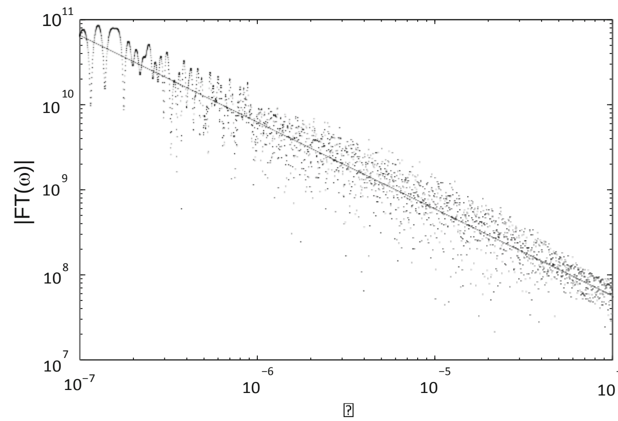


Fig. 2 $|F\{x^*(t)\}|$, and the PL trendline versus ω for the Dax index

Table 2 Parameters $\{p_k, q_k\}$ for the power law trendline of classical Fourier transform for the nineteen indices and a period of $h = 3,840$ days

k	Index	p_k	q_k
1	aex	3.658259992	−0.970742113
2	aord	3.871491253	−0.946546338
3	atx	2.725364749	−0.95316636
4	bfx	4.533784516	−0.94968942
5	bvsp	2.725098806	−0.986078611
6	cac	5.589288961	−0.93360923
7	dax	7.976100882	−0.909161453
8	dji	4.165819773	−0.97331343
9	ftse	6.810398042	−0.938854953
10	hsi	4.821174348	−0.960384906
11	ibex	4.9078229	−0.946868311
12	klse	9.739917234	−0.898420223
13	mxx	1.434360389	−1.036272753
14	ndx	5.907288016	−0.935896616
15	nikkei	4.73173861	−0.959663276
16	nya	5.798348633	−0.936935758
17	sp500	6.045945568	−0.940522354
18	ssmi	6.481462034	−0.932880432
19	sti	4.165971635	−0.966611526

Figure 2 shows the FT spectrum and PL approximation of the Dax index.

Table 2 shows the values of the parameters p_k and q_k for the nineteen indices. In all cases, we get a fractional behavior in between the white ($q = 0$) and the pink ($q = -1$) noises [4,22,23].

The parameters p_k and q_k have different meanings. While parameter p_k is related with the energy of the signal, the parameter q_k reveals the nature of the information content of the signal. Therefore, the closer q_k is to zero the noisier is the time signal. Even so we verify that the values of q are relatively close to the integer $q = -1$ and it is not totally clear if the deviations are merely the result of the numerical fit.

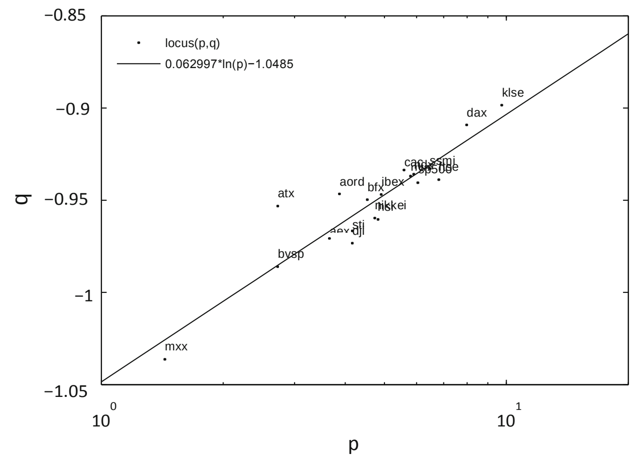


Fig. 3 Locus of the parameters (p_k, q_k), $k = 1, \ldots, 19$ for the FT and the normalized financial indices

Figure 3 shows the locus (p_k, q_k) of PL parameters for all nineteen signal indices. We observe that the parameters fol- low the expression $q_k \approx 0.351\, p_k^{0.063}$ with the mxx and ndx indices in the lower left and upper right corners, respectively. The standard FT is precise in frequency, but not in time.

Small changes in the signal at one time location cause change in the Fourier domain globally. It is of interest to have transformed domains where we can establish a compromise between the precision of the time and frequency domains.

2.2 Windowed Fourier analysis

The WFT provides simultaneous insight in the time and fre- quency behavior of the signal. A window is a function that is zero-valued outside of some chosen interval. When a signal is multiplied by a window function, the product is also zero- valued outside the interval. All that is left is the view through the window. If we time slice the signals and calculate the FT, then for each section of the signal, the resulting spectrum is a smooth curve.

In this line of thought one way of obtaining the time- dependent frequency content of a signal is to take the FT of a function over an interval around an instant τ, where τ is a variable parameter [13]. The Gabor transform accom- plishes this by using the Gaussian window. The WFT, also known as the short time Fourier transform, generalizes the Gabor transform by allowing a general window function [3]. The concept of this mathematical tool is very simple. We multiply the signal to be analyzed $x^*(t)$ by moving window $g(t - \tau)$, and then we compute the FT of the windowed signal $x^*(t)g(t - \tau)$. Each FT gives a frequency domain slice associated with the time value at the window center.

Table 3 Parameters $\{pw_i, qw_i\}$ for the power law trendline of WFT, $i = \{1, 2, \ldots, 5\}$ for the nineteen indices with $t_w = 1,280$ days

k	Index	$p_{1,k}$	$q_{1,k}$	$p_{2,k}$	$q_{2,k}$	$p_{3,k}$	$q_{3,k}$	$p_{4,k}$	$q_{4,k}$	$p_{5,k}$	$q_{5,k}$
1	aex	1.193	−0.952	8.027	−0.826	2.624	−0.923	0.092	−1.111	0.305	−1.071
2	aord	0.299	−1.011	3.476	−0.810	0.070	−1.120	0.089	−1.108	10.603	−0.804
3	atx	0.137	−1.040	0.080	−1.064	0.300	−0.944	0.064	−1.161	0.626	−1.029
4	bfx	2.806	−0.877	0.483	−1.039	5.306	−0.851	2.019	−0.870	2.043	−0.947
5	bvsp	0.121	−1.057	1.334	−0.866	0.476	−0.929	0.523	−0.976	10.782	−0.802
6	cac	2.024	−0.894	9.081	−0.822	2.850	−0.905	0.582	−0.966	6.584	−0.834
7	dax	0.410	−1.029	4.428	−0.880	2.194	−0.925	0.359	−1.013	6.734	−0.840
8	dji	0.792	−0.975	0.374	−1.065	0.657	−1.026	0.071	−1.157	21.338	−0.749
9	ftse	2.039	−0.924	2.335	−0.944	1.639	−0.962	0.618	−0.982	2.688	−0.933
10	hsi	1.905	−0.944	1.550	−0.963	1.258	−0.924	0.251	−1.045	2.531	−0.963
11	ibex	0.244	−1.067	0.410	−1.052	0.568	−0.998	0.296	−1.021	12.346	−0.787
12	klse	800.023	−0.478	0.506	−1.044	0.402	−1.014	0.054	−1.165	0.538	−1.052
13	mxx	0.021	−1.152	0.060	−1.106	0.689	−0.870	0.301	−0.995	9.481	−0.790
14	ndx	9.233	−0.748	5.179	−0.909	1.150	−0.959	0.158	−1.055	1.558	−0.910
15	nikkei	0.280	−1.093	2.806	−0.916	0.489	−1.023	0.164	−1.097	0.730	−1.022
16	nya	1.190	−0.923	0.773	−0.987	0.801	−0.986	0.212	−1.060	22.469	−0.749
17	sp500	1.758	−0.910	0.595	−1.041	0.789	−1.005	0.130	−1.099	17.408	−0.762
18	ssmi	2.723	−0.900	0.481	−1.041	5.340	−0.854	0.343	−1.014	14.320	−0.785
19	sti	0.401	−1.058	1.454	−0.959	1.331	−0.930	0.158	−1.082	1.169	−1.009

Actually, windowing the signal improves local spectral estimates [3]. Considering the window function centered at time τ, the WFT is represented analytically by:

$$\mathcal{F}_\omega(\tau, w) = \int_{-\infty}^{+\infty} x^*(t) g(t-\tau) e^{-j\omega t} \mathrm{d}t \qquad (3)$$

Each window has a width t_w, and the distance between two consecutive windows can be defined in a way so that they become overlapped during a percentage of time β in relation with t_w. The frequencies of the analyzing signal under $1/t_w$ are rejected by the WFT. Diminishing t_w produces a reduction of the frequency resolution and an increase in the time resolution. Augmenting t_w has the opposite effect. Therefore, the choice of the WFT window entails a well-known duration-bandwidth tradeoff. On the other hand, the window can introduce an unwanted side effect in the frequency domain. As a result of having abrupt truncations at the ends of the time domain caused by the window, specially the rectangular one, the spectrum of the FT will include unwanted side lobes. This gives rise to an oscillatory behavior in the FT results called the Gibbs phenomenon [2]. In order to reduce this unwanted effect, a weighting window function that attenuate signals at their discontinuities is generally used. For that reason, there are several popular windows normally adopted in the WFT, for example, Bartlett, Hanning, Hamming, Rectangular, Taylor, Gaussian, and Blackman [16]. If the windows do not overlap, then it is clear that some data are lost. On the other hand, if the windows overlap in a short period of time, a significant part of the time signal is ignored due to the fact that most windows exhibit small values near the boundaries. To avoid this loss of data, overlap analysis must be performed. Having these ideas in mind, the spectra of the signals approximated by the trendlines are analyzed with the WFT. Due to space limitations, we only depict the more relevant features. Among the several possible windows, we adopt the Gaussian window for a sliding-window Fourier transform. The coefficients of a Gaussian window are computed from the equation:

$$g(t) = e^{-\frac{1}{2}\left(\alpha \frac{t}{t_w/2}\right)^2} \qquad (4)$$

where t_w is the window length, $-\frac{t_w}{2} \leq t \leq \frac{t_w}{2}$ and α is the

reciprocal of the standard deviation. The width of the win-
dow is inversely related to the value of α: a larger value of α produces a narrower window.

We now adopt the WFT and we consider $\alpha = 2.5$, $t_w = 1, 280$ days (5 years) and that two consecutive windows are superimposed over a range of window length of $\beta = 50\,\%$. Therefore, consider a total of 5 windows centered at $\tau = 640i$ days for $i = 1, 2, \ldots, 5$.

Table 3 shows the values of the parameters $p_{i,k}$ and $q_{i,k}$, where $i = \{1, 2, \ldots, 5\}$ represents the window number and $k = 1, \ldots, 19$ for the financial indices. Values of q closer to zero indicate larger volatility, while more negative values show a smother time variation.

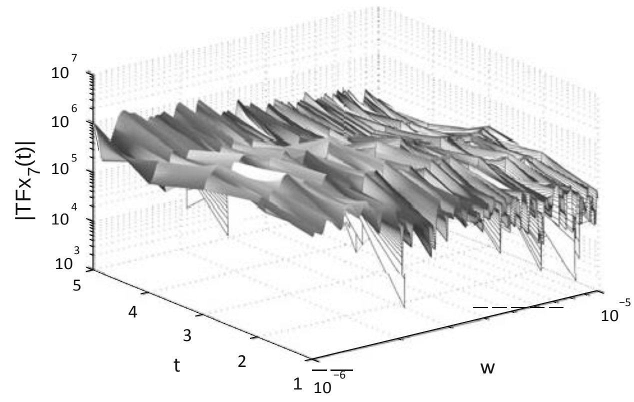


Fig. 4 Windowed Fourier transform $|F_w\{x_7(t)\}|$ for the normalized Dax index with $\alpha = 2.5$, $t_w = 1,280$ days, and $\beta = 50$ %, centered at $\tau = 640i$ for $i = \{1, 2, 3, 4, 5\}$

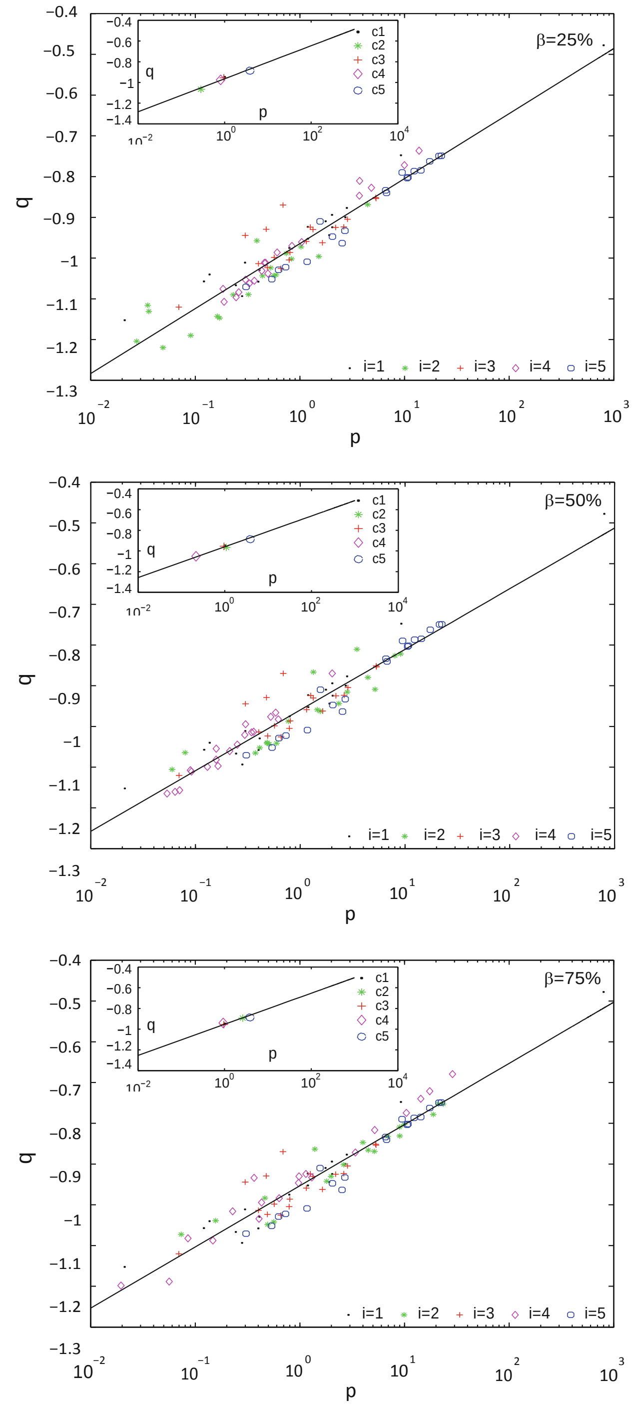
Figure 4 depicts the amplitude of a sliding-window Fourier Transform, $|F_\omega\{x_7(t)\}|$, centered at $\tau = 640i$ for $i = \{1, 2, 3, 4, 5\}$, for the Dax index.

We verify that the WFT ‘sees’ the fractional characteristics, as we had already checked with the FT. This means that now we have not only the frequency response, but also a more precise time information.

2.3 Time and frequency characteristics of the financial indices

We observed that WFT constitutes a good tool for analyzing the fractional dynamics of the SMI in the frequency domain while preserving time information. However, the WFT requires the tuning of several parameters and, there- fore, their influence must be evaluated. In this perspective, we compare the WFT for (i) distinct types of windows $g(t)$, (ii) several overlapping percentages β, and (iii) dif- ferent window widths t_w. We adopt the Dax index ($k = 7$) and, for further comparison, is included a geometric ‘cen- ter’ (c_i, $i = 1, \ldots, i_{\max}$, $i_{\max} = 5$ periods, of each cloud of points with coordinates (p_{av}, q_{av}) given by the geometric and arithmetic averages, respectively. We verify that the four windows produce similar locus and that the PL trendlines $q = a \ln(p) + b$, $a, b \in \mathrm{IR}$ have invariant values for the parameters a and b. A critical comment should be made. The fractional slope of the PLs does not demonstrate the exis- tence of a fractional model. Nevertheless, the results support further efforts in the same perspective.

Figure 5 shows the $(p_{i,k}, q_{i,k})$ locus of WFT amplitude for the nineteen indices, five Gaussian windows with $t_w = 5$ years, $i_{\max} = 5$, and the overlapping $\beta = \{25, 50, 75\}$ %.



We observe again the properties previously identified in Table 3 for the Dax index.

Fig. 5 The $(p_{i,k}, q_{i,k})$ locus of WFT amplitude for the nineteen indi- ces, five Gaussian windows with $t_w = 5$ years, $i_{\max} = 5$, and the overlapping percentages $\beta = \{25, 50, 75\}\%$

Figure 6 depicts the $(p_{i,k}, q_{i,k})$ locus of WFT ampli- tude for the nineteen indices, $i_{\max} = 5$, and the windows {Gaussian, Bartlett, Hamming, Hanning}. There are other type of windows [17,29] but this set, with four types, is suf- ficient. In fact, several experiments demonstrated that other functions $g(t)$ lead to the same type of conclusions and that, as expected, the more abrupt the slicing at the window time

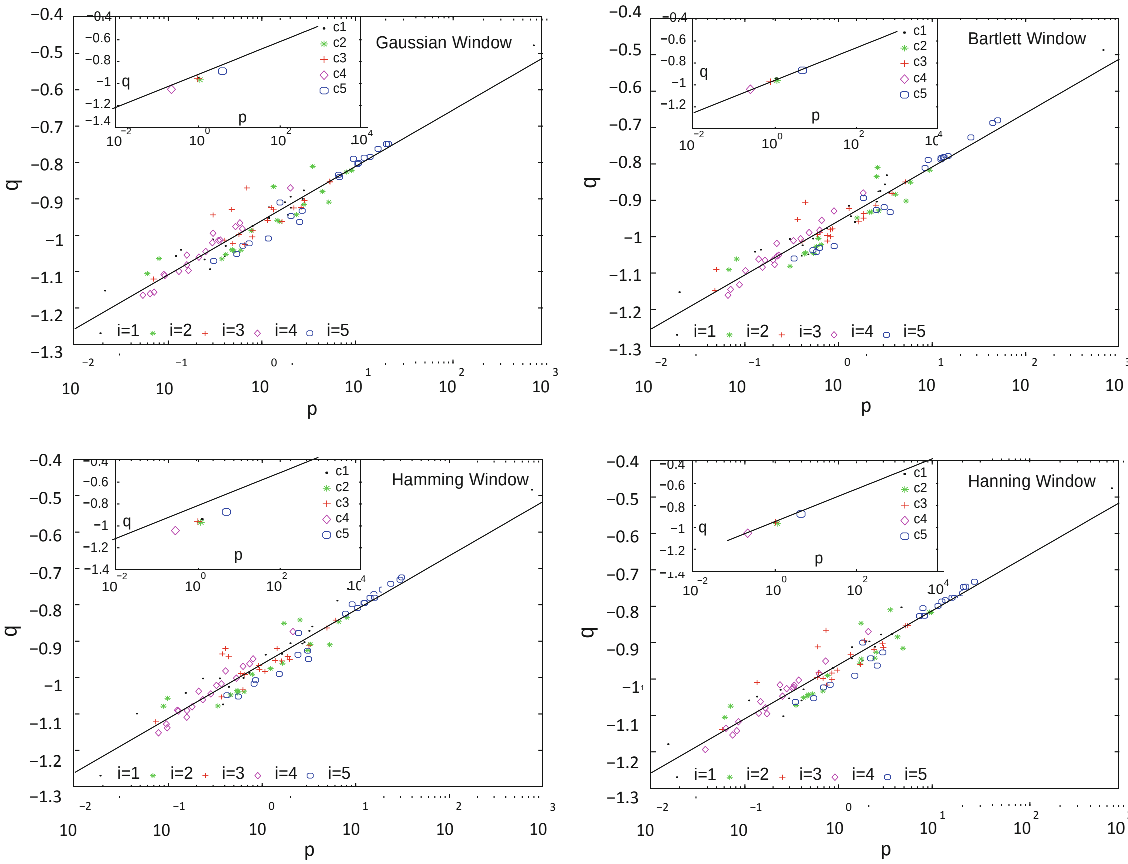


Fig. 6 The $(p_{i,k}, q_{i,k})$ locus of WFT amplitude for the nineteen indices, β = 50 %, t_w = 5 years, with cases $i = 1, \ldots, 5$, and the four windows types {Gaussian, Bartlett, Hamming, Hanning}

extremities (as in the rectangular case), the higher the Gibbs phenomenon.

Figure 7 represents the $(p_{i,k}, q_{i,k})$ locus of WFT amplitude for the nineteen indices, the Gaussian window with the four cases t_w = {2.5, 3.0, 5.0, 7.5} years, with $i_{\max}$ = {11, 9, 5, 3} periods, and β = 50 %. The corresponding trendlines and centers are also included. The main properties, namely the PL trendlines and its parameters, remain invariant, but in this case we observe a larger variability of the 'center' c_i the smaller the value of t_w, particularly for the smaller values, which means that the WFT becomes more sensitive to time information, as expected.

Figures 5, 6, and 7 demonstrate that the choices of window type and the overlap percentage have a small influence upon the (p, q) trendline and its parameters that remain invariant. By other words, the locus is a global property of the proposed method and, as time evolves, the financial indices (i.e., the nineteen SMI) slide along the logarithm trendline, the variability being higher the smaller the time width of the WFT. Therefore, the WFT enables the time analysis and by inspecting the location of the 'center' and comparing it with other cases it is possible to characterize the recession/expansion financial cycle.

In this perspective we can define the distance measure d_{jl} between 'centers' j and l, along the trendline, as:

$$d_{jl} = sgn \times \sqrt{\left[\left(\ln \frac{p_{j,\mathrm{av}}}{p_{l,\mathrm{av}}}\right)^2 + \left(q_{j,\mathrm{av}} - q_{l,\mathrm{av}}\right)^2\right]} \qquad (5)$$

where *sgn* is positive/negative with ‘center’ moving to the right/left, that is, when moving for financial expansion/reces- sion.

For example, Table 4 depicts the distance (d_{jl}), accord- ing to (5), between two consecutive ‘centers’ of the indices, when adopting Gaussian windows with β = 50 % and time lengths t_W = 3.0 years. Positive values reflect expansion, while negative values correspond to recession periods.

In conclusion, we verified that the WFT is a mathematical tool capable of revealing the characteristics of the financial indices, while preserving time information. These proper- ties lead to the development of an analysis procedure that revealed clearly tendencies both of individual and of global SMI.

In fact, when compared with the historical data, the results of the analysis have a good behavior: positive values in expan- sion periods and negative values in recession periods [5,19].

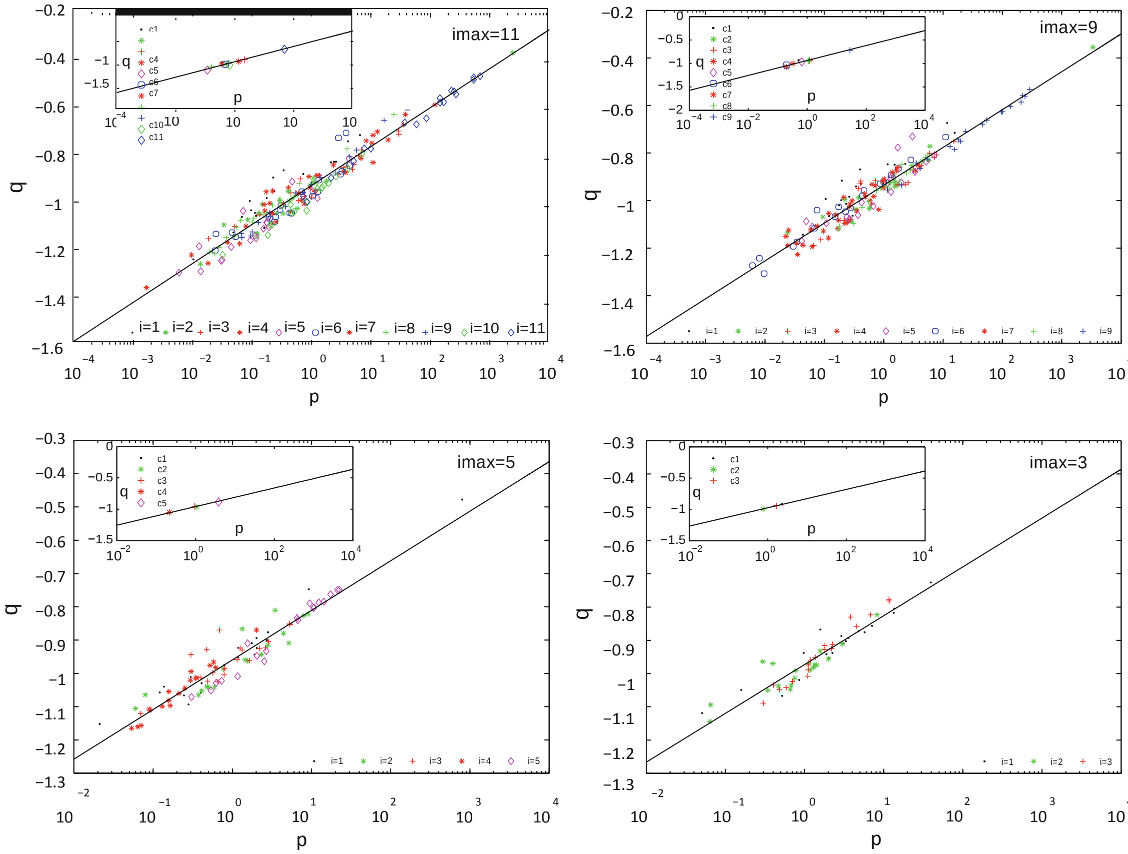


Fig. 7 The $(p_{i,k}, q_{i,k})$ locus of WFT amplitude for the nineteen indices, the Gaussian window with β = 50 %, and the time lengths t_w = {2.5, 3.0, 5.0, 7.5}years, with $i_{\max}$ = {11, 9, 5, 3}

Table 4 Distance, according to (5), between two consecutive 'centers' of the indices with Gaussian windows

Distance	Value	Cycle
d_{12}	+0.0794	Slight expansion
d_{23}	−0.1199	Slight recession
d_{34}	−1.4941	Recession
d_{45}	2.8847	Expansion

3 Conclusions

Financial cycles are the cumulative result of a plethora of different phenomena. Therefore, SMI reveal a complex behav- ior and their dynamical analysis poses problems not usual in other types of systems.

This paper analyzed the evolution of nineteen important signal SMI. An evaluation using the Fourier transform and power law revealed that the indices have characteristics sim- ilar to those of fractional noise, somehow in between the white and pink noises. The results lead to the definition of a distance measure that reflect qualitatively the periods of financial expansion or recession.

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