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STATISTICAL MODELLING OF ROBOT MANIPULATORS

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INTRODUCTION

Robotics is an area encompassing a wide range of industrial, technological and scientific applications. Still, many of them pose challenging issues to the available industrial manipulators. The superior performances of the human arm induced an extensive research towards better mechanical structures. In this line of thought, considerable research as been done, either on the kinematics Tsai and Soni (1), Asada (2) or the dynamics Yoshikawa (3,4), Youcef-Toumi and Asada (5). However, a simple optimization criterion comprising both the kinematics and the dynamics is still lacking. In the present paper we address this problem, and our presentation is organized as follows. In section two the robot manipulator model and the associated optimization criterion are discussed. In section three a new model leading to a natural optimization criterion is formulated and applied to a 2R robot manipulator. Finally, in section four conclusions are drawn.

ON THE MODELLING OF ROBOT MANIPULATORS

The classical modelling of robot manipulators is well known. For the kinematics we have a set of equations relating the joint space and the operational space of the form:

$$q = \alpha(p) \qquad (1a)$$

$$\dot{q} = \theta(p,\dot{p}) \qquad (1b)$$

$$\ddot{q} = \Phi(p,\dot{p},\ddot{p}) \qquad (1c)$$

where $\{q,\dot{q},\ddot{q}\}$ ($\{p,\dot{p},\ddot{p}\}$) are the n-vectors of positions, velocities and accelerations in the joint (operational) space for a n degrees of freedom (d.o.f.) manipulator. Associated with the kinematic model we have a model for the statics which relates the operational space forces Γ with the joint actuator torques T:

$$T = J(q)^T \, \Gamma \qquad (2)$$

where $J(q)$ is the jacobian matrix corresponding to the differential relationship $\dot{p}=J(q)\dot{q}$. Moreover, the dynamics is described by a nonlinear matrix differential equation:

$$T = I(q)\ddot{q}+C(q,\dot{q})+G(q) \qquad (3)$$

having $T_I=I(q)\ddot{q}$, $T_C=C(q,\dot{q})$ and $T_G=G(q)$ as the n-vectors of inertial, Coriolis/centripetal and gravitational torques.

Based on these equations research has been done on issues such as manipulator structure optimization (1-5) and path planning Hollerbach (6) and Sahar and Hollerbach (7). Nevertheless, a more sound consideration of the whole theme reveals that these methods are far from achieving a comprehensive formulation. This observation motivates the re-evaluation of the approachs in use. In fact, expressions (1)-(3) show that the plethora of variables and parameters involved, gives rise to a cumbersome work both in the analysis and design stages. The huge number of possible combinations of values indicates that, in order to overcome implementation problems, alternative concepts are required. Statistics is as a mathematical strategy well adapted to this type of problem. With this method, we lose the "certainty" of a deterministic model, however, we gain a simpler and more intuitive perspective of the phenomena involved. In the sequel we refer to the new approach, as the statistical model Machado and de Carvalho (8) and Galhano et al (9-11) to stress the contrast with the standard method.

Our modelling procedure comprises:

- The statistical description of a set of input variables, that is variables that are free to change independently.
- The statistical description of a set of output variables, that is, variables that are functions of the previous ones.
- A set of parameters which are to be optimized in the design stage.

The above definition allows a considerable freedom in the choice of each set. In the present case, the distribution of the relevant variables through the three referred sets is established as follows:

- $\{p,\dot{p},\ddot{p}\}$ act as input variables of the kinematic system. This option enables a definition of the required kinematic performances on the operational space which are more natural to the designer.
- $\{q,\dot{q},\ddot{q}\}$ act as output variables of the kinematic system, but play the role of input variables set in the dynamic model. Thereby, we arrive at a relationship between kinematics and dynamics in a form amenable to performance optimization criteria as defined in the sequel.
- The set of dynamic output variables consists of the required joint torques $\{T\}$.
- The parameter set consists of link lengths, masses and inertias.

In conclusion, in the kinematics (dynamics), $\{p,\dot{p},\ddot{p}\}$ ($\{q,\dot{q},\ddot{q}\}$) are considered as independent random variables, its probability density functions (p.d.f.'s) being similar to the histograms of a long run sampling, while $\{q,\dot{q},\ddot{q}\}$ ($\{T\}$) are the corresponding random dependent variables. The statistical description of the involved variables, does not consider the (implicit) time variable. Therefore, variables that are related through the time derivative operator are considered independent of each other.

A STATISTICAL MODEL OF THE 2R MANIPULATOR

Let us now adopt the 2R joint-actuated robot manipulator as the support for the development and implementation of the new modelling concepts. In the next sub-section we begin by introducing our approach in the kinematic case. In the second sub-section we shall analyse the dynamic case and in the third sub-section we investigate the properties of the overall (i.e. kinematics + dynamics) system.

TABLE 1 - Categories of $\dot{p}$ and $\ddot{p}$.

	1	2	3	4	5	6	7	8	9
$\sigma_{\dot{p}}$	0.1	0.1	0.1	1	1	1	10	10	10
$\sigma_{\ddot{p}}$	0.1	1	10	0.1	1	10	0.1	1	10

TABLE 2 - Parameters of the 2R robot.

	Length	Mass	Inertia
Link 1	0.3 m	2.16 Kg	0.01755 Kgm²
Link 2	0.3 m	1.68 Kg	0.01324 Kgm²

The Kinematics

The set of kinematic input variables consists of $\{p,\dot{p},\ddot{p}\}$. Therefore, it is necessary to characterize them in statistical terms, namely by defining appropriate p.d.f.'s for each variable. As there is no a priori knowledge about the typical behaviour we start with some reasonable assumptions. For the position variable $p=[x,y]^T$ we consider a bidimensional uniform p.d.f.:

$$f_P(p)=\begin{cases} C & \text{if } (r_1-r_2)^2\leq x^2+y^2\leq(r_1+r_2)^2 \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

with $C=1/\{\pi[(r_1+r_2)^2-(r_1-r_2)^2]\}$. In the sequel we will see how to modify the input p.d.f. in order that the kinematic performances are optimized. In what concerns the p.d.f.'s for $\dot{p}$ and $\ddot{p}$, we decided to use bidimensional Gaussian p.d.f.'s with zero mean:

$$f_{\dot{P}}(\dot{p})=EXP[-(\dot{x}^2+\dot{y}^2)/(2\sigma_{\dot{p}}^2)]/(2\pi\sigma_{\dot{p}}^2) \quad (5)$$
$$f_{\ddot{P}}(\ddot{p})=EXP[-(\ddot{x}^2+\ddot{y}^2)/(2\sigma_{\ddot{p}}^2)]/(2\pi\sigma_{\ddot{p}}^2) \quad (6)$$

Moreover, using these p.d.f.'s we impose some useful properties, such as:
- The random variables p, $\dot{p}$ and $\ddot{p}$ are independent of each other.
- The variables $\dot{p}$ and $\ddot{p}$ are made of two independent components, that is, $\dot{x}$ ($\ddot{x}$) is independent of $\dot{y}$ ($\ddot{y}$).

The "excitation" of the inverse kinematic system produces output random variables $\{q,\dot{q},\ddot{q}\}$ with p.d.f.'s which are related to the previous ones by:

$$f_Q(q) = J_P f_P(p) \quad (7a)$$
$$f_{Q\dot{Q}}(q,\dot{q}) = J_V f_{P\dot{P}}(p,\dot{p}) \quad (7b)$$
$$f_{Q\dot{Q}\ddot{Q}}(q,\dot{q},\ddot{q}) = J_A f_{P\dot{P}\ddot{P}}(p,\dot{p},\ddot{p}) \quad (7c)$$

where the jacobians J_P, J_V and J_A are:

$$J_P=\partial(p)/\partial(q)=r_1r_2S_2 \quad (8a)$$
$$J_V=\partial(p,\dot{p})/\partial(q,\dot{q})=J_P(r_1r_2S_2) \quad (8b)$$
$$J_A=\partial(p,\dot{p},\ddot{p})/\partial(q,\dot{q},\ddot{q})=J_V(r_1r_2S_2) \quad (8c)$$

Each of the expressions (7) is made of two distinct factors:
- Weighting factors - J_P, J_V and J_A - which depend solely on the system kinematic properties.
- The "excitation" p.d.f.'s - $f_P(p)$, $f_{P\dot{P}}(p,\dot{p})$ and $f_{P\dot{P}\ddot{P}}(p,\dot{p},\ddot{p})$ - which are a measure of the task requirements.

Bearing these facts in mind, several experiments were performed, having:
- The total link length constant, L=0.6.
- Seven robot configurations with ratios $\mu=r_1/r_2$ of 0.4, 0.6, 0.8, 1, 1.2, 1.4 and 1.6, respectively.
- Excitation of the kinematic system in the operational space with a numerical random sample obeying p.d.f.'s (4)-(6) for the nine distinct categories of Table 1.
- Analysis of the histograms of the output variables. In order to simplify matters, only marginal p.d.f.'s were considered.

After a large number of experiments using the numerical set of robot parameters depicted in Table 2, we concluded that the shape of the resulting p.d.f.'s varied significantly from variable to variable, but all of them showed symmetry around zero. Consequently, in order to characterize the resulting histograms by a scalar index, we decided to adopt the 95%-inter-percentile range. The numerical results (Fig. 1) reveal that in the majority of the charts we have a minimum about $\mu=1$; yet, this conclusion can be easily inferred, analytically, from expression (7). Indeed, for symmetrical histograms about zero on the x-axis, having a peak on that point, a larger value of the jacobian corresponds to a smaller dispersion of the random variable. This, in turn, means average smaller amplitude requirements posed to that variable. Therefore, we have an optimization criterion which is based on the new statistical modelling concepts. As the maximization of J_P, J_V and J_A requires the same steps, we have for:

$$L = r_1+r_2, \quad \mu = r_1/r_2 \quad (9)$$

that a maximum occurs when:

$$\mu = 1, \quad q_2 = \pi/2 \quad (10)$$

which coincide with the results obtained (using the classical approach) in previous studies (1,3). Moreover, due to (2) the optimization of the kinematics is equivalent to the optimization of the statics.

On the other hand, if further optimization is desired, then the next step will be the selection of an optimum "excitation" p.d.f.. This second step of optimization will define, in a statistical sense, an optimum kinematic class for the manipulator trajectories. Obviously, we can find a multitude of different p.d.f's obeying (10); yet, for the subsequent study a particular choice is of minor importance. Consequently, we decided to adopt the following family of position p.d.f.'s in the operational space (with $K\geq1$):

$$f_P(x,y)=\text{constant}*\{1-[(x^2+y^2-r_1^2-r_2^2)/(2r_1r_2)]^2\}^{(K-1)/2} \quad (11)$$

which, in the joint space, corresponds to:

$$f_Q(q_1,q_2)=\text{constant}*S_2^K \quad (12)$$

As extreme cases, we have that for K=1 it becomes the uniform p.d.f. (4), while for $K\to\infty$ we get Dirac type p.d.f. optimum in the sense of (10). As far as $\dot{p}$ and $\ddot{p}$ are concerned we can see that the kinematic study does not point out any special class of p.d.f.'s. However, these variables are negatively affected by the position deviation from the optimum configuration $q_2=\pi/2$. Therefore, we decided to study the system behaviour, for two alternative situations. In the first case, the operational requirements are described by p.d.f.'s (12), (5) and (6) and, in the second case, we have p.d.f. (12) associated with the "enhanced" (q_2-dependent) p.d.f.'s for $\dot{p}$ and $\ddot{p}$:

$$f_{\dot{P}}(\dot{p},q_2)=EXP\{-(\dot{x}^2+\dot{y}^2)/[2\sigma_{\dot{p}}^2(q_2)]\}/[2\pi\sigma_{\dot{p}}^2(q_2)] \quad (13a)$$

$$\sigma_{\dot{p}}(q_2)=\begin{cases} 2\sigma_{\dot{p}}|q_2|/\pi & \text{if } 0<|q_2|\leq\pi/2 \\ 2\sigma_{\dot{p}}|q_2-\pi|/\pi & \text{if } \pi/2<|q_2|\leq\pi \end{cases} \quad (13b)$$

TABLE 3 - Categories of $\dot{q}$ and $\ddot{q}$ in the joint space.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\sigma_{\dot{q}1}$	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	10	10	10	10	10	10	10	10
$\sigma_{\dot{q}2}$	0.1	0.1	0.1	0.1	10	10	10	10	0.1	0.1	0.1	0.1	10	10	10	10
$\sigma_{\ddot{q}1}$	0.1	0.1	10	10	0.1	0.1	10	10	0.1	0.1	10	10	0.1	0.1	10	10
$\sigma_{\ddot{q}2}$	0.1	10	0.1	10	0.1	10	0.1	10	0.1	10	0.1	10	0.1	10	0.1	10

$$f_{\ddot{p}}(\ddot{p},q_2)=EXP[-(\ddot{x}^2+\ddot{y}^2)/[2\sigma_{\ddot{p}}^2(q_2)]\}/[2\pi\sigma_{\ddot{p}}^2(q_2)] \quad (14a)$$

$$\sigma_{\ddot{p}}(q_2)= \begin{cases} 2\sigma_{\ddot{p}}|q_2|/\pi & \text{if } 0<|q_2|\leq\pi/2 \\ 2\sigma_{\ddot{p}}|q_2-\pi|/\pi & \text{if } \pi/2<|q_2|\leq\pi \end{cases} \quad (14b)$$

In order to test numerically the above conjectures, the previous results for $\mu=1$ are compared with a new case using $\mu=1$ and $K=3$ in expressions (11)-(12). This experiment revealed a remarkable performance improvement as shown by the chart overlaped in Fig. 1, particularly for velocity-dependent requirements.

The Dynamics

The statistical description of the dynamics demands steps similar to those adopted in the kinematics, namely:

- Characterisation of the input variables (q, $\dot{q}$ and $\ddot{q}$) through appropriate p.d.f.'s.
- "Stimulation" of the system behaviour through numerical experiments.
- Analysis of the histograms of the output variables (T).

However, a preliminary observation shows that the dynamic study is much more complex than the kinematic one. Consequently, and in order to gain some insight for the subsequent study we decided to consider, in a first stage, as dynamic output variables, the components of the joint torques, (i.e. T_G, T_C and T_I). Based on this preliminary analysis, then ,in a second stage, we consider the total joint torques.

The preliminary analysis. In the first stage we have:

$$f_G(q) = J_G f_q(q) \quad (15a)$$
$$f_C(q,\dot{q}) = J_C f_{q\dot{q}}(q,\dot{q}) \quad (15b)$$
$$f_I(q,\dot{q},\ddot{q}) = J_I f_{q\dot{q}\ddot{q}}(q,\dot{q},\ddot{q}) \quad (15c)$$

where:

$$J_G=\partial(q)/\partial(T_G) \quad (16a)$$
$$J_C=\partial(q,\dot{q})/\partial(T_G,T_C) \quad (16b)$$
$$J_I=\partial(q,\dot{q},\ddot{q})/\partial(T_G,T_C,T_I) \quad (16c)$$

Unlike the kinematic situation, where the optimization was similar for all the jacobians, now their effects differ according to each dynamic term. Analysing the jacobians (16) we conclude that:

- The maximizing of J_G stipulates that q_1 and q_{12} should have p.d.f.'s with maxima at 0 or π. The observation of histograms resulting from "excitation" p.d.f.'s obeying these conditions showed an interesting result. As expected the (symmetrical) histograms resembled Dirac pulses, but their peaks were located at non-zero values. Indeed, the plots showed sharp symmetrical peaks located at the maxima (positive and negative) values attained by the gravitational torques. This means, thereby, that the optimization procedure must adopt an inverse strategy, that is to say we must minimize J_G.
- The maximizing of J_C implies that q_2 must have a p.d.f. with a maximum on 0 or π. Numerical experiments showed in this case that the resulting histograms of the Coriolis/centripetal terms tended, as desired, towards a Dirac on zero.
- The analytical expression of J_I is more complex. Nevertheless, its analysis revealed a maximizing condition compatible with the previous one.

In conclusion, we may say that J_G defines a "rest region" while J_C and J_I define an "active region" of operation.

The complete analysis. Based on the preliminary analysis now we shall study the (total) dynamics. The direct application of our optimizing method to the dynamics would require the mathematical and numerical treatment of 3n-dimensional p.d.f.'s. In order to avoid this intricate analysis, we decided to integrate the (partial) conclusions pointed out in the first stage (i.e. the guidelines resulting from the separate study of T_G, T_C and T_I) in the formulation of our present investigation. Hence we "excited" the dynamics with four different position p.d.f.'s (having $K=3$):

$$f_q(q_1,q_2)=\text{constant}*S_2^K \quad (17)$$
$$f_q(q_1,q_2)=\text{constant}*(S_1S_2)^K \quad (18)$$
$$f_q(q_1,q_2)=\text{constant}*(S_1S_{12})^K \quad (19)$$
$$f_q(q_1,q_2)=\text{constant}*C_2^K \quad (20)$$

which are suggested by the optimization of the kinematics, a compromise between kinematics and gravitational torques, the gravitational torques, and both the Coriolis/centripetal and inertial torques, respectively. Due to the non-existence of optimization guidelines on $\dot{q}$ and $\ddot{q}$, we considered two gaussian "excitation" p.d.f.'s ($i=1,2$), for the sixteen different categories of Table 3:

$$f_{\dot{q}i}(\dot{q}_i)=EXP[-\dot{q}_i^2/(2\sigma_{\dot{q}i}^2)]/(2\pi\sigma_{\dot{q}i}^2) \quad (21)$$
$$f_{\ddot{q}i}(\ddot{q}_i)=EXP[-\ddot{q}_i^2/(2\sigma_{\ddot{q}i}^2)]/(2\pi\sigma_{\ddot{q}i}^2) \quad (22)$$

Figure 2 depicts the results for T_1 and T_2 when the 95% index is applied to the corresponding histograms. These charts revealed several important properties such as:

- T_1 (T_2) depends strongly on $\dot{q}_2$ ($\dot{q}_1$).
- In a statistical sense, T has low sensitivity to $\ddot{q}$ requirements.
- The suggestions pointed out by the first stage are compatible with these last results. In fact, for "rest" (or "non-active") requirements, p.d.f. (19) is the more appropriate, while for the "active" (or "non-rest") situation p.d.f. (20) is the optimal.

The Total System

We discussed the kinematics and dynamics separately, however, in the real manipulator, these systems can not be taken apart. Therefore, the statistical description of the total system (i.e. both the kinematics and dynamics) will have cross-coupling effects and its influence must be evaluated. To test these effects, the total system was numerically "excited" through random samples according to position p.d.f.'s (17)-(20) combined with the two alternative $\dot{q}$ and $\ddot{q}$ p.d.f.'s, namely (5)-

(6) or (13)-(14). For the categories of operational requirements represented in Table 1, Fig. 3 reveals that:
- For low $\dot{p}$ and $\ddot{p}$ (category 1), the 95% index gives almost similar results for all p.d.f.'s, because the gravitational torques predominate.
- $\dot{p}$ has a much stronger influence than $\ddot{p}$.
- Kinematic effects prevail over the dynamic ones; therefore, the best results come from the "kinematic-dependent" p.d.f.'s (17)-(18).

In conclusion, the statistical analysis shows that the kinematics and dynamics have different effects upon the robot system. However, mechanical joint-actuated manipulators are much more sensitive to $\dot{p}$ than to $\ddot{p}$. These facts indicate that we are dealing with "position and acceleration machines" rather than "velocity machines". Although obvious, this aspect has been somewhat overlooked. Moreover, it points out that standard robot actuators are not well adapted to robotic applications. Alternative solutions, such as muscle like actuators Kuribayashi (12), Tatara (13) and Caldwell and Taylor (14) will allow more efficient robot structures Galhano et al (15).

CONCLUSIONS

A new method for modelling robot manipulators was presented. System descriptions are usually based on a set of differential equations which, though leading to very precise results and strategies, are very complex and hard to tackle. This motivates the need of models based on alternative concepts having distinct characteristics, towards which the proposed statistical scheme is a valid contribution. The new method provides a framework giving clear guidelines on the robot structure optimization. Therefore, the manipulator design procedure, both kinematic and dynamic, leads to simple and intuitive conclusions. Previously proposed models gave similar results for the simple kinematic case; nevertheless, they are difficult to apply in the (more complex) dynamic stage. With the statistical method both situations appear as natural and clear extensions of a common and systematic methodology. Moreover, the inherent use of histograms allows not only fast calculation procedures but, also, the use of experimental data, therefore avoiding complex dynamic modelling exercises. Furthermore, it should be highlighted the results pointing out some characteristics of the trajectory planning block, such as optimal rest and active regions, and ideal-actuator properties. This observation is of utmost importance as it gives a net basis to new mechanical robot manipulator structures, with performances close to the muscle-actuated biological systems.

REFERENCES

1. Tsai, Y.C. and Soni, A.H., 1981, "Accessible Region and Synthesis of Robot Arms", ASME J.Mechanical Design, 103, 803-811.

2. Asada, H., 1983, "A Geometrical Representation of Manipulator Dynamics and its Application to Arm Design", ASME J. Dynamic Syst., Meas., Contr., 105, 131-142.

3. Yoshikawa, T., 1985, "Manipulability of Robotic Mechanisms", The Int. J. Robotics Research, 4, 3-9.

4. Yoshikawa, T., 1986, "Analysis and Design of Articulated Robot Arms from the View-point of Dynamic Manipulability", Robotics Research, The Third Int. Symp., MIT Press, 273-279.

5. Youcef-Toumi, K. and Asada, H., 1987, "The Design of Open-Loop Manipulator Arms with Decoupled and Configuration-Invariant Inertia Tensors", ASME J. Dynamic Syst., Meas., Contr., 109, 268-275.

6. Hollerbach, J.M., 1984, "Dynamic Scaling of Manipulator Trajectories", ASME J. Dyn. Syst., Meas., Contr., 106, 102-106.

7. Sahar, G. and Hollerbach, J.M., 1984, "Planning of Minimum-Time Trajectories for Robot Arms", The Int. J. Robotics Research, 5, 90-100.

8. Machado, J.A.T. and de Carvalho, J.L.M., 1988, "A Statistical Approach to the Analysis and Design of Robot Manipulators", IEEE Int. Workshop on Intelligent Robots and Systems, Tokyo, Japan, 471-476.

9. Galhano, A.M.S.F., Machado, J.A.T. and de Carvalho, J.L.M., 1990, "On the Analysis and Design of Robot Manipulators: A Statistical Approach", 11th IFAC World Congress, Tallinn, USSR.

10. Galhano, A.M.S.F., Machado, J.A.T. and de Carvalho, J.L.M., 1990, "Statistical Modelling of Robot Manipulators", IEEE Int. Workshop on Intelligent Motion Control, Istanbul, Turkey, 811-816.

11. Galhano, A.M.S.F., Machado, J.A.T. and de Carvalho, J.L.M., 1990, "The Statistical Study of Robot Manipulators", Proc. 5th IEEE Int. Symp. on Intelligent Control, Philadelphia, USA, 424-429.

12. Katsutoshi Kuribayashi, 1986, "A New Actuator of a Joint Mechanism Using TiNi Alloy Wire", The Int. J. Robotics Research, 4, 47-58.

13. Tatara, Y., 1987, "Mechanochemical Actuators", Advanced Robotics, 2, 69-85.

14. Caldwell, D.G. and Taylor, P.M., 1988, "Artificial Muscles as Robotic Actuators", IFAC Symp. on Robot Control SYROCO'88, Karlsruhe, FRG.

15. Galhano, A.M.S.F., Machado, J.A.T. and de Carvalho, J.L.M., 1990, "The Statistical Study of Biomechanical Arms", Proc. 5th IEEE Int. Symp. on Intelligent Control, Philadelphia, USA, 824-829.

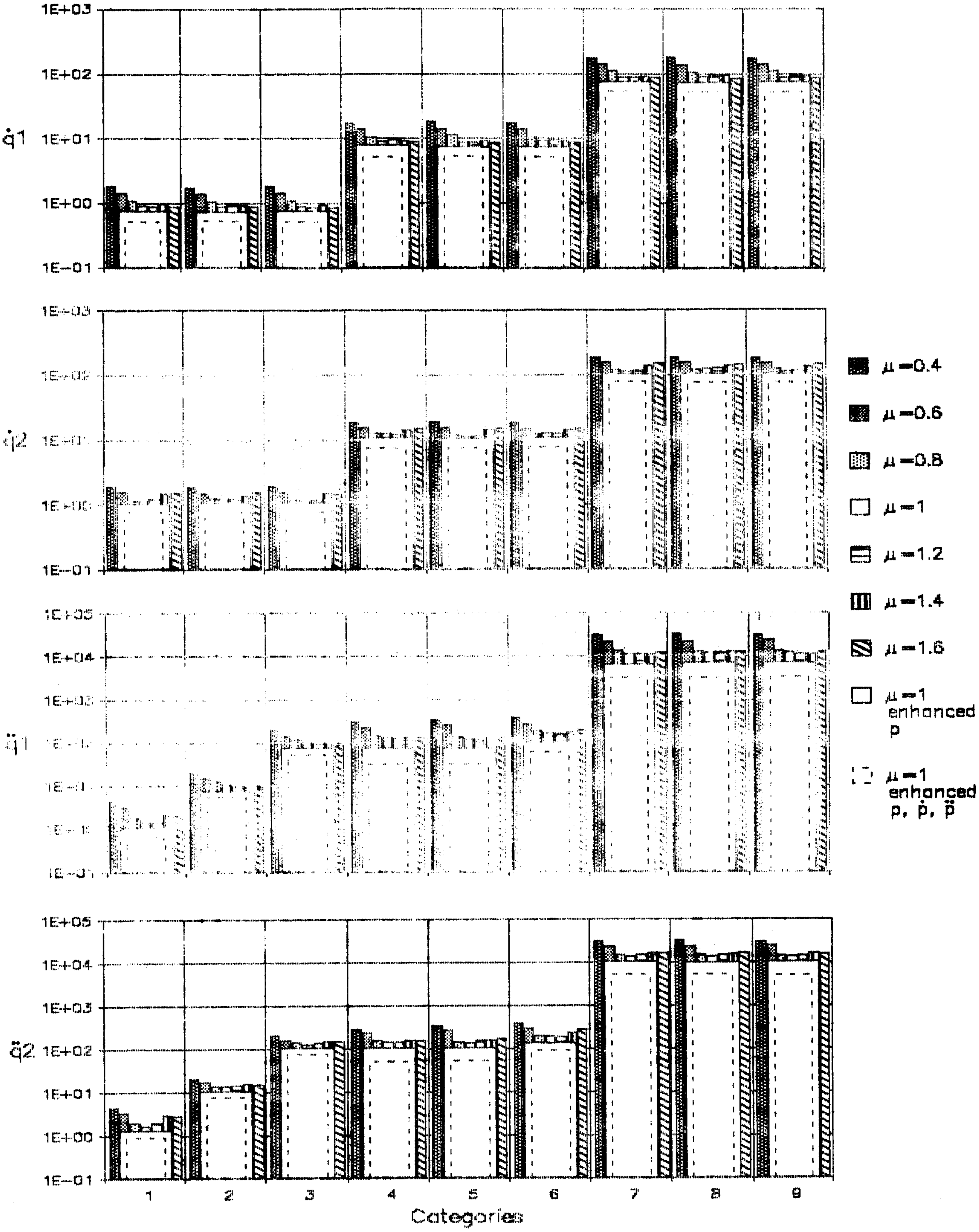

Figure 1 Kinematic performances of the 2R manipulator. The narrow back columns correspond to seven geometric configurations excited with p.d.f.'s (4, 5, 6). The wider front columns correspond to the optimum geometric configuration $\mu=1$ excited with the "enhanced" p.d.f. (12) together with p.d.f.'s (5, 6) for the solid borders and the "enhanced" p.d.f.'s (12, 13, 14) for the dotted borders.

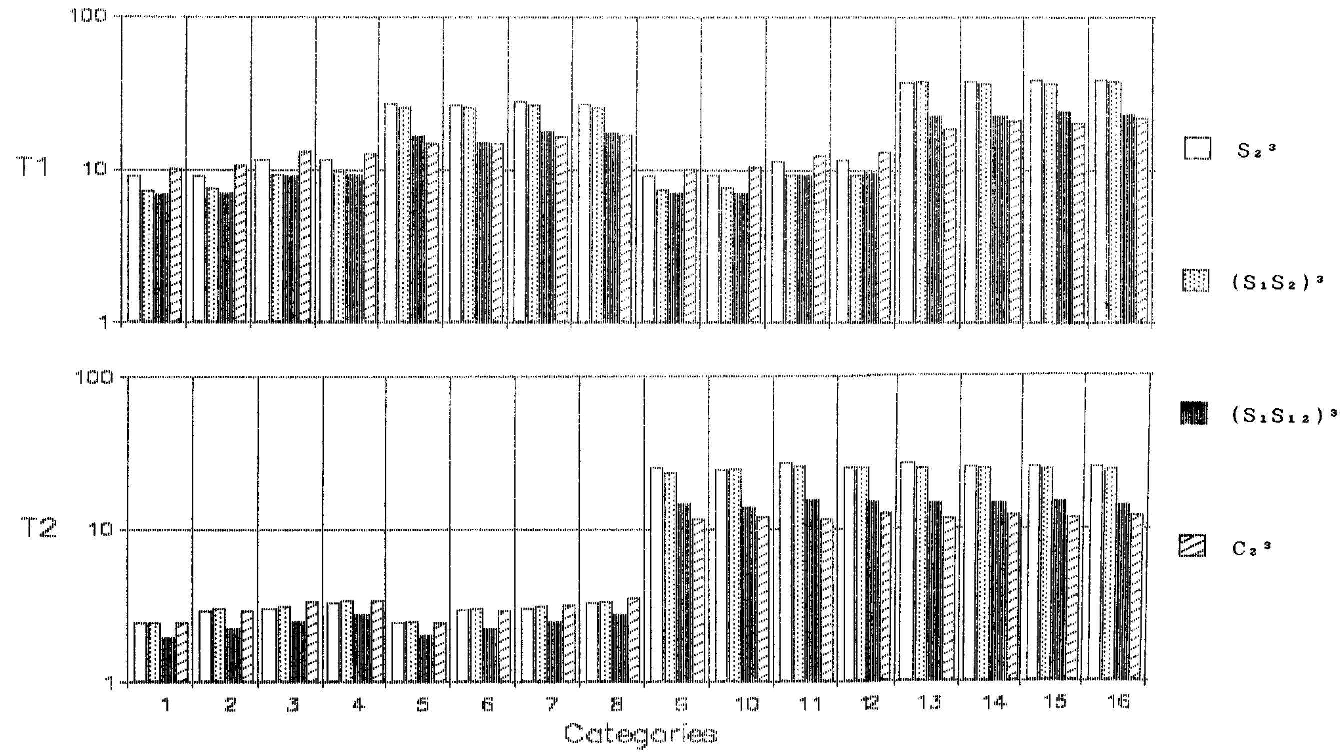

Figure 2 Dynamic performances of the 2R manipulator (μ=1) excited with the p.d.f.'s:
1st column: $f_\varphi(q_1,q_2)=\text{constant}*S_2^3$ 3rd column: $f_\varphi(q_1,q_2)=\text{constant}*(S_1S_{12})^3$
2nd column: $f_\varphi(q_1,q_2)=\text{constant}*(S_1S_2)^3$ 4th column: $f_\varphi(q_1,q_2)=\text{constant}*C_2^3$

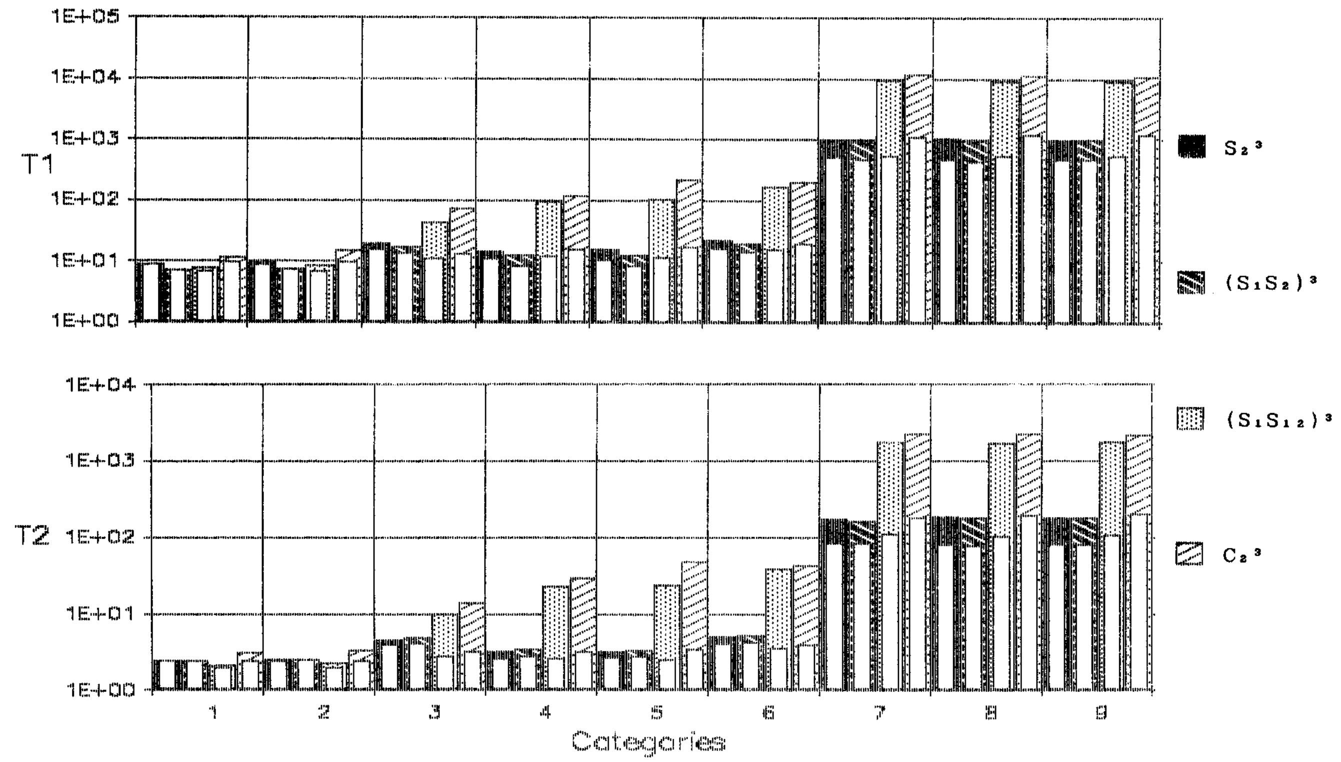

Figure 3 Kinematic/dynamic performances of the 2R manipulator (μ=1) excited with the p.d.f.'s:
1st column: $f_\varphi(q_1,q_2)=\text{constant}*S_2^3$ 3rd column: $f_\varphi(q_1,q_2)=\text{constant}*(S_1S_{12})^3$
2nd column: $f_\varphi(q_1,q_2)=\text{constant}*(S_1S_2)^3$ 4th column: $f_\varphi(q_1,q_2)=\text{constant}*C_2^3$
The back columns correspond to p.d.f.'s (5, 6) and the front white columns correspond to the "enhanced" p.d.f.'s (13, 14).