Double power laws, fractals and self-similarity

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A B S T R A C T

Power law (PL) distributions have been largely reported in the modeling of distinct real phenomena and have been associated with fractal structures and self-similar systems. In this paper, we analyze real data that follows a PL and a double PL behavior and verify the relation between the PL coefficient and the capacity dimension of known fractals. It is to be proved a method that translates PLs coefficients into capacity dimension of fractals of any real data.

Keywords:
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PL coefficient
Fractals
Correlation dimension

1. Introduction

Power law (PL) distributions, also known as heavy tail distributions, firstly appeared in the literature in the 19th century. In 1896 [1], Vilfredo Pareto applied a PL to model the distribution of individuals' incomes (this PL was later called Pareto law). He found that the relative number of individuals with an annual income larger than a certain value x was proportional to a power of x. From then on, studies of applications of PLs to real world phenomena have largely increased.

The most well known examples of PL distributions are the Pareto [1] and the Zipf [2,3] laws. The later is also known as rank-size rule. Let X be a non-negative discrete random variable following a PL distribution. Then, its complementary cumulative distribution function is of the form $F(x) = P(X \geqslant x) = \frac{C}{\alpha-1}x^{-(\alpha-1)}$, where $\alpha > 0,\ C > 0$. In the text, we will consider $\tilde{\alpha} = \alpha - 1$ and $\widetilde{C} = \frac{C}{\tilde{\alpha}}$. The probability function of a discrete random variable following Pareto distribution is given by:

$$P(X = x) = Cx^{-\alpha} \qquad (1)$$

Zipf law is a special case of the Pareto law with exponent $\tilde{\alpha} = 1$.

Application of PL behavior in natural or human-made phenomena usually comes with a log–log plot, where the axes represent the size of an event and its frequency. The log–log plot is asymptotically a straight line with negative slope. We can consider, for example, a country, such as United States (US) or Portugal (PT), and order the cities by population. In the case of US, New York appears first, Los Angeles, second, and so on. Analogously for PT, Lisbon is the first, Oporto the second. Then, we can plot the logarithm of the rank on the y-axis and the logarithm of the city size on the x-axis. New York and Lisbon both have log rank ln 1, and Los Angeles and Oporto have log rank ln 2. The graphs are straight lines with negative slopes. The same thing happens for popularity. We all know that popularity is an extreme imbalanced phenomenon. Only a few people have access to the glare of the spotlight and fewer still manage that his/her name be recorded in history. Most of us go through life being known just to the people in our social circle. How is popularity related to city sizes? They have in common the power law behavior, that is, exhibiting, in a log–log scale plot, a straight line. In fact, power laws seem to reign in the study of phenomena where popularity of some kind is present. Another such example is popularity of web sites. It is found that the number of web pages that have k in-links follows a PL distribution [4]. The usefulness of PLs can be at the level of controlling the outcome of some phenomena. For example, in computer networks. Some typical behavior in these networks is individual agents acting in their own best interest, giving rise to a global power law. This can be changed by giving agents incentives to modify their conduct. Of course this type of strategy would not apply to earthquakes. Nevertheless, it could be used in an extremly important event, such as stock markets. Gopikrishnan et al. [5] described a PL behavior of stock market returns. Thus, comprehending PLs may be key to the understanding of stock market crashes [6], and many other important real life events. More examples are in wealth distribution and expenditure [7,8], city size distribution [9,3,10], number of articles' citations and scientific production [11,12], number of hits in webpages [4], number of victims in wars, terrorist attacks, and earthquakes [13–18], words' frequency [19,3] and occurence of personal names [20]. Interesting reviews on PL behavior and applications can be found in [21–23].

Most of the PLs application seen in the literature use a single PL to model the studied events. Nevertheless other PLs, such as double PLs also appear and are said to be a better fit in some cases [24–27].

Detecting or proving the existence of a PL behavior in natural or human-made systems can be a very difficult task. The modeling of PLs has been primarily theoretical. A different approach to find a more complete model should consider contributions from statistics, control theory, and economics [28].

Self-similar systems are characterized by being scale-free. This translates, in day-to-day English, in looking exactly the same, despite a closer or a more distant look. Fractals are ubiquitous in nature, appearing everywhere, from plant structures, body parts, such lungs, coastlines, mountain ranges, condensed-matter systems including polymers, composite materials, porous media, and other natural phenomena [29–32]. Self-similarity, self-invariance and fractal dimension are properties of fractals.

Mandelbrot [33], in 1982, wrote that.

Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line.

Mandelbrot turned mathematicians, physicists, biologists, and other scientists', attentions to fractal patterns. The term "fractal dimension" [34] is frequently defined as the exponent D of the expression (2) given by:

$$n(\epsilon) = k\epsilon^{-D} \qquad (2)$$

where n is the minimum number of open sets of diameter ϵ needed to cover the set and k is a constant that depends on the fractal size. This is also known as the capacity dimension of the fractal. There are other dimensions used to characterize fractals, being the Hausdorff's and Kolmogorov's dimensions, the ones that are more accurate, but also harder to use. Computing the fractal dimension of the length of a country's coastline is an extremely difficult task, that depends on the length of the ruler used in the measurements. The shorter the ruler the bigger the length, since a shorter ruler measures more accurately the sinuosity of bays and inlets. Doing a log–log plot of the length of the ruler s versus the measured length L of the coastline, a straight line, with slope between 1 and 2, is obtained. Mandelbrot computed this slope to be $1 - D$, so the analytical expression of the straight line is $\log L = (1 - D)\log s + b$. Last expression can be rewritten as $L = \tilde{b}s^{1-D}$, that is a PL, and $b = \log \tilde{b} \in \mathbf{R}$. This feature suggests an analogy between any phenomenon characterized by a PL distribution and the fractal dimension, having for the variables in the x- and y-axes the same role as ϵ and $n(\epsilon)$, and for the PL slope the fractal dimension D. One can say that $\tilde{\alpha}$ is analogous to D and, therefore, we can interpret the phenomenon in the perspective of geometric fractals. For example, if for a real case the PL reveals $\tilde{\alpha} \simeq 0.63$, one may say that the phenomenon has, in some way, similarities with the ternary Cantor fractal, whose fractal dimension is $D = \log 2/\log 3$ and that each object, or entity, in the phenomenon, is related with two smaller objects having $1/3$ the size each. Nevertheless, during the flow of recursion situations, the recursion scheme may vary. In that case the log–log plot changes and the different slopes reflect the distinct recursive laws. We will return to this subject with an example in Section 4.

PLs are extremely important in the study of systems that are self-similar or fractal-like over many orders of magnitude. PL behavior allows extrapolation and prediction over a wide range of scales. The study of scaling reveals itself as a powerful tool of simplifying systems complexity and of understanding the basic principles ruling those systems. Moreover, experimental data from self-similar systems cannot be described by any other statistical distributions, as Normal or exponential, since order in complex systems relies heavily on correlations between different levels of scale. For the sake of completeness, we remark that recently, work by Sornette [35] has driven attention to a new type of extreme events, labeled *dragon-kings*, that could not be predicted by the extrapolation of power law distributions. The detection of these *wild* events or outliers depends on the phenomena that is under observation, there is not a unique methodology to find them. Nevertheless, this is not the focus here in this work.

2. Fatal events

Understanding victimology patterns arising in fatal events, such as wars, terrorists attacks and tornadoes, is extremely important due to political, cultural, historical and geographical issues. Many researchers have attempted explanations in the last decades [13,14,36,15–17,37,18]. Nevertheless, understanding these patterns is still far from complete.

War is probably the most fatal event, concerning the number of victims. Everyone remembers the millions of victims in big conflicts such as the two World Wars (WWI and WWII). Nevertheless, the nature of war has been slowly changing in the last 50 or 60 years. Guerilla forces, insurgent groups, and terrorists opposing incumbent governments, are nowadays events that cause fewer victims in each episode/event, (but sum up to millions too). These events seem to be random attacks to vulnerable targets of opportunity, and give little insight in predicting future events. Nevertheless, one can observe a common underneath behavior considering the number of casualties and frequency of these natural and human-made disasters. Large casualties are associated with low frequency phenomena, WWI and WWII are two examples, and are lesser frequent than other wars, not so harmful in terms of preserving human lives [14,36]. Analogously, the frequency of occurrence of terrific earthquakes, that cause a large number of victims, is much lower than that of smaller earthquakes with few casualties [13].

Other fatal events are domestic and international cases of violence, studied by Richardson [14]. He divided those into five logarithmic categories, from 3 up to 7, that corresponded to casualties measured in powers of 10. Later, in 1960 [36], the same author showed that as events increased by powers of 10 in severity, their frequency decreased by a factor close to three.

Johnson et al. [37] studied war and global terrorism patterns, and developed a theory that tried to explain their similar dynamical evolution, invariant to underlying ideologies, motivations and the terrain in which they operated. They considered each insurgent force as a generic, self-organizing system, which evolved dynamically through the continual coalescence and fragmentation of its constituent groups. Researchers have used wars in Iraq and Afghanistan, and long-term guerrilla war in Colombia, as examples. On global terrorism, attacks to London, Madrid, and New York (September 11) were main choices. Results obtained showed a PL behavior for Iraq, Colombia and Afghanistan, with coefficient value (close to) $\tilde{\alpha} = 2.5$. This value of the coefficient equalized the coefficient value characterizing non-G7 terrorism. This result suggested that PL patterns would emerge within any modern asymmetric war, fought by loosely-organized insurgent groups. In 2007, Clauset et al. [38] plotted a log–log chart for the frequency versus the severity of terrorist attacks, since 1968, and found a straight line, denoting PL behavior. Economic development, the type of weapon used in the attack, or short time scales, did not affect the results. The same authors presented a toy model of the frequency of severe terrorist attacks, that reproduced the heavy-tail behavior observed in the log–log plot of real data.

Bohorquez et al. [18] were interested in the quantitative relation between human insurgency, global terrorism and ecology. Universal patterns occurring across wars (distribution of the number of victims and the timing of within-conflict events) were explained by a unified model of human insurgency. The insurgent populations were considered self-organized groups that dynamically evolved through decision-making processes. Authors computed the Pareto coefficient to a value close to $\tilde{\alpha} = 2.5$, agreeing with previous work on Iraq and Colombia wars, and with the other insurgent wars studied in this paper (Peru, Sierra Leone, Afghanistan, amongst others). Data of the Spanish and American Civil Wars was best fitted by PL distribution with coefficient value $\tilde{\alpha} = 1.7$. Moreover, for the later, a lognormal distribution might be applied to fit the data. As a consequence of these findings, researchers claimed that insurgent wars were qualitatively distinct from traditional wars. A coefficient value of $\tilde{\alpha} = 2.5$ was in concordance with the coefficient value of $\tilde{\alpha} = 2.48 \pm 0.07$ obtained by Clauset et al. [38] on global terrorism.

3. Other events

Self-similarity or self-criticality has also been observed in other phenomena, such as forest or city fires, language, amongst others.

The study of city and forest-fire distributions may allow to take measures beforehand in view of possible hazards, thus saving natural resources and animal and human lives. Considering as a measure of the size of a fire, either a city or a forest fire, its burned area, then the frequency-size distribution of fires follows a PL behavior. Drossel et al. [39] proposed a stochastic cellular automaton to model this PL behavior. The authors have improved a model introduced by Bak et al. [40], in which the forest was denoted by a *d*-dimensional lattice, with size *L*, and *Ld* sites. Tree growth and ignition and spread of fires were simulated in a random way in the model. The model steady state was defined by frequency-size distribution of fires following a PL. In 1998, Malamud et al. [41] computed the log–log plot of the frequency of occurrence vs the burned area in USA and Australian fires. They observed a straight line, denoting a PL behavior, over distinct orders of magnitude, and invariant with time. Analogously behavior was found by Ricotta et al. [42], for 9164 wildfire records of a northern region of Italy (Liguria), from 1986 to 1993. The PL coefficient was computed to be $\tilde{\alpha} = 0.723$. Song et al. [43], improved the model of Drossel et al. [39], by including the effects of rain and human fire fighting efforts, such as firebreaks. Application of the model to data from Chinese fires, supported the claim that small effects could preserve trees and prevent large, hazardous forest fires. In 2006, Weiguo et al. [44] explored three distinct PL behaviors seen in frequency-size, frequency-interval, and frequency-density-of-population distributions of forest fires. They considered the environmental differences (weather, tree species, etc.) between different countries (China, USA, Japan) and human-related variables the main factors contributing to the distinct exponent values.

City fires are more intricate systems to study, due to a large variety of structures in the cities, from industrial facilities, to buildings and parks, and to the greater human interactions. In 2003, Song et al. [45] studied fire distribution in Chinese and Swiss cities. The authors computed the frequency loss plot and the rank-size plot and verified validity of a PL to these plots. The frequency loss was the frequency of fires with loss *L*, that is, fire loss *L* converted into Chinese YUAN. The rank was computed by sorting city fires from large to small, and considering the largest with rank 1. The PL distribution was invariant for scale and time, meaning that fire distribution is common for different places and times.

Brown et al. [31] studied one restricted class of emergent ecological phenomena characterized by scaling relationships, that were self-similar over a wide range of spatial or temporal scales. Some features appeared to be universal, occurring in virtually all taxa of organisms and types of environments. Mathematically, they translated that behavior by PLs. Authors contributed to the understanding of the mechanisms that generated those PLs, and for explaining the diversity of species and complexity of ecosystems in terms of physical and biological factors.

Ma et al. [46] proposed a self-similar packing of atomic clusters for medium-range order in metallic glasses. They showed that this medium-range order had fractal properties exhibiting fractal dimension of 2.31. At medium-range length-scale, the patterns were described by a PL correlation function.

4. Analysis of real world cases

PLs are present in many natural and man-made systems and, for certain cases, a single PL distribution holds over the entire data range. As an example, Fig. 1 represents the normalized rank/frequency log–log plot of the largest private

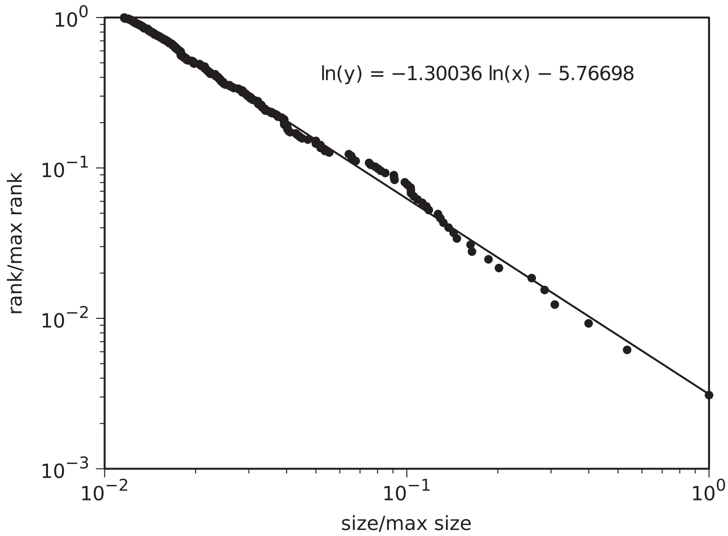

Fig. 1. Rank/frequency log–log plot of the size of the largest American companies in 1997 (max size = 56; max rank = 324).

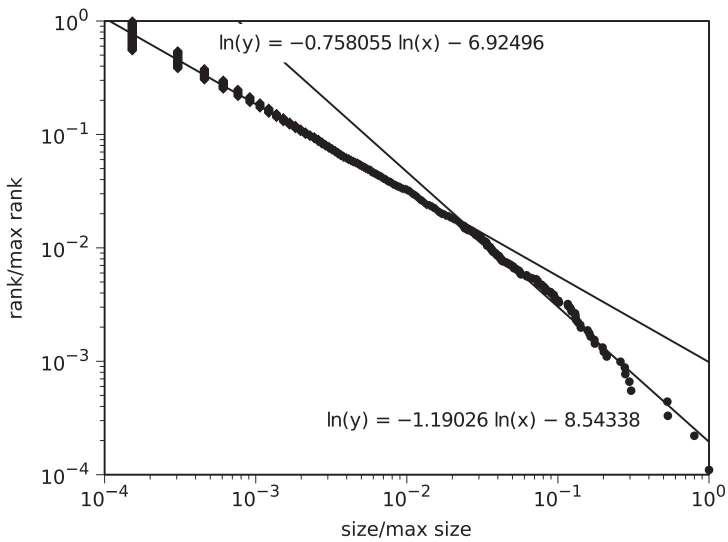

Fig. 2. Rank/frequency log–log plot of a variable associated to man-made systems following a dual PL: number of words in the Homer's epic poem Odyssey, English version (max size = 6561; max rank = 9077).

American companies, with respect to their annual revenue, in the year 1997, according to Forbes (http://www.forbes.com/). To construct the plot, we first sorted the companies in descending order, according to their size, and numbered them, consecutively, starting from one [47]. Then a normalization of the values was carried out, meaning that, the companies' sizes (x-axis) were divided by the corresponding highest value, and the ranks (y-axis) were divided by the rank of the smallest company. Finally, a PL was adjusted to the data using a least squares algorithm. All the log–log plots presented in this paper were made using a similar procedure.

As can be seen in Fig. 1, a PL behavior distribution with parameters $(\widetilde{C}, \tilde{\alpha}) = (0.0031, 1.3004)$ holds over the entire range of the companies' annual revenue.

In other real applications, different PLs, characterized by distinct PL parameters, might also be observed. In the sequel, several cases of such behavior are illustrated. In Fig. 2 the cumulative distribution of the number of words in the Homer's epic poem Odyssey (in an English version) is represented. The words were counted using the free software TextSTAT (http://neon.niederlandistik.fu-berlin.de/en/textstat/). The PL parameters are $(\widetilde{C}_1, \tilde{\alpha}_1) = (0.0010, 0.7581), (\widetilde{C}_2, \tilde{\alpha}_2) = (0.0002, 1.1903)$.

We use Fig. 2 to interpret the phenomenon of the number of words in a text in the perspective of geometric fractals. Fig. 2 presents two PL behaviors, that switch at $x = 0.024$, namely with $\tilde{\alpha}_1 \simeq 0.76$ and $\tilde{\alpha}_2 \simeq 1.19$.

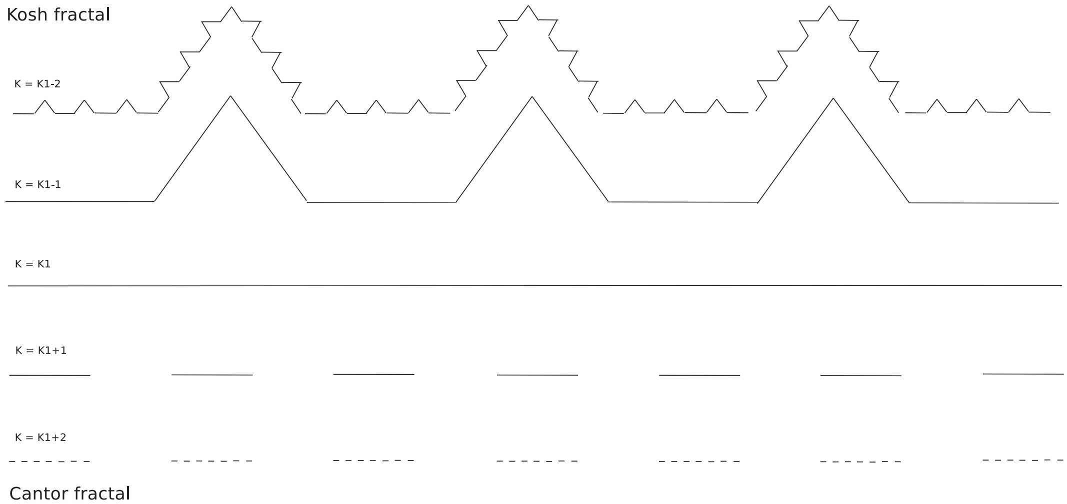

Fig. 3. Joining of two generalized fractals, the Cantor set and the Kosh curve, at iteration $k = k_1$.

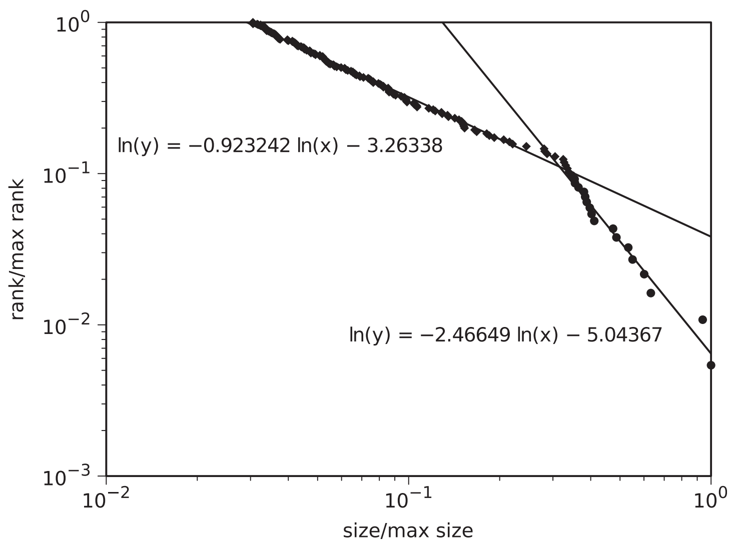

Fig. 4. Rank/frequency log–log plot of a system presenting dual PL behavior: size of forest fires in Portugal, year 2001 (max size = 3272.5; max rank = 185).

Let us now consider joining two fractals at iteration $k = k_1$, namely the generalized Cantor set and the generalized Kosh Curve, as represented in Fig. 3.

In the Cantor recursion is considered a division in $N_1 = 13$ pieces, while in the Kosh fractal is considered a division in $N_2 = 7$ objects. Therefore, we get $D_1 = \frac{\log N2}{\log N_1} = 0.75$ and $D_2 = \frac{\log 10}{\log 7} = 1.18$, which are values close to those observed in the real case. We should highlight that we have chosen N_1 and N_2 to these particular values only to match the case under analysis. A second observation is that the 'switching' between distinct fractal structures is easier to design for $k = k_1$ and a straight line, but other cases seem possible to be constructed. Nevertheless, it is not required to have a switching between dimensions smaller/higher than one. For example, adopting two different Cantor fractals leads to a switching between two dimensions smaller than 1.0. Therefore, preserving the rest of the analogy, we can say that large/small words have a recursion similar to generalized Kosh/Cantor fractals, being the switching at k_1 in correspondence to value $x = 0.024$.

Fig. 4 shows the cumulative distribution function of the size of forest fires in Portugal, over the year 2001. The adopted measure for size is the total burned area. Only fires greater than 100 ha in total burned area are considered. The data is available on the Portuguese National Forest Authority (AFN) website (http://www.afn.min-agricultura.pt/). For this case,

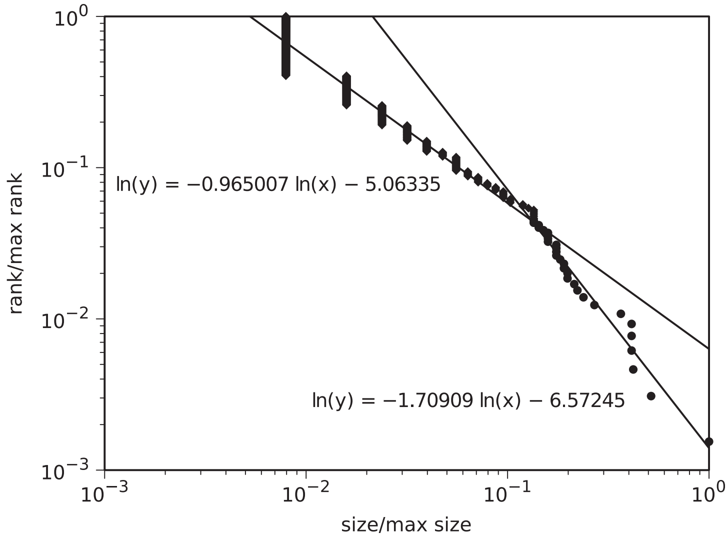

Fig. 5. Rank/frequency log–log plot of a system presenting dual PL behavior: severity of worldwide terrorist attacks in 2003 (max size = 126; max rank = 648).

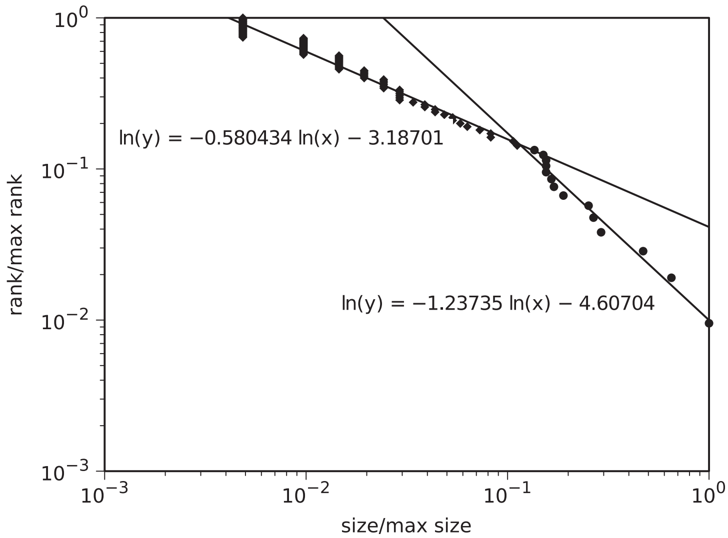

Fig. 6. Rank/frequency log–log plot of a system presenting dual PL behavior: severity of tornadoes in the USA in 2003 (max size = 206; max rank = 105).

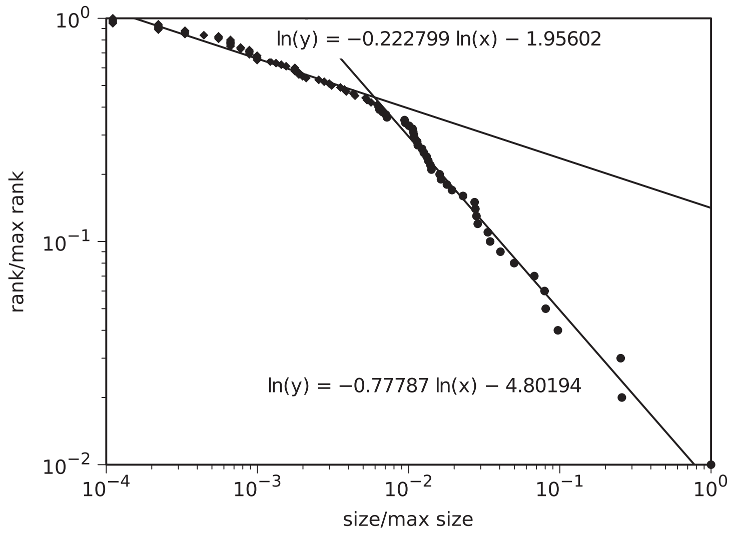

Fig. 7. Rank/frequency log–log plot of a variable associated to man-made systems following a dual PL: number of hits received by the websites in category 'hobby', according to uCoz Web Services (max size = 9045; max rank = 100).

two distinct PLs with parameters $(\widetilde{C}_1, \tilde{\alpha}_1) = (0.0383, 0.9232)$ and $(\widetilde{C}_2, \tilde{\alpha}_2) = (0.0065, 2.4665)$ fit the data. The change in the behavior occurs at the relative value of 0.35, approximately.

Two additional related similar cases are depicted in Figs. 5 and 6. The former refers to the rank/frequency log–log plot of the severity of worldwide terrorist attacks, in 2003. The number of people killed and injured in an event is used to quantify its severity. The data is available in the RAND Database of Worldwide Terrorism Incidents (RDWIT) (http://smapp.rand.org/rwtid/). The latter (Fig. 6) represents the severity of tornadoes in the USA, during 2003. The total number of human victims (killed and injured) directly related to a given occurrence is used to quantify its severity. The used data is available at the U.S National Oceanic and Atmospheric Administration (http://www.noaa.gov/), National Weather Service, Storm Prediction Center website (http://www.spc.noaa.gov/). Both cases reveal a dual PL behavior with parameters $(\widetilde{C}_1, \tilde{\alpha}_1) = (0.0063, 0.9650)$, $(\widetilde{C}_2, \tilde{\alpha}_2) = (0.0014, 1.7091)$ and $(\widetilde{C}_1, \tilde{\alpha}_1) = (0.0413, 0.5804)$, $(\widetilde{C}_2, \tilde{\alpha}_2) = (0.0100, 1.2374)$. The change in the behavior occurs at $x = 0.13$ and $x = 0.11$, respectively.

The cumulative distribution of the number of hits received by the websites in category 'hobby', according to uCoz Web Services (http://top.ucoz.com/), is depicted in Fig. 7. The data was collected on the 20th January, 2012. Only in English written websites were considered. It is clear that a dual PL behavior holds. The PL parameters are $(\widetilde{C}_1, \tilde{\alpha}_1) = (0.1414, 0.2228)$, $(\widetilde{C}_2, \tilde{\alpha}_2) = (0.0082, 0.7779)$, respectively. The two PLs switch at $x = 0.006$.

In conclusion, this study presented several examples of real world phenomena where a double PL behavior can be observed. We have discussed, with an analogy, how geometric fractals may be used to interpret phenomena defined by PLs.

5. Conclusion

In this paper, we focused on the self-similar, fractal-like and PL behaviors of sets of real data. We presented examples of data that was fitted well by a single straight line and examples that were best described by two distinct PL distributions. We have related the fractal dimension with the exponent of some PL distributions, and have given an analogy for this case. More work is needed in order to find a general method that translates between PL coefficients and fractal dimensions in distinct real phenomena.

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