Complex order van der Poloscillator

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Abstract In this paper a complex-order van der Pol oscillator is considered. The complex derivative $D^{\alpha \pm j\beta}$, with α, $\beta \in \mathbf{R}^+$ is a generalization of the con- cept of integer derivative, where $\alpha = 1$, $\beta = 0$. By applying the concept of complex derivative, we ob- tain a high-dimensional parameter space. Amplitude and period values of the periodic solutions of the two versions of the complex-order van der Pol oscillator are studied for variation of these parameters. Fourier transforms of the periodic solutions of the two oscilla- tors are also analyzed.

Keywords Van der Pol oscillator · Complexorder derivative·Dynamicalbehavior

1 Introduction

The van der Pol (VDP) oscillator is an ordinary differ- ential equation that has arisen as a model of electrical circuits containing vacuum tubes [45] (re-edited [8]). It produces self-sustaining oscillations in which en- ergy is fed into small oscillations while it is removed from large oscillations. This is the first relaxation os- cillator appearing in the literature [44, 46]. It is given by the following second order differentialequation:

$$\ddot{x} + \mu\left(x^2 - 1\right)\dot{x} + x = 0 \qquad (1)$$

Parameter μ controls the way the voltage flows through the system. For $\mu = 0$ this is just a simple linear oscillator. For large values of μ, that is for $\mu \gg 1$, the system exhibits a relaxation oscillation. This meansthattheoscillatorhastwodistinctphases: a slow recovery phase and a fast release phase (vac- uum tubes quickly release or relax their voltage after slowly building uptension).

This equation has been used in the design of var- ious systems, from biology, with the modeling of the heartbeat [17, 22, 30], the generation of action poten- tials [20, 21], up to acoustic systems [2] and electrical circuits [3, 12]. The VDP oscillator has also been used in the context of chaos theory [9, 12, 14–16].

Fractional calculus (FC) has been an important re- search issue in the last few decades. FC is a generaliza- tion of the ordinary integer

differentiation and integra- tion to an arbitrary, real or complex, order [28, 34, 43]. Applications of FC have been emerging in differ- ent and important areas of physics and engineering [5, 10, 26, 27, 31, 32, 35, 39, 41, 42]. Fractional order behavior has been found in areas such as fluid mechanics [29], mechanical systems [13], electrochem- istry [33], and biology [1, 11], namely in the modeling of the central pattern generators for animal locomotor rhythms [36, 37].

There are several definitions of fractional deriva- tives of order $\alpha \in \mathbf{R}$, three of the most important being the Riemann–Liouville, the Grünwald–Letnikov, and the Caputo ones, given by

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \quad n-1 < \alpha < n \tag{2}$$

$$D_t^\alpha f(t) = \lim_{h \to 0} \frac{1}{h^\alpha} \sum_{h=0}^{\left[\frac{t-a}{h}\right]} (-1)^k \binom{\alpha}{k} f(t-kh) \tag{3}$$

$$D_t^\alpha f(t) = \frac{1}{\Gamma(\alpha-n)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \quad n-1 < \alpha < n \tag{4}$$

where *r()* is the Euler gamma function, [] means the integer part of *x*, and *h* represents the step time incre- ment. It is also possible to generalize the results based on transforms, yielding

$$L\{D_t^\alpha f(t)\} = s^\alpha L\{f(t)\} - \sum_{k=0}^{n-1} s^k D_t^{\alpha-k-1} f(0^+) \tag{5}$$

where *s* and *L* represent the Laplace variable and op- erator, respectively.

The definitions demonstrate that fractional deriva- tives capture the history of the variable, or, in other words, ones that have memory, contrary to integer derivatives, which are local operators. The Grünwald–Letnikov formulation inspires the numerical calcula- tion of the fractional derivative based on the approxi- mation of the time increment *h* through the sampling period *T* and the series truncation at the *r* th term. This method is often denoted as Power Series Expansion (PSE) yielding the equation in the *z* domain:

$$Z\{D^\alpha x(t)\} \approx \left[\frac{1}{T^\alpha} \sum_{k=0}^{r} \frac{(-1)^k \Gamma(\alpha+1)}{k! \Gamma(\alpha-k+1)} z^{-k}\right] X(z) \tag{6}$$

where $X(z) = Z\{x(t)\}$ and *z* and *Z* represent the *z*-transform variable and operator, respectively. In fact, the expression (3) represents the Euler (or first back- ward difference) approximation in the $s \to z$ dis- cretization scheme, the Tustin approximation being another possibility. The most often adopted general- ization of the generalized derivative operator consists in $\alpha \in \mathbf{R}$. The case of having a fractional derivative of complex order $\alpha \pm j\beta \in \mathbf{C}$ leads to complex out- put valued results and imposes some restrictions be- fore a practical application. To overcome this problem, recently [6, 18, 19] the association of two complex- order derivatives was proposed. In fact, there are sev- eral arrangements that produce real valued results. For example, with the real part of two complex conjugate derivatives $D^{\alpha \pm j\beta}$ we get

$$\begin{aligned} &Z\left\{\frac{1}{2}\left[D^{\alpha-j\beta}x(t) + D^{\alpha+j\beta}x(t)\right]\right\} \\ &\quad \approx \frac{1}{T^\alpha}\left\{\sin\left[\beta \ln\left(\frac{1}{T}\right)\right]\left[\beta z^{-1} + \frac{1}{2}\beta(1-2\alpha)z^{-2} + \cdots\right]\right. \\ &\quad\quad + \cos\left[\beta \ln\left(\frac{1}{T}\right)\right]\left[-1 + \alpha z^{-1}\right. \\ &\quad\quad \left.\left. - \frac{1}{2}\beta(\alpha^2 - \alpha - \beta^2 + \cdots)\right]\right\} X(z) \end{aligned} \tag{7}$$

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Other combinations and the adoption of a Padé fraction, instead of the series for the approximation, are also possible. Nevertheless, in the sequel the case of expression (7) is explored.

We must remark that a scheme of discretized frac- tional derivative is adopted commonly accepted in en- gineering, but not yet fully investigated from a math- ematical view point. Let us recall the basic definitions of Riemann–Liouville (R-L) and Caputo (C) deriva- tives, in (2) and (4), based on integral expressions, and of Grünwald–Letnikov (G-L), in (3), based on a series expression suitable for discretization. The C derivative is a regularization of the R-L derivative and their G-L derivative is the series representation of the R-L deriv- ative. In (5) the Laplace transform of the R-L deriva- tives differs from that of the C derivative, as shown by Gorenflo and Mainardi [23, 24], subsequently pointed out in the well-known treatises on Fractional Calcu- lus by Podlubny [38] and Kilbas-Srivastava and Tru- jillo [25]. So, in practice the classical R-L derivative of real order is generalized to complex orders. In case of adopting the Caputo derivative (in view of standard initial conditions), there should be adopted a differ- ent Laplace transform (see Gorenflo et al. [23, 24]), and also a modified G-L series representation, because the C derivative is a regularization of the R-L deriv- ative in the time origin. For real order less than one the G-L representation of the C derivative has been adopted in the paper by Gorenflo et al. [24]. As a con- sequence, the *Z* representation in (6) in the complex domain should be modified accordingly for the Caputo derivative. While tackling these matters is not straight- forward, we must note that the simulations carried out in the sequel correspond to steady state responses and that the initial conditions have a minor impact in the resulting charts.

Having these ideas in mind, this paper is organized as follows. In Sect. 2, we introduce the two approx- imations of the complex-order van der Pol oscillator (CVDP) and we present results from numerical simu- lations. In Sect. 3 we outline the main conclusions of this study.

2 Complex-order van der Pol system

Fractional VDP systems have been studied by many authors [4, 7, 9, 14–16, 40]. Their work differs in the approaches considered to express the fractional deriv- ative. Chen et al. [9] considered a forced van der Pol equation with fractional damping of the form:

$$\ddot{x} + \mu\left(x^2 - 1\right)D^{\alpha}x(t) + x(t) = a\sin(\omega t) \tag{8}$$

where μ is an endogenous damping parameter, a de- notes the amplitude of a periodic forcing, and ω is the forcing frequency.

Barbosa et al. [7] considered the following modi- fied version of the van der Pol equation:

$$x^{(1+\lambda)} + \alpha\left(x^2 - 1\right)x^{(\lambda)} + x = 0 \tag{9}$$

with $0 < \lambda < 1$.

Tavazoei et al. [40] determined the parametric range for which the fractional VDP system studied by Barbosa et al. [7] can perform as an undamped os- cillator. They also showed that, contrary to the integer given by

$$\begin{aligned} \frac{d^{\alpha}x_1}{dt^{\alpha}} &= x_2 \\ \frac{d^{\beta}x_2}{dt^{\beta}} &= -x_1 - \mu\left(1 - x_1^2\right)\left(c - ax_1^2\right)x_2 + b\sin t \end{aligned} \tag{10}$$

where α and β are fractional numbers.

To the best of the knowledge of the authors, little attention has been given to CVDP oscillators. In this paper, we consider the following two complex-order state-space models of the VDP oscillator:

$$\begin{bmatrix} \frac{1}{2}\left(D^{\alpha+j\beta} + D^{\alpha-j\beta}\right)x_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -\mu\left(x_1^2 - 1\right) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{11}$$

$$\begin{bmatrix} \dot{x}_1 \\ \frac{1}{2}\left(D^{\alpha+j\beta} + D^{\alpha-j\beta}\right)x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -\mu\left(x_1^2 - 1\right) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{12}$$

where $D^{\alpha \pm j\beta}$, $\alpha, \beta \in \mathbf{R}^{+}$, is a generalization of the concept of the integer derivative that corresponds to $\alpha = 1$ and $\beta = 0$.

We adopt the PSE method for the approximation of the complex-order derivative in the discrete time numerical integration. Several experiments demonstrated that a slight adaption to the standard approach based on a simple truncation of the series is required. In fact, since our objective is to generate limit cycles, the trun- cation order VDP, trajectories in a fractional VDP oscillator do not converge to a unique cycle.

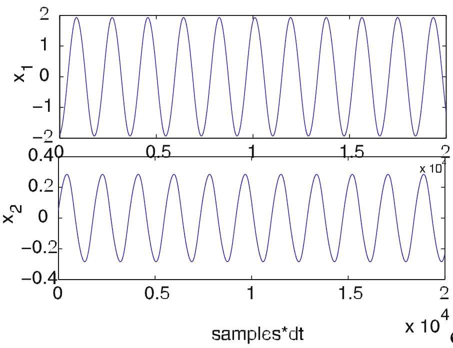
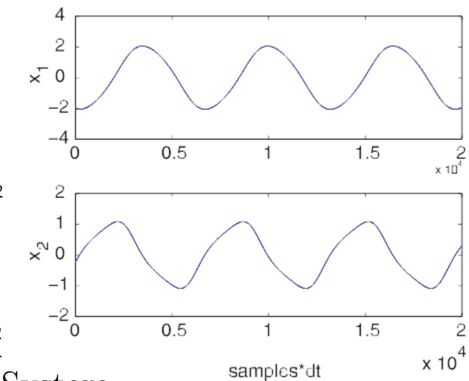
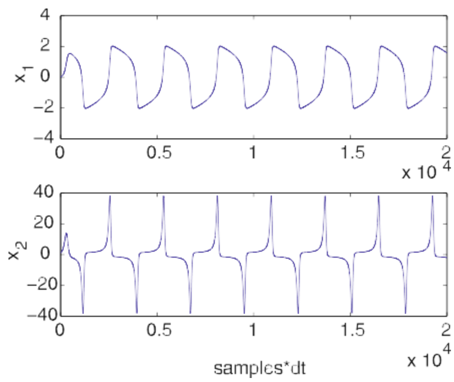
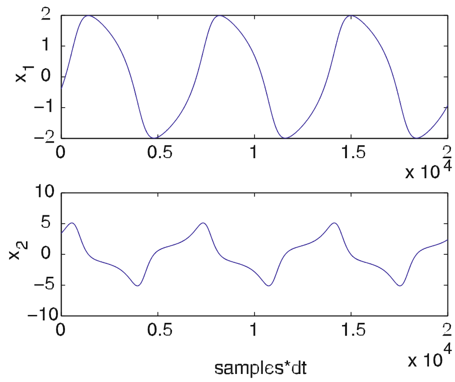
Ge et al. [14] studied the autonomous and non-autonomous fractional van der Pol oscillator. The non-
autonomous system in state-space oscillator model is

corresponds to a diminishing of the gain [42] and, consequently, leads to difficulties in the promo- tion of periodic orbits. Therefore, in order to overcome this limitation, we decided to include a gain adjust- ment factor corresponding to the sum of the missing truncated series coefficients.

The discretization of the CVDP oscillators (11) and (12) leads to, respectively:

$$\begin{aligned} x_1(k+1) &= \frac{1}{\psi(\beta, \Delta t)}\left(H\left(x_1(k)\right) + (\Delta t)^{\alpha} x_2(k)\right) \\ x_2(k+1) &= x_2(k) + \Delta t\left(-x_1(k)\right. \\ &\quad \left. - \mu\left(x_1^2(k) - 1\right)x_2(k)\right) \end{aligned} \tag{13}$$

Table 1 Periodic solutions of the CVDP systems (13)–(14) for $\beta = 0.8$, $\mu = 0.5$ and $\alpha \in \{0.4,\ 0.8\}$

CVDP	$\alpha = 0.4$	$\alpha = 0.8$
System (13)		
System (14)		

$$x_1(k+1) = x_1(k) + \Delta t\, x_2(k)$$

$$x_2(k+1) = \frac{1}{\psi(\beta, \Delta t)}\Big(H\big(x_2(k)\big) + (\Delta t)^{\alpha}\big(-x_1(k) - \mu\big(x_1^2(k) - 1\big)x_2(k)\big)\Big) \tag{14}$$

where $t = 0.0005$ is the time increment, $\psi(\beta,\ t) =$ $\cos[b \log(\frac{1}{t})]$ and function $H(x_i)$, $i = 1, 2$, results from the Taylor series expansion truncation.

In Table 1 we depict periodic solutions of sys- tems (13)–(14) for $\alpha = 0.4$, $\beta = 0.8$ and $\mu = 0.5$. One can observe the appearance of the relaxation oscilla- tion phenomena as α increases (first row of the table). Note that in the case of system (14) this phenomenon is already present for $\alpha = 0.4$ and is emphasized for $\alpha = 0.8$ (second row of the table).

In Figs. 1–2, we show the phase portraits of solu- tions $(x_1(t), x_2(t))$ of systems (13)–(14) for $\beta = 0.8$, $\mu \in \{0.5, 2\}$ and different values of α. As expected, the larger the value of μ, the more nonlinear the oscilla- tion becomes. We verify also that we can control the period of the oscillation by varying α.

We now simulate the ordinary differential sys- tems given by expressions (13)–(14) for $\beta = 0.8$, $\alpha \in \{0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0\}$, $\mu = \{0.5, 1.0, 1.5, 2\}$, and we measure the amplitude and the period of the solutions. Values of $\alpha \in [0.0,\ 0.2[$ were also considered in the simulations. For system (13) we found stable periodic solutions nevertheless for sys- tem (14) simulation results led to unstable solutions, so we have decided to omit these values.

Here are adopted the initial conditions $x_1(1) = 0.0$, $x_1(2) = 0.005$, $x_1(3) = 0.010$, $x_1(4) = 0.015$, $x_1(5) = 0.02$, $x_2(1) = 1.0$, $x_2(2) = 1.005$, $x_2(3) = 1.010$, $x_2(4) = 1.015$, $x_2(5) = 1.02$.

Each simulation is executed until a stable periodic solution is found. The amplitude and the period of the solutions versus α are depicted in Figs. 3–4. We find

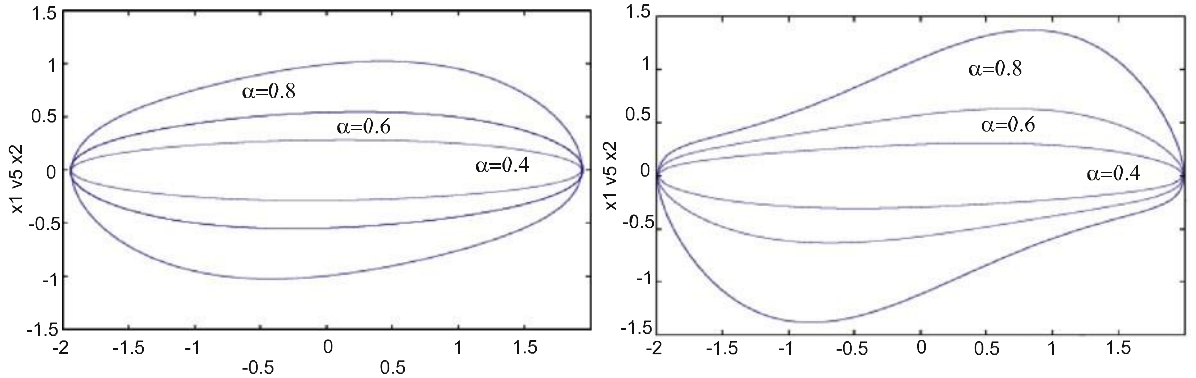


Fig. 1 Phase-space solutions $(x_1(t), x_2(t))$ of the CVDP system (13) for $\alpha \in \{0.4, 0.6, 0.8\}$, $\beta = 0.8$ and $\mu = 0.5$ (*left*) and $\mu = 2.0$ (*right*)

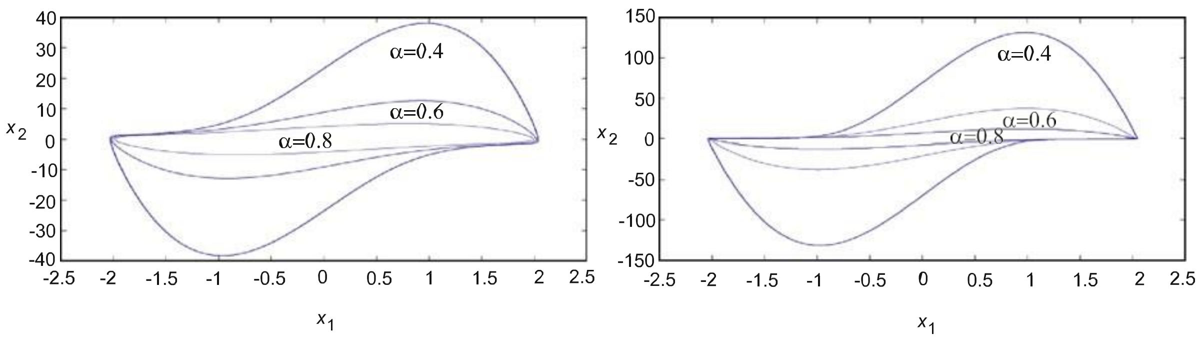


Fig. 2 Phase-space solutions $(x_1(t), x_2(t))$ of the CVDP system (14) for $\alpha \in \{0.4, 0.6, 0.8\}$, $\beta = 0.8$ and $\mu = 0.5$ (*left*) and $\mu = 2.0$ (*right*)

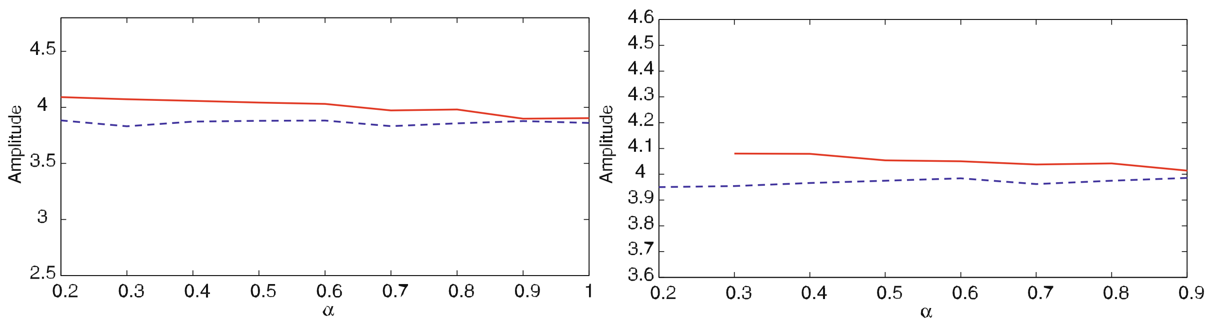


Fig. 3 Amplitude of the periodic solutions $x_1(t)$ produced by the CVDP oscillators (13) (*dashed*) and (14) (*line*) for $\beta = 0.8$, $\alpha \in \{0.2, 0.3, 0.4, 0.5, 06, 0.7, 0.8, 0.9, 1.0\}$ and $\mu = 0.5$ (*left*) and $\mu = 2.0$ (*right*)

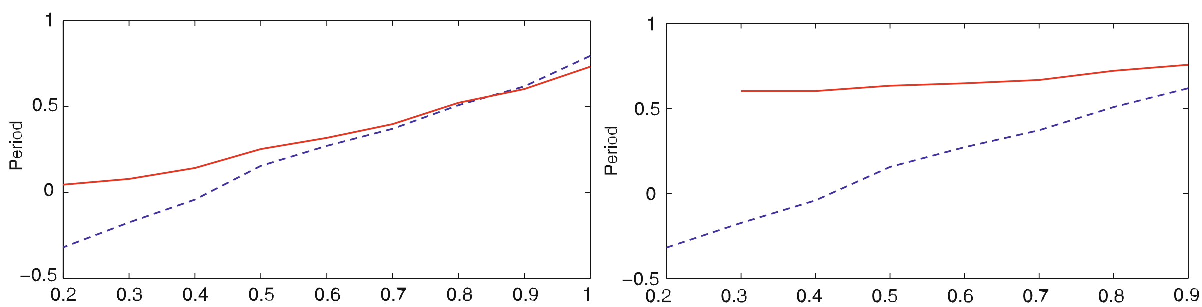


Fig. 4 Period of the solutions $x_1(t)$ produced by the CVDP oscillators (13) (*dashed*) and (14) (*line*) for β = 0.8, $\alpha \in \{0.2, 0.3, 0.4, 0.5, 06, 0.7, 0.8, 0.9, 1.0\}$ and $\mu = 0.5$ (*left*) and $\mu = 2.0$ (*right*)

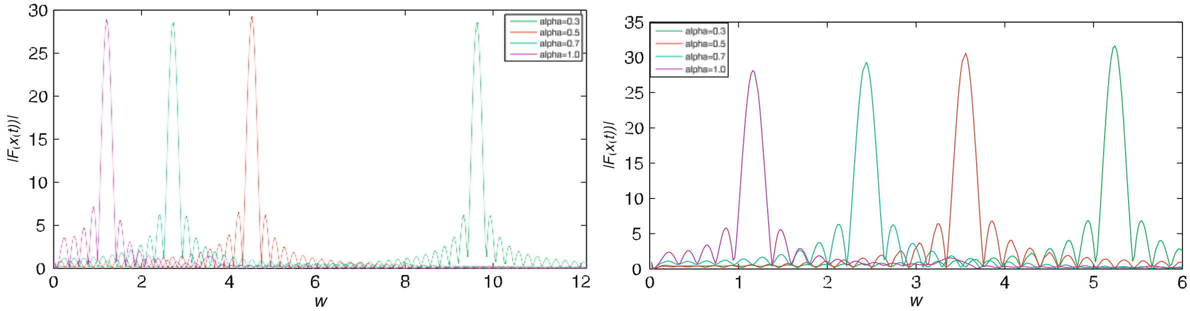


Fig. 5 Amplitude of the Fourier transform $|F\{x_1(t)\}|$ vs. ω for the CVDP oscillators (13) (*left*) and (14) (*right*) with β = 0.8, $\alpha \in \{0.3, 0.5, 0.7, 1.0\}$ and $\mu = 0.5$

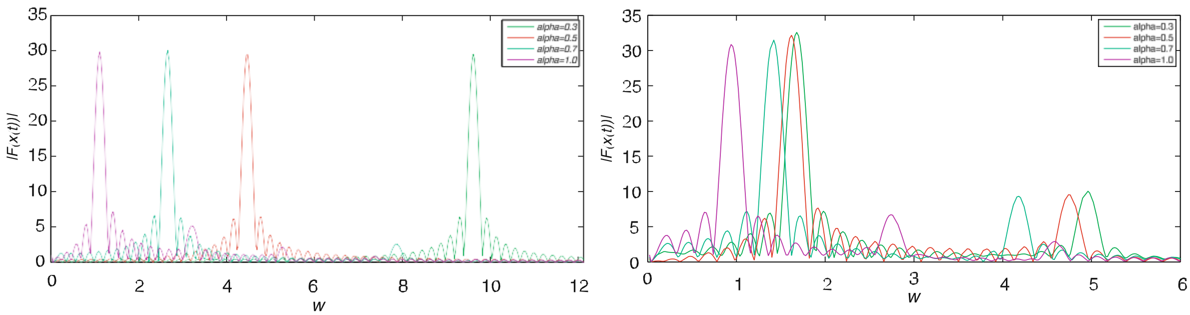


Fig. 6 Amplitude of the Fourier transform $|F\{x_1(t)\}|$ vs. ω for the CVDP oscillators (13) (*left*) and (14) (*right*) with β = 0.8, $\alpha \in \{0.3, 0.5, 0.7, 1.0\}$ and $\mu = 2.0$

that the period increases as α goes from 0.2 to 1.0, in both systems (13)–(14). On the other hand, the amplitude is almost constant. To be precise, the ampli- tude shows a very tiny increase in system (13) and a very tiny decrease in system (14), as α increases to one.

We now compute the Fourier transforms $|F\{x_1(t)\}|$ of the periodic solutions of systems (13) and (14), for $\beta = 0.8$ and $\alpha = \{0.3, 0.5, 0.7, 1.0\}$. Figures 5–6 de- pict the amplitude of the Fourier transforms vs. the frequency ω. The charts demonstrate that the main part of the signal energy is concentrated in the funda- mental frequency ω_0. The remaining energy, located in the higher harmonics increases with μ. Further- more, it is also observed that the fundamental fre- quency of the oscillations ω_0 varies with α and μ. For the range of values tested, the numerical fitting leads to exponential and rational fraction approxima- tions, for systems (13) and (14), respectively, given by

$$\omega_0 \approx 23.032\exp(-3.0686\alpha) \qquad (15)$$

and

$$\omega_0 \approx \frac{1}{0.0214 + 0.29\mu + (0.7437 - 0.1202\mu)\alpha^{2.3158+0.8522\mu}} \qquad (16)$$

We verify that in the first case the frequency of oscillation is independent of μ, while in the second case it is related both with α and μ.

3 Conclusions

In this paper two complex-order approximations to the well-known van der Pol oscillator were proposed. The amplitude and the period of solutions produced by these two approximations were then measured. The imaginary part was fixed while the real component was varied, for two distinct values of parameter μ. It was observed that the waveform period increases as α varies between 0.2 and one. On the other hand, the amplitude values are almost constant as α varies. Moreover, it seems there is a tiny increase in the amplitude of solutions for system (13) and a tiny decrease for system (14) as α approaches one.

The Fourier transform $|F\{x_1(t)\}|$ for systems (13) and (14) was also calculated. It was verified that the main part of the signal energy is concentrated in the fundamental frequency ω_0. The remaining energy, located in the higher harmonics, increases with μ. It is also observed that the fundamental frequency of the oscillations ω_0 varies with α and μ. For the range of values tested, the numerical fitting leads to exponential and rational fraction approximations for systems (13) and (14), respectively.

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