On the central Chi-square distribution with even degrees of freedom and correlated complex components

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Resumo: This contribution presents the derivation of alternate expressions for the central Chi-square distribution with even degrees of freedom. The complex Gaussian components of the chi-square distribution are modelled with a linear correlation model using different statistics for each component. We focus on the specific expressions for the probability density function (PDF) and complementary cumulative density function (CCDF). Unlike previous approaches, we use a frequency domain interpretation that allows us to derive a closed form expression for the characteristic function (CF) as an inverse polynomial equation. Using the roots of this polynomial equation, it is possible to rewrite the CF as a partial fraction expansion (PFE). This allows us to obtain a simple expression for both the PDF and CCDF by simply using the inverse transform of the CF. The statistics derived here have a much lower complexity than the expressions obtained from conventional non-frequency domain methods at the expense of the complexity of the polynomial root solution scheme. In scenarios where the average statistics of the components do not change over some periods of time, the proposed expressions provide the lowest possible complexity, as the polynomial rooting process needs to be conducted only once and potentially offline for the period where the average statistics of the components remain constant.
The central chi-square distribution (see [1]) has many different types of applications in science and engineering. In communications, the chi-square distribution is mainly used for modelling of fading channels and multiple antenna receivers (see [2]-[4]). The main statistics for the chi-square distribution are well known and have been studied for long time. However, under the assumptions of correlation and multivariate components, the analysis becomes complex and relatively less known [5].
Consider the Gaussian complex circular variable h_m with variance γ_m. For convenience, all variables will be written as follows:

$$h_m = \sqrt{\gamma_m}(\sqrt{1-\lambda_m^2} Z_m + \lambda_m G),$$

where the variables Z_m, G are identically and independently distributed (*i.i.d.*) zero-mean complex circular symmetrical Gaussian random variables with unit variance. The correlation coefficient can be defined as follows $\rho_{m,\tilde{m}} = \frac{E[h_m^* h_{\tilde{m}}]}{\gamma_m \gamma_{\tilde{m}}} = \frac{\lambda_m^* \lambda_{\tilde{m}}}{\gamma_m \gamma_{\tilde{m}}}$. The conditional characteristic function (CF) of ξ can be thus written as: $\Psi_{\xi|G}(i\omega) = \prod_{m=1}^{M} \frac{1}{(1-i\omega\tilde{\gamma}_m)} e^{\frac{i\omega|\tilde{\lambda}_m G|^2}{1-i\omega\tilde{\gamma}_m}}$, where $\tilde{\gamma}_m = \gamma_m(1-\lambda_m^2)$ and

$\tilde{\lambda}_m = \sqrt{\gamma}_m \lambda_m$. Averaging the previous expression over the PDF of the random variable G yields the expression for the unconditioned characteristic function:

$$\Psi_\xi(i\omega) = \frac{1}{\sum_{m=1}^{M} i\omega|\tilde{\lambda}_m|^2 \prod_{\tilde{m}=1;\tilde{m}\neq m}^{M}(1 - i\omega\tilde{\gamma}_{\tilde{m}}) - \prod_m (1 - i\omega\tilde{\gamma}_m)}.$$

Using partial fraction expansion, the previous expression becomes $\Psi_\xi(i\omega) = \sum_{k=1}^{M} \frac{A_k}{i\omega - \breve{\gamma}_k}$. where $A_k = \prod_{\tilde{k}\neq k} (\breve{\gamma}_k - \breve{\gamma}_{\tilde{k}})^{-1}$, and $\breve{\gamma}_k$ is the kth root of the polynomial function of the denominator. The back-transform of (the last expression yields a complementary cumulative distribution function (CCDF) given by: $\bar{F}_{\hat{\Gamma}}(y) = \sum_{k=1}^{M} A_k e^{\breve{\gamma}_k y}$. This concludes the derivation of the statistics of the chi-square distribution with correlated complex components.

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