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A Statistical Approach to the Harmonic Modelling of Mechanical Manipulators

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Abstract

A new approach to the analysis and design of robot manipulators is presented. The novel feature resides on a non-standard formulation to the modelling problem. Usually, system descriptions are based on a set of differential equations which, in general, require laborious computations and may be difficult to analyse. These facts motivate the need of alternative models based on different mathematical concepts. The proposed statistical approach to the Fourier modelling gives clear guidelines towards the optimisation of the robot kinematics and point out structural characteristics of the trajectory planning algorithms.

1. Introduction

Mechanical manipulators are developed according to engineering and scientific principles which are based on fundamental concepts such as those arising from mathematics and physics. Based on these formulations, the first step on the study of a physical phenomena is the development of an adequate model. Usually the fundamental concepts are the differential and matrix calculus and the classical newtonian physics, while the model consists on a set of differential equations. Nevertheless, several phenomena, such as quantum physics and thermodynamics, may be studied through different mathematical tools namely using statistical methods. These facts suggest that, for a given problem, we may adopt different mathematical models, each with its own merits and drawbacks. The second step on the study of the physical phenomena is the analysis of the properties revealed by the model. For a model consisting on a set of linear differential equations we can adopt simple strategies (e.g. the Fourier analysis) but, for a non-linear model these tools are not adequate and the analysis becomes complex and difficult to generalise. In fact, experience demonstrates that for a large number of cases, such as the kinematic and dynamic models arising in robotics, efficient tools capable of rendering clear results are still lacking.

This paper presents a framework where it is developed a new modelling formalism based on the embedding of statistical and Fourier transform concepts. These concepts are then illustrated on several experiments. The examples reveal not only the capabilities of the new method but also the limitations of standard robot structures and path planning algorithms. Consequently, in order to develop the new formalism the paper is

organised as follows. Section two starts by presenting the fundamental modelling concepts. Based on the new concepts, section three illustrates its application on the kinematic and trajectory planning analysis of mechanical manipulators. Finally, section four outlines the main conclusions.

2. Embedding Statistical and Fourier Transform Towards the Modelling of Robot Manipulators

For a robot having n degrees of freedom (dof) the classical direct kinematic model is described by a set of equations $\mathbf{p} = \psi(\mathbf{q})$ relating the operational and the joint

spaces, where $\mathbf{p} = [p_1, \dots, p_n]^T$ and $\mathbf{q} = [q_1, \dots, q_n]^T$

are the $n \times 1$ vectors of position in the operational and joint spaces, respectively. Based on these equations considerable research has been done on issues such as the optimisation of the manipulator structure [1,2] and the development of efficient path planning algorithms [3,4]. However, the kinematic equations usually are non-linear and reveal a plethora of variables and parameters that give rise to a cumbersome work both in the analysis and design stages. Therefore, in order to overcome these problems alternative concepts are required. Statistics is a mathematical tool well adapted to handle a large volume of data that has already been used in some restricted classes of robotic problems [5-7]. Nevertheless, for the kinematic modelling, statistics is not capable of dealing with time-dependent relations. Therefore, to overcome the limitations of statistics [7-9], the new method will also take advantage of the Fourier transform by embedding both tools in a broader formalism. In this line of thought, the first stage of the new modelling formalism starts by comprising:

- A set of input variables (*ivs*), that is, variables that are free to change independently.
- A set of output variables (*ovs*), that is, variables that depend on the *ivs*.
- A set of parameters which are to be optimised in the design stage.

As usually, in the direct kinematics the *ivs* and *ovs* are established by the relation $\psi: \{\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}\} \rightarrow \{\mathbf{p}, \dot{\mathbf{p}}, \ddot{\mathbf{p}}\}$, while for the inverse kinematics we get the reverse relation $\psi^{-1}: \{\mathbf{p}, \dot{\mathbf{p}}, \ddot{\mathbf{p}}\} \rightarrow \{\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}\}$. In both cases, the set of parameters depend on the manipulator structure and the time/space evolution of the trajectories.

The second stage of the formalism consists on the embedding of the statistical analysis into the Fourier transform through the algorithm:

- i) A statistical sample for the variables is obtained by driving the manipulator through a large number of trajectories (generated according with adequate statistics) having appropriate time/space evolutions. All the variables (*i.e.* the *ivs* and the *ovs*) are calculated, sampled in the time domain, and the resulting numerical values are stored in arrays.
- ii) For each of the previous arrays, the Fourier transform is computed (numerically) and the corresponding frequency spectrum is stored in a second class of arrays.
- iii) After concluding the statistical sample of trajectories, for all the variables and for each frequency within the spectrum range under study, several statistical indices (*e.g.* percentiles) of the amplitudes and/or phases of the arrays obtained in ii) are calculated. The statistics of the frequency spectrum is stored in a third class of arrays.
- iv) For all the variables and for each frequency, the values of the statistical indices calculated in iii) are collected on a 'composite' frequency spectrum and stored in a fourth class of arrays.
- v) The procedure i) to iv) is repeated for different numerical values of the link lengths. The numerical results for the fourth class of arrays obtained in iv) are compared and analytical expressions, that fit the

numerical data, are extrapolated.

- vi) The algorithm i) to v) is repeated for different time/space trajectories.
- vii) The algorithm i) to vi) is repeated for different robot structures.
- viii) The partial conclusions drawn by the analytical expressions obtained in v), vi) and vii) are integrated and final conclusions are drawn.

In order to illustrate the new method in the next section we analyse the kinematics of planar manipulators and we compare the results with the classical direct and inverse kinematic equations (Table 1).

3. A New Model for the Kinematics of Planar Robots

The application of the formalism defined in the previous section requires the development of numerical calculations for the statistics. Therefore, before proceeding, we need to establish the different experiments according with the following guidelines:

- i) Modelling case:
 - IK = {Inverse Kinematics}
 - DK = {Direct Kinematics}
- ii) Type of trajectory
 - ii.1) Time acceleration profile:
 - O = {On/Off acceleration},
 - T = {Triangular acceleration},
 - P = {Parabolic acceleration},
 - S = {Sinusoidal acceleration}

Table 1 Classical Direct and Inverse Kinematic Models of Planar Robots

Robot RR	Robot RP
$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} l_1 C_1 + l_2 C_2 \\ l_1 S_1 + l_2 S_2 \end{bmatrix}$	$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} q_2 C_1 \\ q_2 S_1 \end{bmatrix}$
$\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} -l_1 S_1 - l_2 S_{12} & -l_2 S_{12} \\ l_1 C_1 + l_2 C_{12} & l_2 C_{12} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$	$\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} -q_2 S_1 & C_1 \\ q_2 C_1 & S_1 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$
$\begin{bmatrix} \ddot{p}_1 \\ \ddot{p}_2 \end{bmatrix} = \begin{bmatrix} -l_1 S_1 & -l_2 S_{12} \\ l_1 C_1 & l_2 C_{12} \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} - \begin{bmatrix} l_1 C_1 & l_2 C_{12} \\ l_1 S_1 & l_2 S_{12} \end{bmatrix} \begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{bmatrix}$	$\begin{bmatrix} \ddot{p}_1 \\ \ddot{p}_2 \end{bmatrix} = \begin{bmatrix} -q_2 S_1 & C_1 \\ q_2 C_1 & S_1 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} - \begin{bmatrix} -q_2 C_1 & -2S_{12} \\ -q_2 S_1 & 2C_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_1 \dot{q}_2 \end{bmatrix}$
$q_2 = \cos^{-1} \left(\frac{p_1^2 + p_2^2 - l_1^2 - l_2^2}{2l_1 l_2} \right)$	$q_1 = \tan^{-1} \left(\frac{p_2}{p_1} \right)$
$q_1 = \tan^{-1} \left(\frac{p_2}{p_1} \right) - \tan^{-1} \left(\frac{l_2 S_2}{l_1 + l_2 C_2} \right)$	$q_2 = \sqrt{p_1^2 + p_2^2}$
$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \frac{1}{l_1 l_2 S_2} \begin{bmatrix} l_2 C_{12} & l_2 S_{12} \\ -l_1 C_1 - l_2 C_{12} & -l_1 S_1 - l_2 S_{12} \end{bmatrix} \begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix}$	$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \frac{1}{q_2} \begin{bmatrix} -p_2/q_2 & p_1/q_2 \\ p_1 & p_2 \end{bmatrix} \begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix}$
$\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} = \frac{1}{l_1 l_2 S_2} \begin{bmatrix} l_2 C_{12} & l_2 S_{12} \\ -l_1 C_1 & -l_1 S_1 \end{bmatrix} \begin{bmatrix} \ddot{p}_1 \\ \ddot{p}_2 \end{bmatrix} +$ $+ \frac{1}{l_1 l_2 S_2} \begin{bmatrix} l_1 l_2 C_2 & l_2^2 \\ -l_1^2 & -l_1 l_2 C_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{bmatrix}$	$\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} = \frac{1}{q_2} \begin{bmatrix} -p_2/q_2 & p_1/q_2 \\ p_1 & p_2 \end{bmatrix} \begin{bmatrix} \ddot{p}_1 \\ \ddot{p}_2 \end{bmatrix} + \begin{bmatrix} -(2/q_2) \dot{q}_1 \dot{q}_2 \\ q_2 \dot{q}_1^2 \end{bmatrix}$

ii.2) Total time definition:

$MAL = \{\text{Maximum Acceleration Limitation}\}$,
 $MVL = \{\text{Maximum Velocity Limitation}\}$,
 $RAL = \{\text{Random Acceleration Limitation}\}$

ii.3) Space evolution:

$SL = \{\text{Straight Line}\}$,
 $DP = \{\text{Direct Parabolic}\}$,
 $IP = \{\text{Inverse Parabolic}\}$

iii) Type of robot mechanical structure:

$RR = \{\text{joint 1 Rotational, joint 2 Rotational}\}$,
 $RP = \{\text{joint 1 Rotational, joint 2 Prismatic}\}$

The type of "time acceleration profile" leads to different formulae for the "total time definition" according with:

Acceleration $O \propto MAL: t_{\max} = 2\sqrt{\text{dist}/A_{\max}}$;

$MVL: t_{\max} = 2 \text{dist}/V_{\max}$;

$RAL: t_{\max} = 2\sqrt{\text{dist}/\text{random}(A_{\max})}$

Acceleration $T \propto MAL: t_{\max} = \sqrt{8\text{dist}/A_{\max}}$;

$MVL: t_{\max} = 2 \text{dist}/V_{\max}$;

$RAL: t_{\max} = \sqrt{8 \text{dist}/\text{random}(A_{\max})}$

Acceleration $P \propto MAL: t_{\max} = \sqrt{8\text{dist}/A_{\max}}$;

$MVL: t_{\max} = 2 \text{dist}/V_{\max}$;

$RAL: t_{\max} = \sqrt{8 \text{dist}/\text{random}(A_{\max})}$

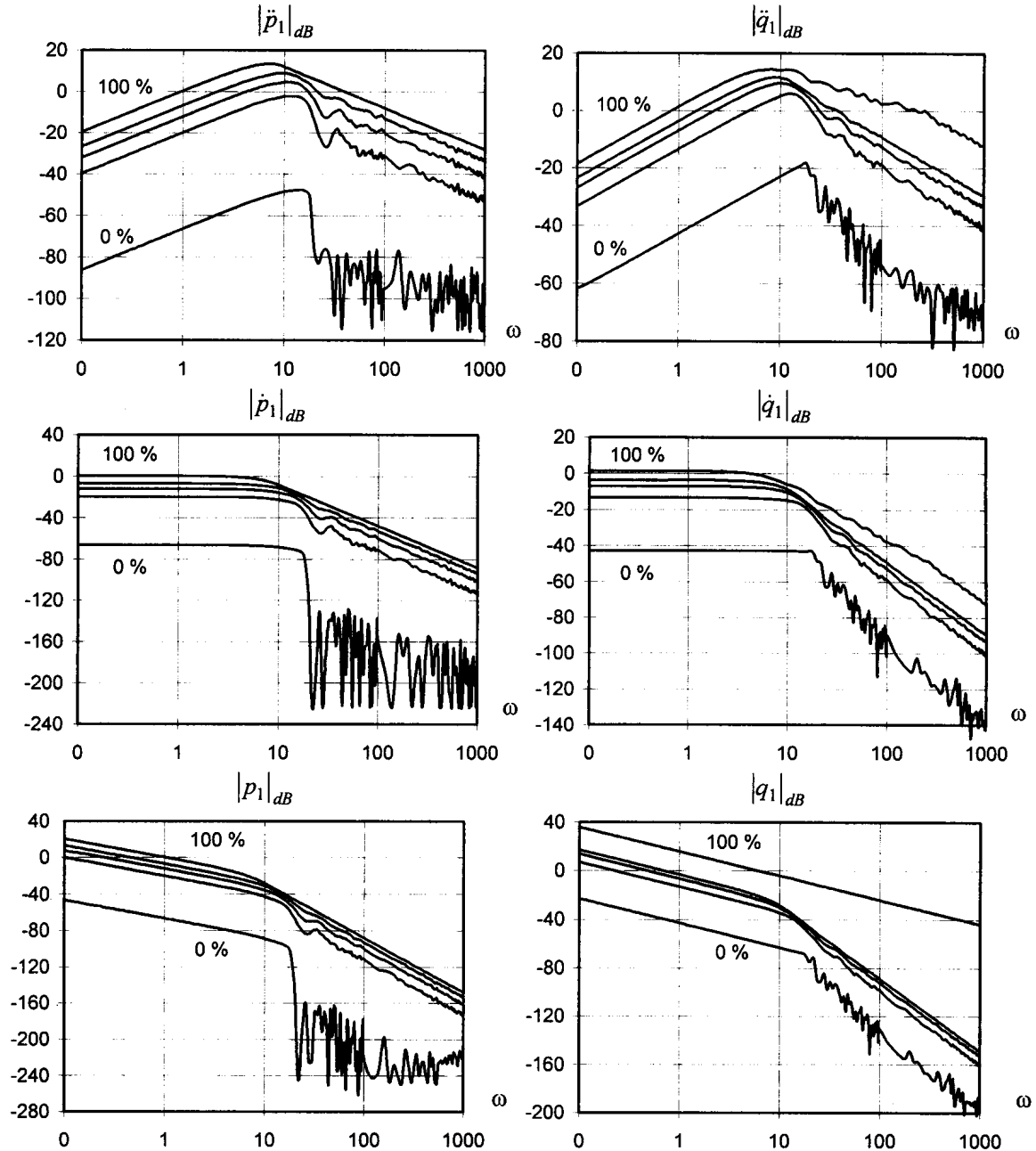


Fig. 1 Percentiles of {0,25,50,75,100} of the SHC-amplitude versus the frequency ω for several ivs and ovs in the experiment $\{IK-RR,O,MAL,SL\}$ with $l_1 = 1, l_2 = 0.1, A_{\max} = 10$.

$$\text{Acceleration } S \propto \text{MAL: } t_{\max} = \sqrt{2\pi \text{dist} / A_{\max}} ;$$

$$\text{MVL: } t_{\max} = \text{dist} / V_{\max} ;$$

$$\text{RAL: } t_{\max} = \sqrt{2\pi \text{dist} / \text{random}(A_{\max})}$$

where $0 \leq \text{random}(A_{\max}) \leq A_{\max}$, dist is the total distance along the trajectory, t_{\max} is the total time of duration of the movement and A_{\max} and V_{\max} are the maximum allowed values for the acceleration and the velocity, respectively.

For example, Fig. 1 shows the Statistics of the Harmonic Content (SHC) of several variables for a sample of 500 trajectories and the experiment $\{IK\text{-RR}, O, \text{MAL}, \text{SL}\}$. The analytical expressions that can be derived from the different experiments are depicted on Table 2. Observing the results and applying a frequency-domain identification algorithm [10] to estimate the SHC's, we conclude the following properties:

- Numerical convergence - After repeating a large number of numerical experiments the charts with the SHC of the variables do not change significantly.
 - Derivative/integral sensitivity - Although being composite curves, the SHC still obey the 'standard' $j\omega$ operator for variables that are related by the derivative operator in the time domain.
 - Analytical coherence - The numerical data that results from the experiments 'fits' the analytical expressions that lead to clear conclusions. For example, the DK , that calculates the trajectories in the joint space, leads to expressions for the poles that do not depend on the length of the workspace, while the IK , that calculates the trajectories in the operational space, leads to poles that are sensitive to the length of the workspace. Therefore, trajectory planning algorithms on the joint space are more 'robust', that is, lead to a constant bandwidth.
 - Generality - While for the classical deterministic models we can not find clear relations between different robot structures, with the SHC general characteristics are highlighted. For example, joint 1 of the RR robot and joint 1 of the RP structure reveal almost similar SHC's which reflects that, in both cases, we have a rotational joint. On the other hand, for joint 2, we have different mechanical articulations in the two cases and, therefore, the experiments lead to distinct results.
 - Compatibility - The conclusions based on the analysis of the SHC are coherent with the results of previous studies using different mathematical tools [1,2,9]. For example, if $l_1 + l_2 = \text{constant}$ we verify that the maximum gain and bandwidth of the SHC occurs for $l_1 = l_2$. Also, the 'lower-elbow' and 'upper-elbow' solutions for the IK where analysed revealing, as expected, similar properties in both cases.
- In conclusion, the new method leads to promising results, consistent with previous knowledge, and points out future research on the analysis of the kinematics, dynamics and path planning algorithms.

4. Conclusions

A new method to the analysis and design of robot manipulators was announced. The novel feature resides on a non standard approach to the modelling problem. Usually, system descriptions are based on a set of differential equations which can be very complex and hard to tackle. This motivates the need of models based on alternative concepts having distinct characteristics. The proposed method, by embedding the statistical analysis into the Fourier transform, is a step in that direction which has been shown to enable a new formalism. Therefore, the new method provides a framework giving clear guidelines towards the optimisation both of the path planning algorithm and the robot structure and a deeper understanding of the actuator limitations and requirements.

5. References

- [1] Y.C. Tsai and A.H. Soni, "Accessible Region and Synthesis of Robot Arms," ASME J. Mech. Design, Vol. 103, pp. 803-811, 1981.
- [2] T. Yoshikawa, "Manipulability of Robotic Mechanisms," The Int. J. Robotics Research, Vol. 4, pp. 3-9, 1985.
- [3] G. Sahar and J.M. Hollerbach, "Planning of Minimum-Time Trajectories for Robot Arms," The Int. J. R. Research, Vol. 5, pp. 90-100, 1986.
- [4] V. Scheinman and B. Roth, "On the Optimal Selection and Placement of Manipulators," Proc. RoManSy, Udine, Italy, 1984.
- [5] B. W. Mooring and T. J. Pack, "Aspects of Robot Repeatability," Robotica, Vol. 5, pp. 223-230, 1987.
- [6] J. R. Singh, J. Rastegar and S. Snee, "Determination of Manipulator Kinematic Characteristics: A Probabilistic Approach," Int. Journal of Robotics and Automation, Vol. 7, N. 4, pp.161-170, 1992.
- [7] Alexandra M. S. F. Galhano, J. L. M. de Carvalho and J. A. T. Machado, "The Statistical Study of Robot Manipulators," IEEE Int. Symp. on Intelligent Control, Philadelphia, Pennsylvania, 1990.
- [8] Alexandra M. S. F. Galhano, J. L. M. de Carvalho and J. A. T. Machado, "On the Statistical Modelling of Mechanical Manipulators," IFAC/IFIP/IMACS Symp. on Robot Control SYROCO'91, Vienna, Austria, 1991.
- [9] J.A. Tenreiro Machado and Alexandra M.S.F. Galhano, "Statistical Modelling of Robotic Manipulators", SAMS - Journal Systems Analysis-Modelling-Simulation, Gordon & Breach Science Publishers, Vol. 12, pp. 101-123, USA, 1993.
- [10] P. Lopes dos Santos and J. L. Martins de Carvalho, "Automatic Transfer Function Synthesis from a Bode Plot", 29th IEEE Conf. On Decision and Control, Honolulu, Hawaii, USA, 1990.

Table 2 Characterisation of the SHC for the Kinematics of Planar Manipulators

	Experiment	$F\{\ddot{x}\}, F\{\ddot{y}\}$	$F\{\ddot{q}_1\}$	$F\{\ddot{q}_2\}$	
1	<i>IK-RR</i> <i>O, MAL, SL</i>	$\frac{\sqrt{A_{\max}}}{\sqrt[4]{l_1 l_2}} (0.492 \pm j1.042)$	$\sqrt{A_{\max}} \left(\frac{0.644}{\sqrt[4]{l_1 l_2}} \pm j \frac{1.396}{\sqrt[6]{l_1^2 l_2}} \right)$	$\frac{\sqrt{A_{\max}}}{\sqrt[4]{l_1 l_2}} (0.545 \pm j1.827)$	Poles
		$0.738 \sqrt{l_1 l_2}$	$1.421 - 0.202 \frac{l_1 - l_2}{l_1 + l_2} - 1.071 \left \frac{l_1 - l_2}{l_1 + l_2} \right $	0.678	Gain
2	<i>IK-RR</i> <i>S, MAL, SL</i>	$\frac{\sqrt{A_{\max}}}{\sqrt[4]{l_1 l_2}} (0.401 \pm j1.803),$ $\frac{\sqrt{A_{\max}}}{\sqrt[4]{l_1 l_2}} 0.564$	$\sqrt{A_{\max}} \left(\frac{0.408}{\sqrt[4]{l_1 l_2}} \pm j \frac{2.169}{\sqrt[6]{l_1^2 l_2}} \right),$ $\frac{\sqrt{A_{\max}}}{\sqrt[6]{l_1^2 l_2}} 0.695$	$\frac{\sqrt{A_{\max}}}{\sqrt[4]{l_1 l_2}} (0.392 \pm j2.367),$ $\frac{\sqrt{A_{\max}}}{\sqrt[4]{l_1 l_2}} 1.231$	Poles
		$0.888 \sqrt{l_1 l_2}$	$1.623 - 0.287 \frac{l_1 - l_2}{l_1 + l_2} - 1.158 \left \frac{l_1 - l_2}{l_1 + l_2} \right $	0.758	Gain
3	<i>IK-RR</i> <i>T, MAL, SL</i>	$\frac{\sqrt{A_{\max}}}{\sqrt[4]{l_1 l_2}} (0.445 \pm j1.660),$ $\frac{\sqrt{A_{\max}}}{\sqrt[4]{l_1 l_2}} 0.583$	$\sqrt{A_{\max}} \left(\frac{0.469}{\sqrt[4]{l_1 l_2}} \pm j \frac{2.001}{\sqrt[6]{l_1^2 l_2}} \right),$ $\frac{\sqrt{A_{\max}}}{\sqrt[6]{l_1^2 l_2}} 0.794$	$\frac{\sqrt{A_{\max}}}{\sqrt[4]{l_1 l_2}} (0.419 \pm j2.149),$ $\frac{\sqrt{A_{\max}}}{\sqrt[4]{l_1 l_2}} 1.261$	Poles
		$0.883 \sqrt{l_1 l_2}$	$1.611 - 0.290 \frac{l_1 - l_2}{l_1 + l_2} - 1.154 \left \frac{l_1 - l_2}{l_1 + l_2} \right $	0.759	Gain
4	<i>IK-RR</i> <i>P, MAL, SL</i>	$\frac{\sqrt{A_{\max}}}{\sqrt[4]{l_1 l_2}} (0.566 \pm j0.824),$ $\frac{\sqrt{A_{\max}}}{\sqrt[4]{l_1 l_2}} (0.938 \pm j3.176)$	$\sqrt{A_{\max}} \left(\frac{0.612}{\sqrt[4]{l_1 l_2}} \pm j \frac{1.072}{\sqrt[6]{l_1^2 l_2}} \right),$ $\sqrt{A_{\max}} \left(\frac{1.069}{\sqrt[4]{l_1 l_2}} \pm j \frac{4.133}{\sqrt[6]{l_1^2 l_2}} \right)$	$\frac{\sqrt{A_{\max}}}{\sqrt[4]{l_1 l_2}} (0.575 \pm j1.406),$ $\frac{\sqrt{A_{\max}}}{\sqrt[4]{l_1 l_2}} (1.194 \pm j3.941)$	Poles
		$0.806 \sqrt{l_1 l_2}$	$1.456 - 0.237 \frac{l_1 - l_2}{l_1 + l_2} - 1.068 \left \frac{l_1 - l_2}{l_1 + l_2} \right $	0.691	Gain
5	<i>IK-RR</i> <i>O, MVL, SL</i>	$\frac{V_{\max}}{\sqrt{l_1 l_2}} (0.525 \pm j0.843)$	$V_{\max} \left(\frac{0.614}{\sqrt{l_1 l_2}} \pm j \frac{1.061}{\sqrt[3]{l_1^2 l_2}} \right)$	$\frac{V_{\max}}{\sqrt{l_1 l_2}} (0.522 \pm j1.431)$	Poles
		$0.795 \sqrt{l_1 l_2}$	$1.465 - 0.221 \frac{l_1 - l_2}{l_1 + l_2} - 1.082 \left \frac{l_1 - l_2}{l_1 + l_2} \right $	0.709	Gain
6	<i>IK-RR</i> <i>O, RAL, SL</i>	$\frac{\sqrt{A_{\max}}}{\sqrt[4]{l_1 l_2}} (0.338 \pm j0.600)$	$\sqrt{A_{\max}} \left(\frac{0.472}{\sqrt[4]{l_1 l_2}} \pm j \frac{0.855}{\sqrt[6]{l_1^2 l_2}} \right)$	$\frac{\sqrt{A_{\max}}}{\sqrt[4]{l_1 l_2}} (0.441 \pm j1.161)$	Poles
		$0.741 \sqrt{l_1 l_2}$	$1.444 - 0.244 \frac{l_1 - l_2}{l_1 + l_2} - 1.062 \left \frac{l_1 - l_2}{l_1 + l_2} \right $	0.713	Gain

Table 2 Characterisation of the SHC for the Kinematics of Planar Manipulators (continuation)

	Experiment	$F\{\ddot{x}\}, F\{\ddot{y}\}$	$F\{\ddot{q}_1\}$	$F\{\ddot{q}_2\}$	
7	<i>DK-RR</i> <i>O, MAL, SL</i>	$\sqrt{A_{\max}} (0.388 \pm j1.459)$	$\sqrt{A_{\max}} (0.353 \pm j0.841)$	$\sqrt{A_{\max}} (0.337 \pm j0.830)$	Poles
		$\sqrt{l_1 l_2} \left(0.924 + 2.093 \frac{ l_1 - l_2 ^3}{ l_1 + l_2 } \right)$	1.665	0.827	Gain
8	<i>IK-RR</i> <i>O, MAL, DP</i>	$\ddot{x} \sim \frac{\sqrt{A_{\max}}}{\sqrt[4]{l_1 l_2}} (0.495 \pm j0.884),$ $\ddot{y} \sim \frac{\sqrt{A_{\max}}}{\sqrt[4]{l_1 l_2}} (0.348 \pm j1.373)$	$\sqrt{A_{\max}} \left(\frac{0.632}{\sqrt[4]{l_1 l_2}} \pm j \frac{1.147}{\sqrt[6]{l_1^2 l_2}} \right)$	$\frac{\sqrt{A_{\max}}}{\sqrt[4]{l_1 l_2}} (0.479 \pm j2.065)$	Poles
		$\ddot{x} \sim 2.24 \sqrt{l_1 l_2},$ $\ddot{y} \sim 1.373 \sqrt{l_1 l_2}$	$2.469 - 1.586 \frac{ l_1 - l_2 }{ l_1 + l_2 }$	0.749	Gain
9	<i>IK-RR</i> <i>O, MAL, IP</i>	$\ddot{x} \sim \frac{\sqrt{A_{\max}}}{\sqrt[4]{l_1 l_2}} (0.348 \pm j1.373),$ $\ddot{y} \sim \frac{\sqrt{A_{\max}}}{\sqrt[4]{l_1 l_2}} (0.495 \pm j0.884)$	$\sqrt{A_{\max}} \left(\frac{0.632}{\sqrt[4]{l_1 l_2}} \pm j \frac{1.147}{\sqrt[6]{l_1^2 l_2}} \right)$	$\frac{\sqrt{A_{\max}}}{\sqrt[4]{l_1 l_2}} (0.479 \pm j2.065)$	Poles
		$\ddot{x} \sim 1.373 \sqrt{l_1 l_2},$ $\ddot{y} \sim 2.24 \sqrt{l_1 l_2}$	$2.469 - 1.586 \frac{ l_1 - l_2 }{ l_1 + l_2 }$	0.749	Gain
10	<i>IK-RP</i> <i>O, MAL, SL</i>	$\frac{\sqrt{A_{\max}}}{\sqrt[4]{l_1 l_2}} (0.510 \pm j1.028)$	$\frac{\sqrt{A_{\max}}}{\sqrt[4]{l_1 l_2}} (0.580 \pm j1.188)$	$\frac{\sqrt{A_{\max}}}{\sqrt[4]{l_1 l_2}} (0.464 \pm j1.800)$	Poles
		$0.772 \sqrt{l_1 l_2}$	$1.441 - 1.289 \frac{ l_1 - l_2 }{ l_1 + l_2 }$	$\frac{1}{\sqrt{l_1 l_2}} \left(0.470 - 0.343 \frac{ l_1 - l_2 }{ l_1 + l_2 } \right)$	Gain
11	<i>DK-RP</i> <i>O, MAL, SL</i>	$\sqrt{A_{\max}} (0.447 \pm j1.535)$	$\sqrt{A_{\max}} (0.457 \pm j0.948)$	$\sqrt{A_{\max}} (0.378 \pm j0.883)$	Poles
		$\sqrt{l_1 l_2} \left(0.704 + 2.502 \frac{ l_1 - l_2 ^3}{ l_1 + l_2 } \right)$	1.671	$\frac{1}{\sqrt{l_1 l_2}} \left(0.521 - 0.435 \frac{ l_1 - l_2 }{ l_1 + l_2 } \right)$	Gain

Note: For the *RP* robot $l_{\max} = l_1 + l_2$ and $l_{\min} = |l_1 - l_2|$ where l_{\max} and l_{\min} are the maximum and minimum lengths of the workspace, respectively.
For the *RR* robot l_1 and l_2 are the lengths of the first and second links, respectively.