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Fractional Derivatives and Their Applications

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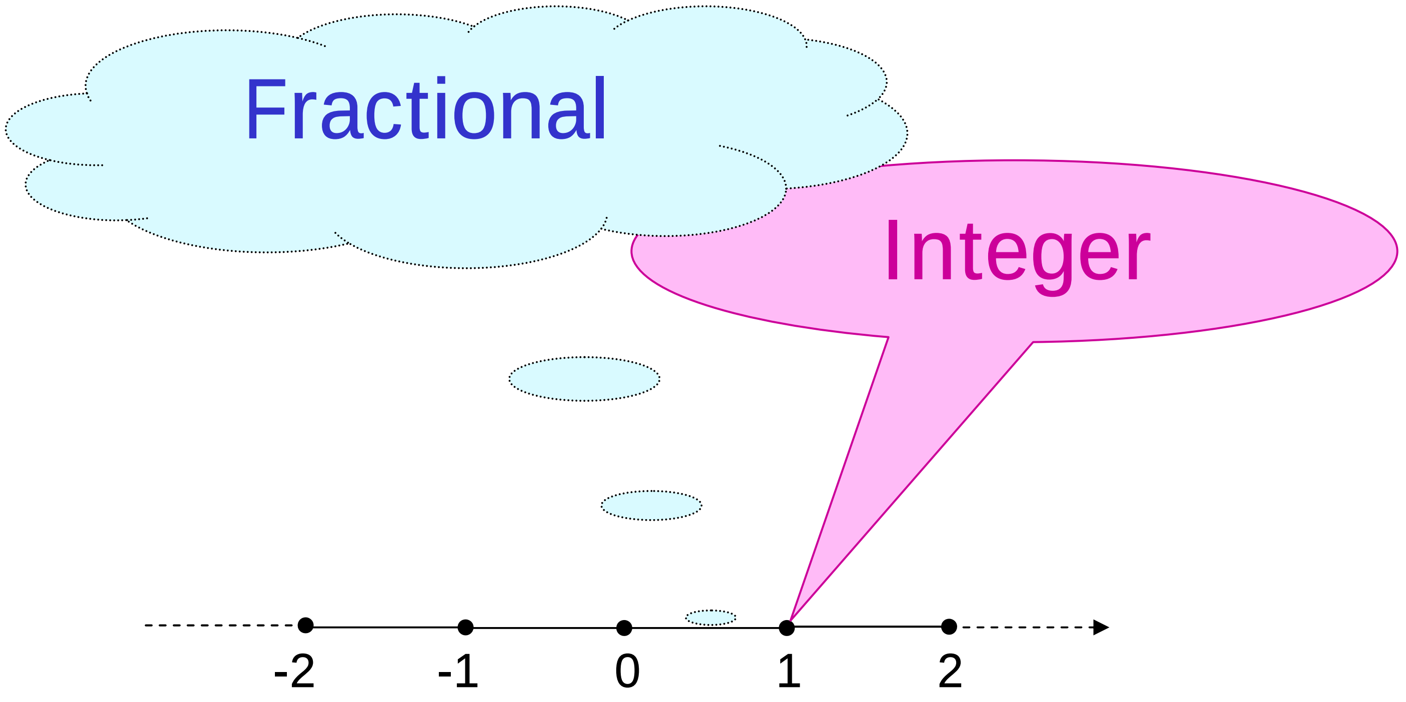
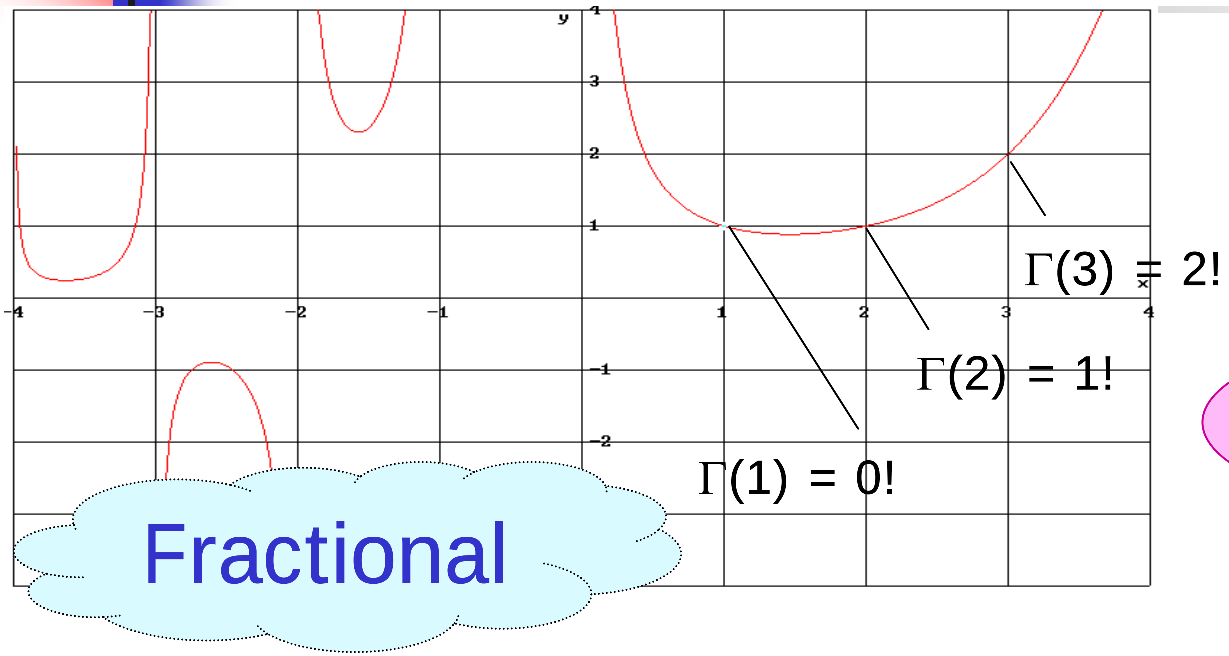
Integer *vs* fractional numbers



Factorial *vs* Gamma function



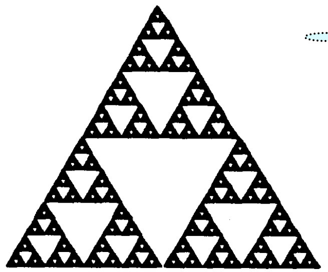
Integer

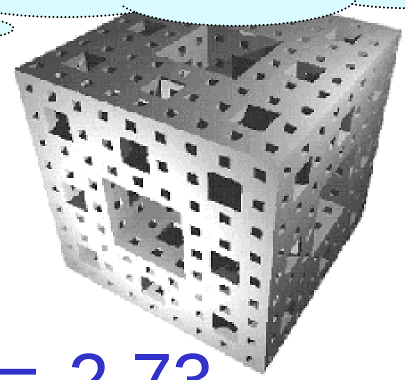
$$\Gamma(n) = \int_0^\infty t^{n-1} e^{-t} dt \Rightarrow \Gamma(n+1) = n(n-1)\cdots 1 = n!$$

Integer *vs* Fractal dimension

Fractional

$d = 0.63$

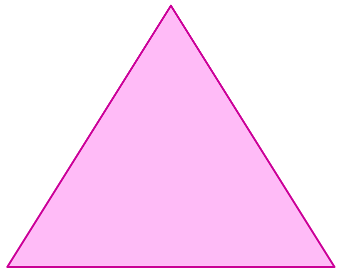


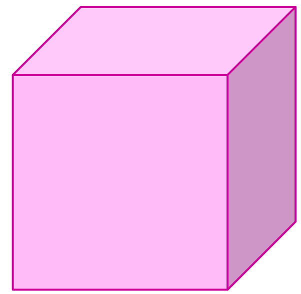


$d = 1$

$d = 1.58$

$d = 2.73$





$d = 2$

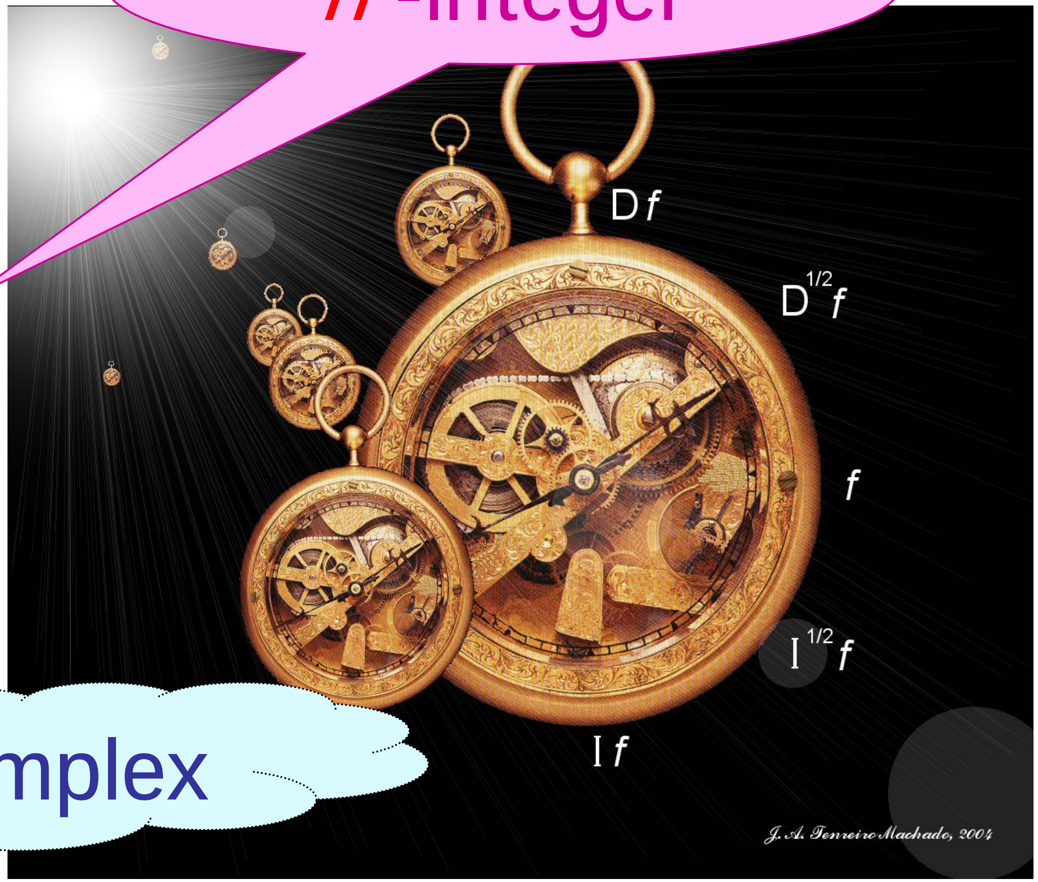
$d = 3$

Integer

Integer *vs* Fractional derivative

- $D^{1}(e^{ax}) = a\ e^{ax}$
- $D^{2}(e^{ax}) = a^{2}\ e^{ax}$
- $D^{3}(e^{ax}) = a^{3}\ e^{ax}$
- ….
- $D^{n}(e^{ax}) = a^{n}\ e^{ax}$
- $D^{\alpha}(e^{ax}) = a^{\alpha}\ e^{ax}$

n -integer



α - complex

Guillaume de l'Hôpital

(1661–1704)

What is the meaning of $D^{½}y$?

1695

intimate connection between derivatives and infinite series

Gottfried Wilhelm Leibniz

(1646–1716)

This is an apparent paradox from which, one day, useful consequences will be drawn...

Fractional Calculus

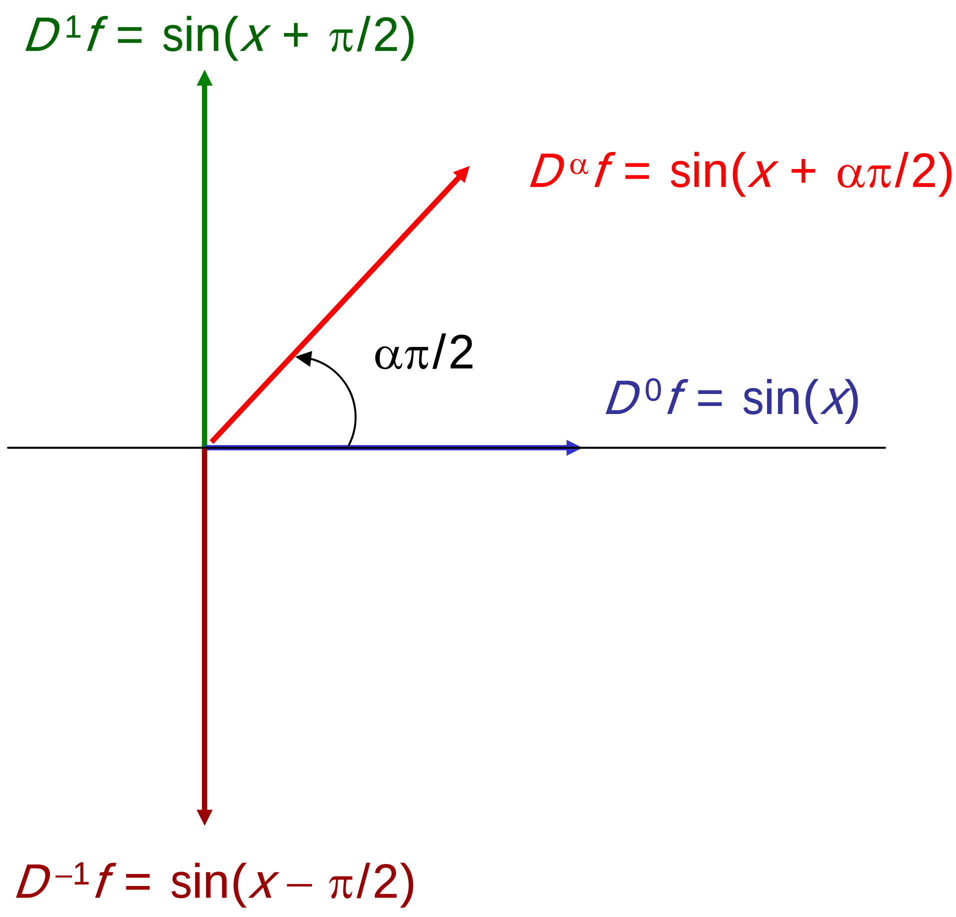
- The name "fractional calculus" is actually a misnomer
- The designation "integration and differentiation of arbitrary order" is more appropriate

Motivation: *sin(ax)* function

- $D^1[\sin(ax)] = a^1 \sin(ax + 1\,\pi/2)$
- $D^2[\sin(ax)] = a^2 \sin(ax + 2\,\pi/2)$
- $D^3[\sin(ax)] = a^3 \sin(ax + 3\,\pi/2)$
- $D^4[\sin(ax)] = a^4 \sin(ax + 4\,\pi/2)$
- ...
- $D^\alpha[\sin(ax)] = a^\alpha \sin(ax + \alpha\,\pi/2)$

Weyl derivative ${}_{-\infty}D_t^\alpha$

Vector interpretation of D^α for the function $f = \sin(x)$



Definitions of fractional derivatives-1

Bernhard Riemann Joseph Liouville

(1826–1866) (1809–1882)

Riemann-Liouville definition

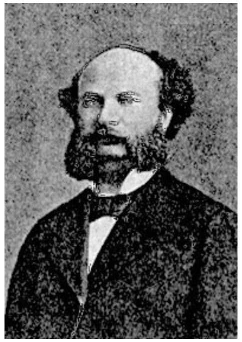
$$ {}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^n \int_a^t \frac{f(\tau)d\tau}{(t-\tau)^{\alpha-n+1}} $$

$$ n-1 < \alpha < n $$

Grünwald-Letnikov definition

$$ {}_aD_t^{\alpha} f(t) = \lim \frac{1}{h^{\alpha}} \sum_{k=0}^{\left[\frac{t-a}{h}\right]} (-1)^k \binom{\alpha}{k} f(t-kh) $$

$[x]$ – integer part of x



Aleksey Letnikov Anton Grünwald

(1837-1888) (1838-1920)

Definitions of fractional derivatives -2

Michele Caputo

Caputo definition

$$ {}_{a}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)d\tau}{(t-\tau)^{\alpha-n+1}} $$

$$ n-1 \le \alpha < n $$

Pierre-Simon Laplace

(1749-1827)

Laplace definition

$$ D^{\alpha} x(t) = L^{-1}\left\{ s^{\alpha} X(s) - \sum_{k=0}^{n-1} s^k D^{\alpha-k-1} x(t)\Big|_{t=0} \right\} $$

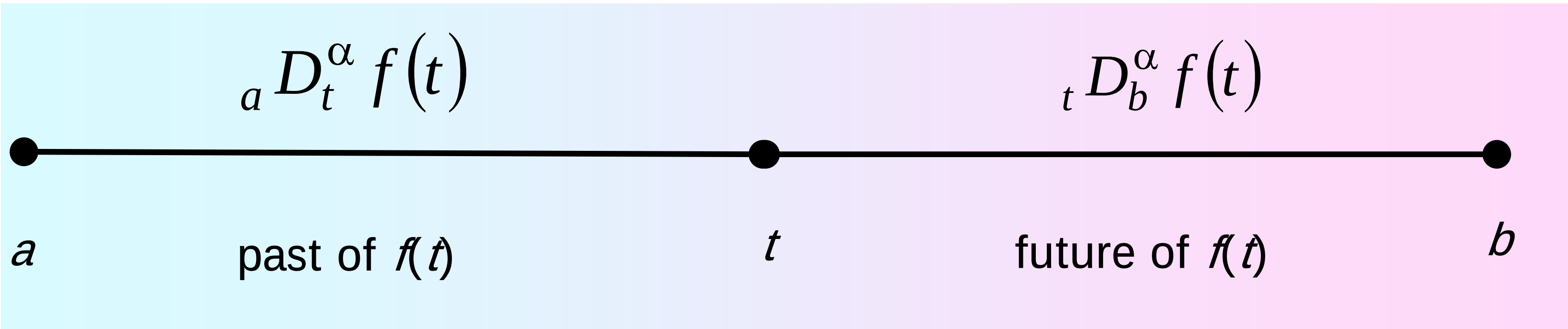
Left and Right fractional derivatives

- Left-sided

$$ {}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^n \int_a^t \frac{f(\tau)\,d\tau}{(t-\tau)^{\alpha-n+1}} $$

- Right-sided

$$ {}_tD_b^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dt} \right)^n \int_t^b \frac{f(\tau)\,d\tau}{(t-\tau)^{\alpha-n+1}} $$



Grünwald-Letnikov definition

$$D^1[f(x)] = \lim_{h \to 0} \frac{f(x) - f(x-h)}{h}$$

$$D^2[f(x)] = \lim_{h \to 0} \frac{f(x) - 2f(x-h) + f(x-2h)}{h^2}$$

$$D^3[f(x)] = \lim_{h \to 0} \frac{f(x) - 3f(x-h) + 3f(x-2h) - f(x-3h)}{h^3}$$

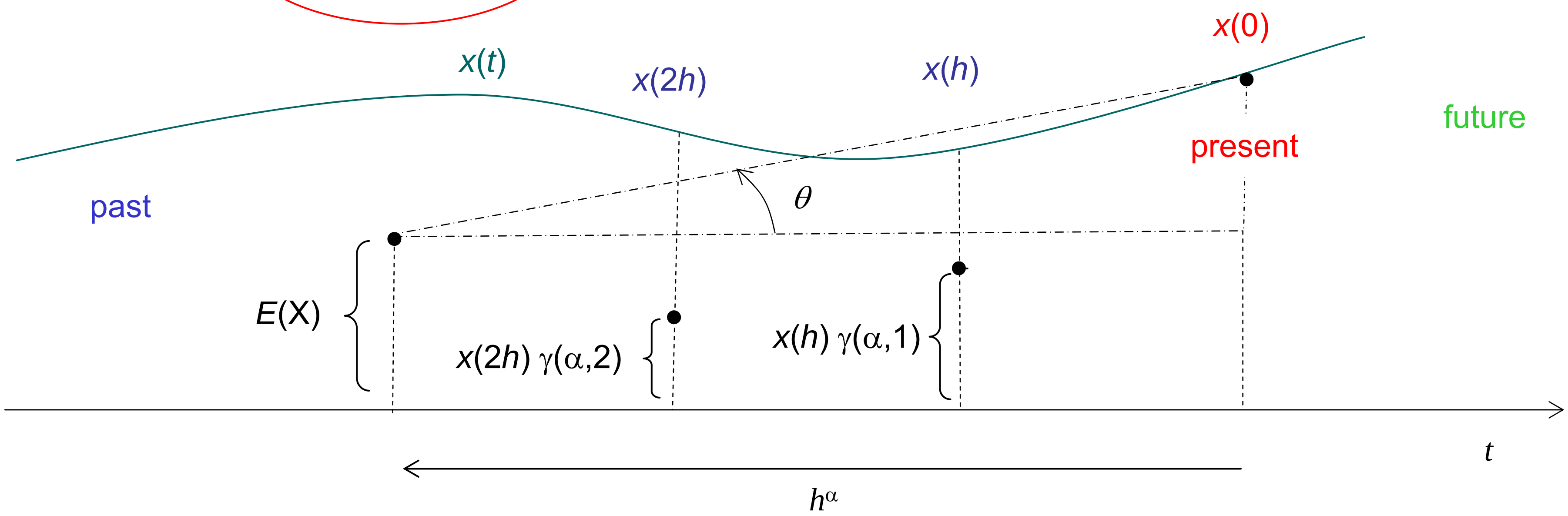
$$D^{1/2}[f(x)] = \lim_{h \to 0} \frac{f(x) - 1/2\, f(x-h) - 1/8\, f(x-2h) - 1/16\, f(x-3h) - \cdots}{h^{1/2}}$$

A probabilistic perspective...

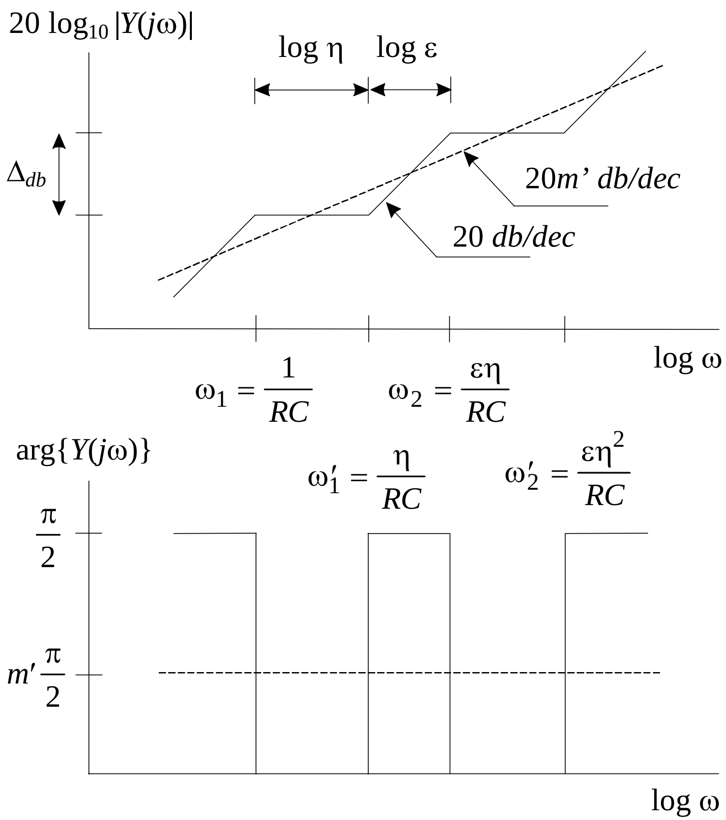
$$D^{1/2}[f(x)] = \lim_{h\to 0} \frac{f(x) - 1/2\, f(x-h) - 1/8\, f(x-2h) - 1/16\, f(x-3h) - \cdots}{h^{1/2}}$$

$$\frac{1}{2} + \frac{1}{8} + \frac{1}{16} + \cdots = 1$$

non uniform time variation



Frequency-based FD approximation



- Recursive relationships of pole/zero frequencies:

$$\frac{\omega'_{i+1}}{\omega'_i} = \frac{\omega_{i+1}}{\omega_i} = \varepsilon\eta \qquad \frac{\omega_i}{\omega'_i} = \varepsilon \qquad \frac{\omega'_{i+1}}{\omega_i} = \eta$$

- Average slope:

$$m' = \frac{\log \varepsilon}{\log \varepsilon + \log \eta}$$

- Approach to D^α $(0 < \alpha < 1)$:

$$m' = \alpha$$

Discrete-time FD approximation

Grünwald-Letnikov definition:

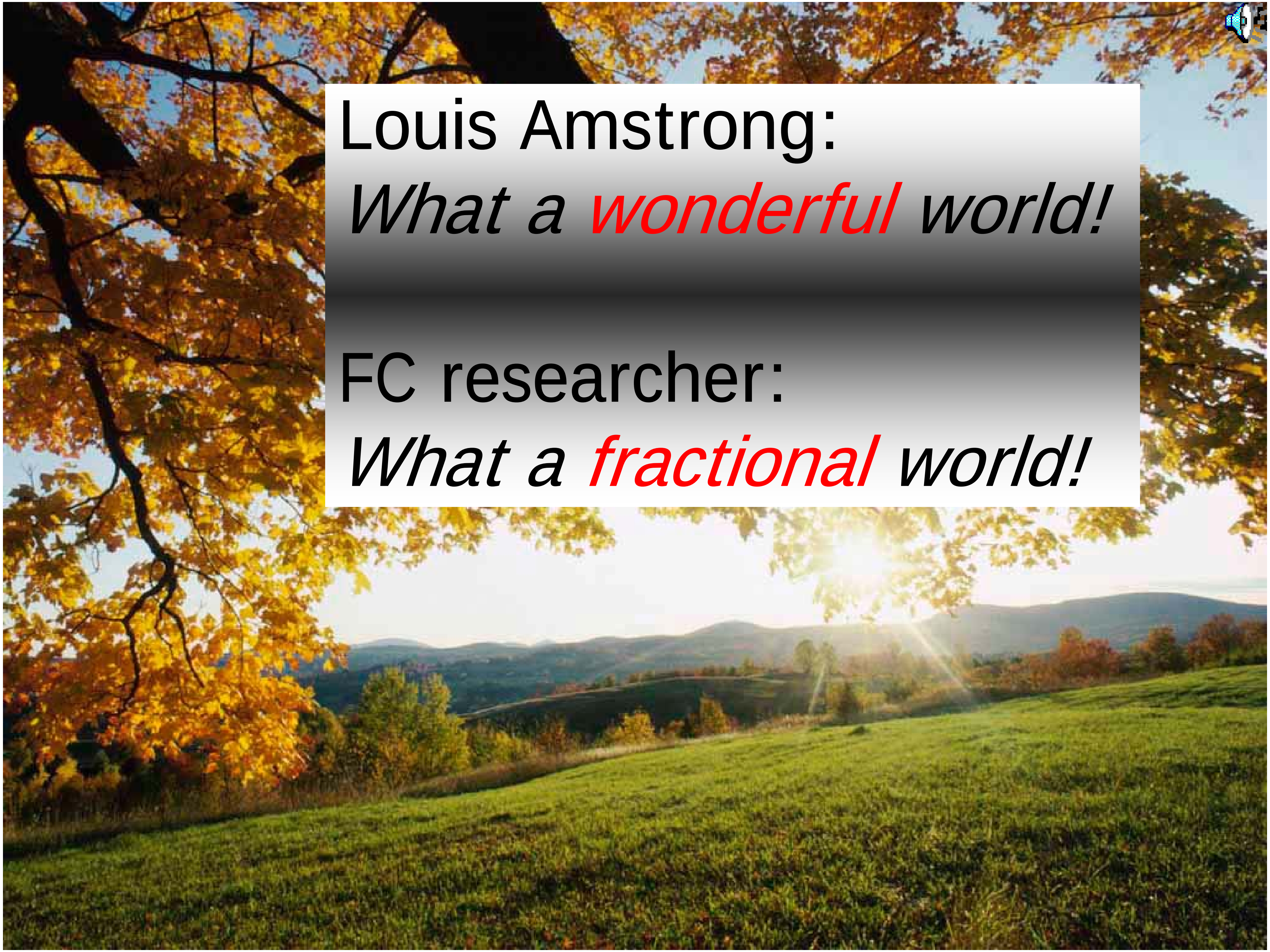
$$D^{\alpha} x(t) = \lim_{h \to 0} \left[\frac{1}{h^{\alpha}} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)} x(t-kh) \right]$$

$h \approx T$, T - sampling period:

$$\frac{Z\{D^{\alpha} x(t)\}}{X(z)} \approx \frac{1}{T^{\alpha}} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(\alpha+1)}{k!\, \Gamma(\alpha-k+1)} z^{-k} = \left(\frac{1-z^{-1}}{T} \right)^{\alpha}$$

Fraction approximation

$$\left(\frac{1-z^{-1}}{T} \right)^{1/2} \approx \frac{1}{T^{1/2}} \frac{-\frac{7}{64} z^{-3} + \frac{7}{8} z^{-2} - \frac{7}{4} z^{-1} + 1}{-\frac{1}{64} z^{-3} + \frac{3}{8} z^{-2} - \frac{5}{4} z^{-1} + 1}$$


Louis Amstrong:
What a wonderful world!
FC researcher:
What a fractional world!

Fractional physics...

- Spring
  - Hooke law

$$F = kx$$

- Viscous friction
  - Newton fluid

$$F = k\dot{x}$$

- Mass
  - Newton 2nd law

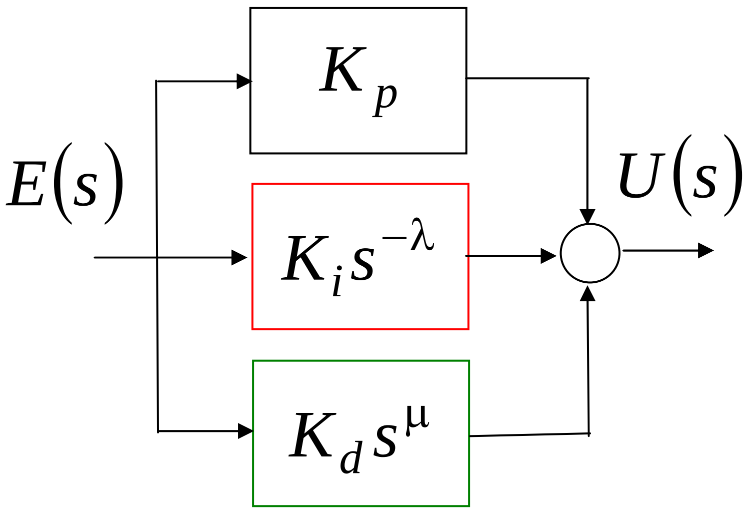
$$F = k\ddot{x}$$

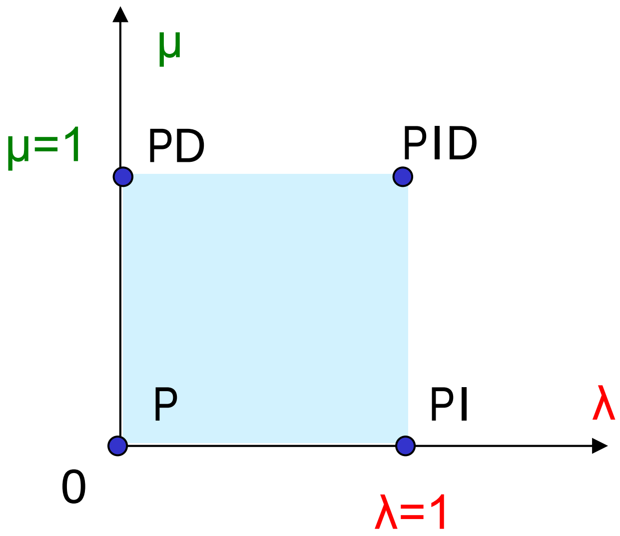
$$F = kx^{(\alpha)}$$

Fractional-Order Controllers

- The fractional-order PI$^{\lambda}$D$^{\mu}$ controller:

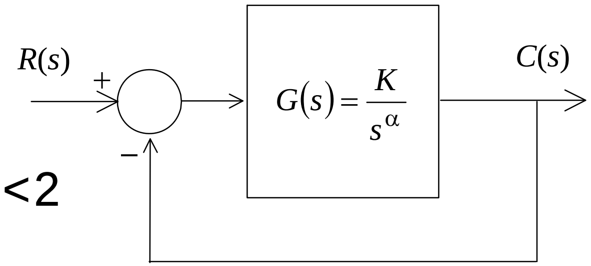
$$C(s) = \frac{U(s)}{E(s)} = K_p + K_i s^{-\lambda} + K_d s^{\mu}$$





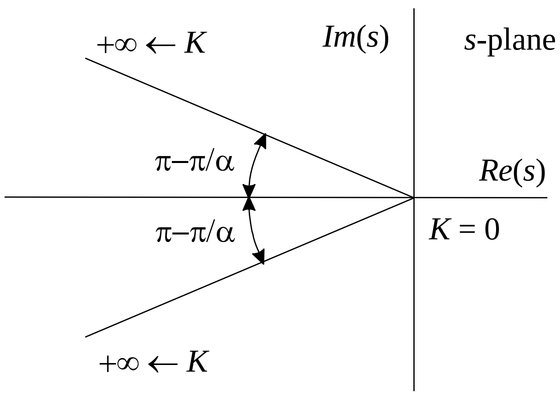
Control robustness

Feedback control system

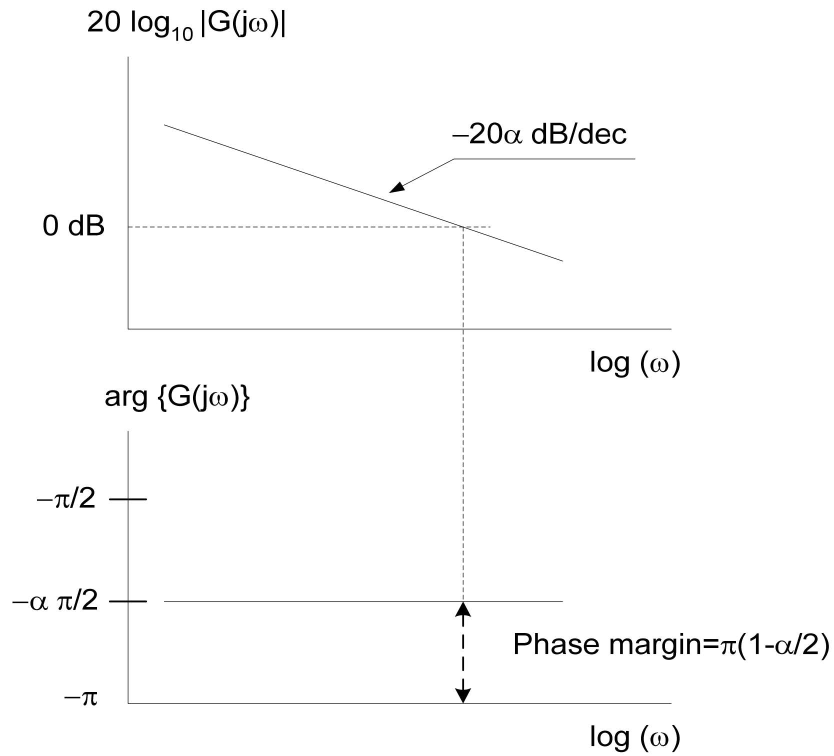


$1<\alpha<2$

Root-locus

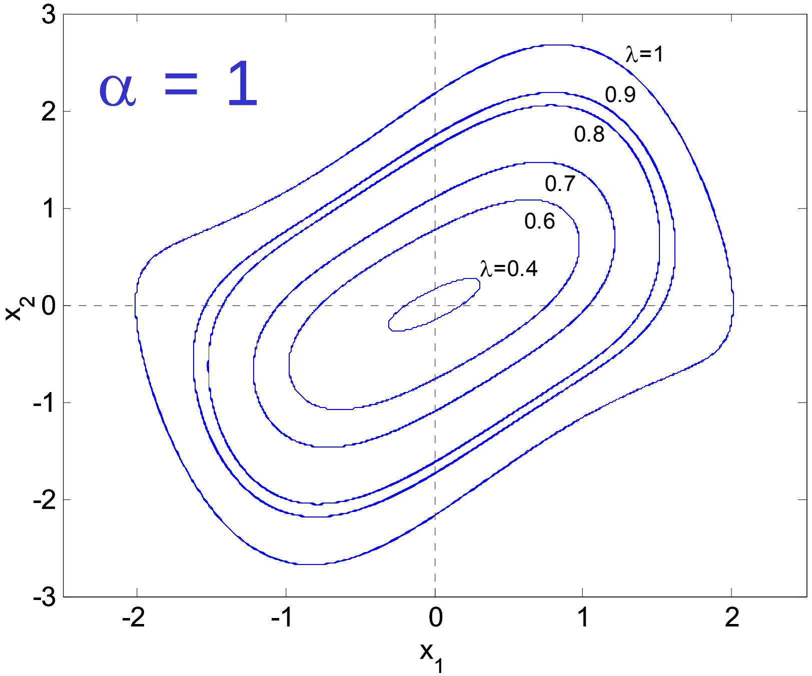


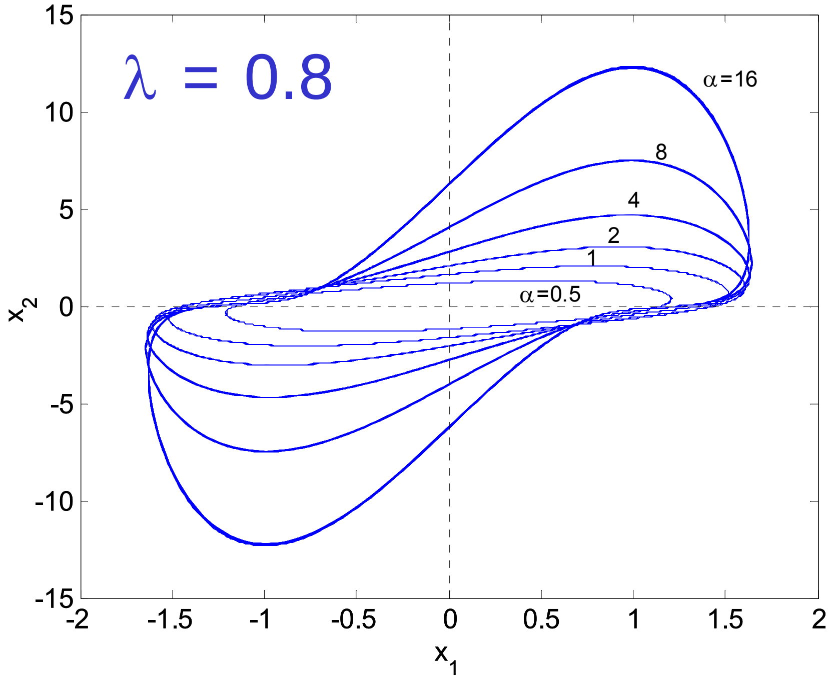
Bode diagrams



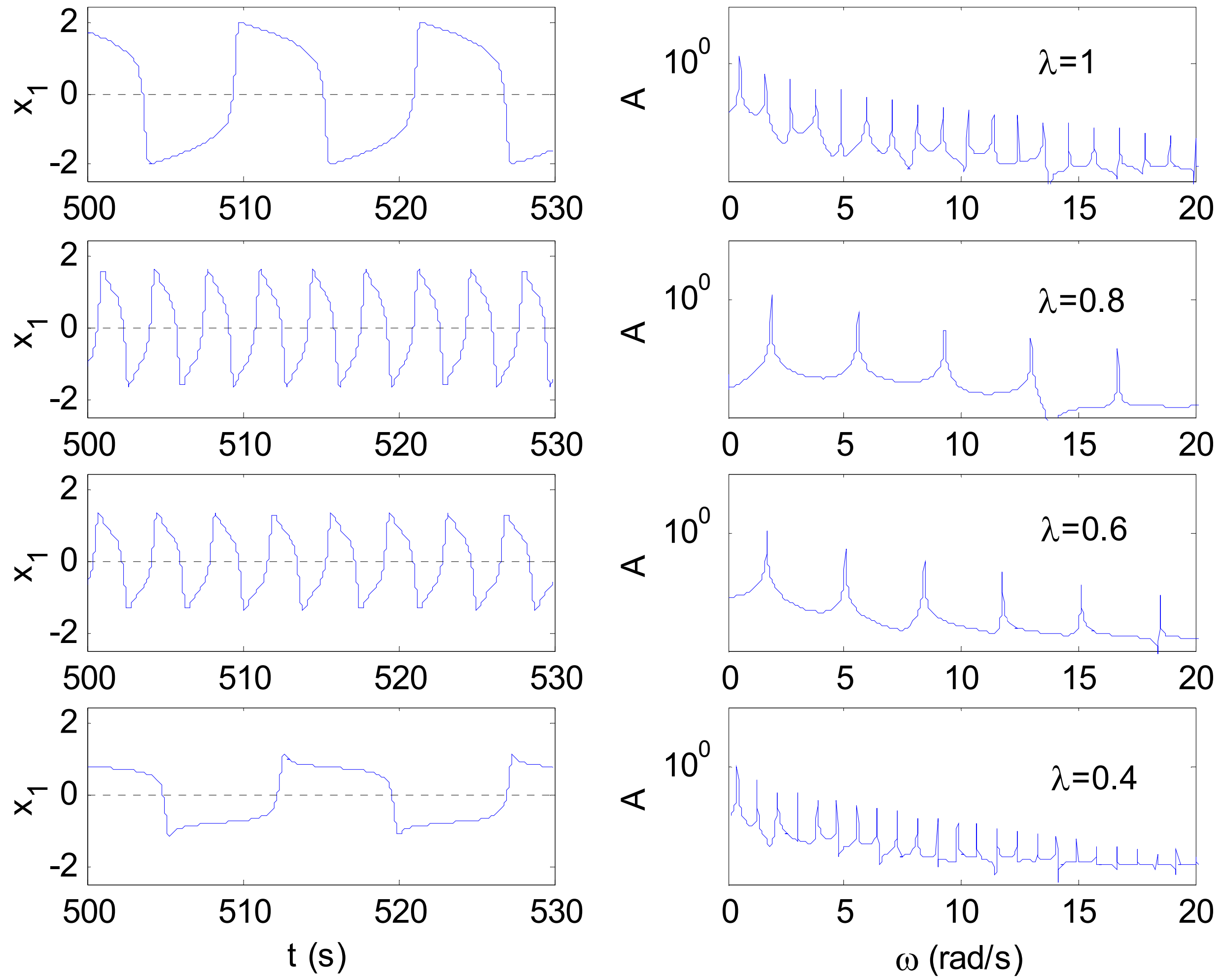
Van der Pol oscillator of fractional order

$$x^{(1+\lambda)} + \alpha\left(x^2 - 1\right)x^{(\lambda)} + x = 0, \quad 0 < \lambda < 1$$



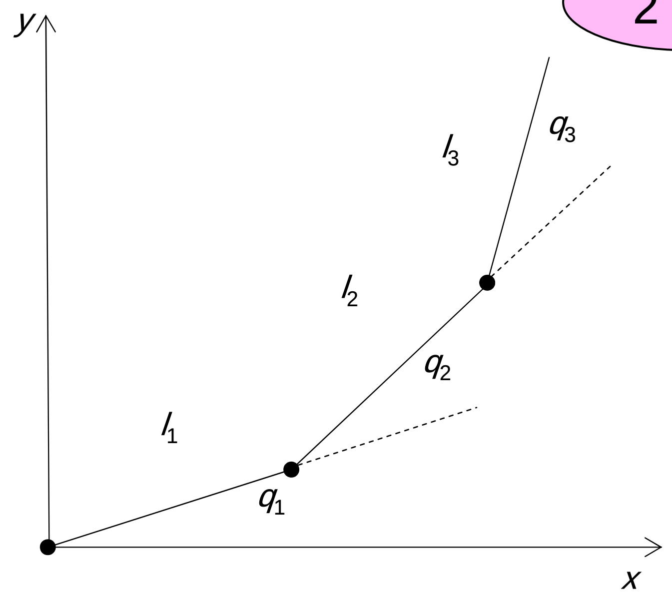


Time and frequency responses

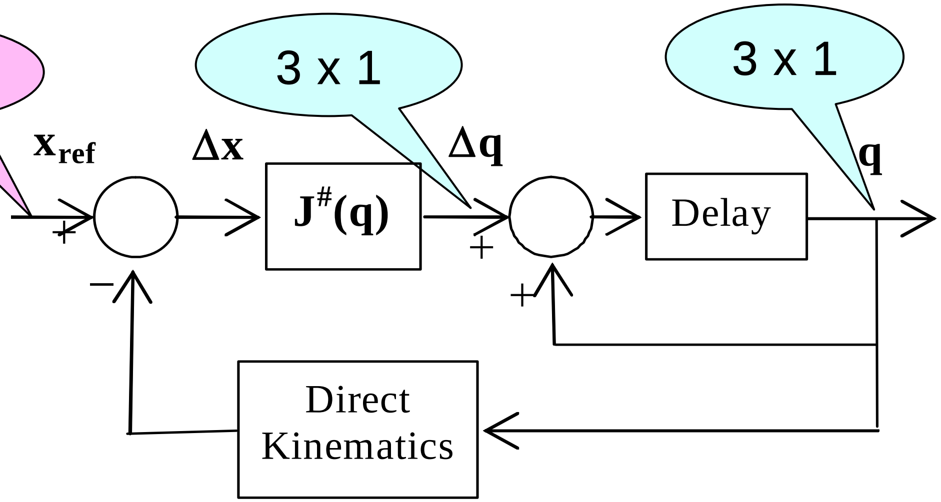


Redundant manipulators: Trajectory planning

- $3R$ planar redundant manipulator

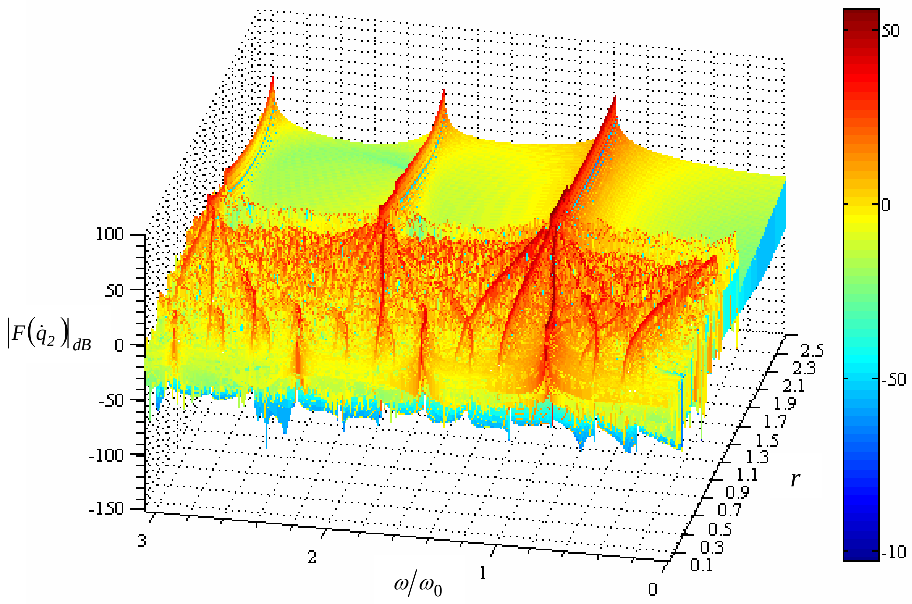


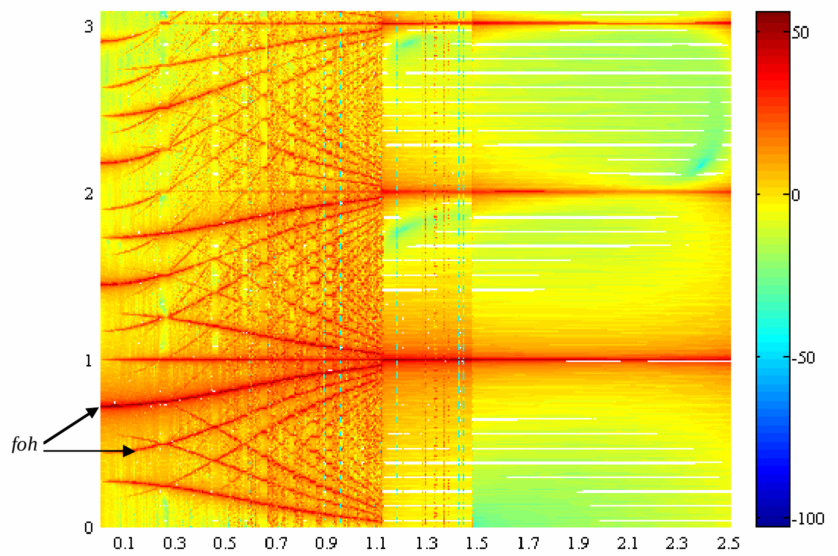
- Computation of joint positions



$\mathbf{J}^{\#}$ – generalized inverse

F{dq_2/dt} *vs* (r ,ω/ω_0) for ρ=0.5





FC - Books

- K. B. Oldham, J. Spanier, *The Fractional Calculus*, Academic Press, 1974.
- A. Oustaloup, *La Commande CRONE: Commande Robuste d'Ordre Non Entier*, Editions Hermès, 1991.
- K. S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley & Sons, 1993.
- S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional Integrals and Derivatives*. Gordon and Breach, 1993.
- I. Podlubny, *Fractional Differential Equations*, Academic Press, 1999.
- Ed. Jocelyn Sabatier, Om Agrawal, J. Tenreiro Machado (ed), *Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering*, Springer, 2007.

FC- Special Issues

- Nonlinear Dynamics, vol. 29, 1-4, July 2002.
  J. T. Machado (ed.)
- Signal Processing, vol. 83, 11, Nov. 2003.
  M. Ortigueira, J. T. Machado (ed.)
- Nonlinear Dynamics, vol. 38, 1-4, Dec. 2004.
  Om Agrawal, J. T. Machado, J. Sabatier (eds.)
- Signal Processing, vol. 86, 10, Oct. 2006.
  M. Ortigueira, J. T. Machado (ed.)
- ASME J. Computational and Nonlinear Dynamics, vol. 3, 2, April 2008.
  J. T. Machado, A. Luo (eds.)
- Journal of Vibration and Control, 2008.
  J. T. Machado, R. Barbosa (eds.)
- Journal Européen des Syst. Automatisés, 2008.
  J. Sabatier, P. Melchior, J. T. Machado, B. Vinagre (eds.)

FC - Conferences

- IFAC Workshops Fractional Differentiation and Their Applications
  - 1st FDA04, Bordeaux, France, 2004
  - 2nd IFAC FDA06, Porto, Portugal, 2006
  - 3rd IFAC FDA08, Ankara, Turkey, 2008
- ASME Symp. Fractional Derivatives and Their Applications
  - 1st FDTA, Chicago, USA, 2003
  - 2nd FDTA, Long Beach, USA, 2005
  - 3rd FDTA, Las Vegas, USA, 2007
- ASME Classic and Fractional Dynamics on Continuous and Discontinuous Vector Fields, Las Vegas, USA, 2007
- ENOC Mini Symp. Fractional Derivatives and Their Applications
  - 1th FDTA, Eindhoven, Netherlands, 2005
  - 2nd FDTA, Saint Petersburg, Russia, 2008