Fractional Calculus
& Applied Analysis
An International Journal for Theory and Applications
VOLUME 7, NUMBER 2 (2004) ISSN 1311-0454

PID CONTROLLER TUNING USING FRACTIONAL CALCULUS CONCEPTS

Ramiro S. Barbosa *, **J. A. Tenreiro Machado** *,
Isabel M. Ferreira **

Abstract

In this paper, we present a new approach for tuning PID controllers. The proposed method is based on the application of basic fractional calculus concepts. In fact, the controller specifications include the desired gain crossover frequency and the slope at that frequency (which is equivalent to prescribing a specific phase margin) of a fractional-order integrator inserted in the forward path of a unit feedback control system. The PID parameters are obtained by minimizing the integral of square error (ISE) between the step responses of the ideal closed-loop system (with the fractional-order integrator) and that of the actual closed-loop system with the PID controller. The obtained closed-loop system is robust to gain variations with step responses exhibiting an iso-damping property. Simulation examples are given to illustrate the effectiveness and applicability of the proposed scheme.

Mathematics Subject Classification: 26A33 (main), 93C15, 93C55, 93C80
Key Words and Phrases: fractional calculus, fractional-order systems, control theory, PID tuning, ISE optimization

1. Introduction

In the recent years, fractional calculus (FC) has been a fruitful field of research with application in many areas of science and engineering [1] - [4]. In fact, many scientific areas are currently paying attention to the FC concepts and we can refer its adoption in viscoelasticity and damping, diffusion and wave propagation, electromagnetism, chaos and fractals, heat transfer, biology, electronics, signal processing, system identification, percolation, modelling and identification, chemistry, irreversibility, physics, economy and finance [1] - [6]. In what concerns the area of automatic control systems the application of the FC concepts is still scarce and only in the last two decades appeared the first applications [5] - [11].

The type of controller usually adopted in industrial systems is the Proportional plus Integral plus Derivative (PID) controller. Among the various existing schemes for tuning PID controllers, the Ziegler-Nichols (Z-N) method is the most popular and is still extensively used for the determination of the PID parameters. It is well known that the compensated systems, with controllers tuned by this method, have generally a step response with a high percent overshoot. Moreover, the Z-N heuristics are only suitable for plants with monotonic step response. Therefore, other approaches were developed such as root-locus based techniques [12] and methods based on the minimization of some performance criterion [12], [17]. Also, many of these alternative schemes do not impose any constraint on the maximum value for the response overshoot.

In this paper, we propose a new methodology for tuning PID controllers such that the response of the compensated system has an almost constant overshoot defined by a prescribed value. The proposed method is based on the minimization of the integral of square error (ISE) between the step responses of a unit feedback control system, whose open-loop transfer function is given by a fractional-order integrator, and of the PID compensated system. The controller specifications consist on the gain crossover frequency and the slope at that frequency of the fractional-order integrator. In this perspective, we can ensure the nearly flatness of the phase curve around the gain crossover frequency of the compensated system. This implies that the system will be more robust to gain variations, exhibiting step responses with an almost constant overshoot, that is, with an iso-damping property.

Having these ideas in mind, the article is organized as follows. Section 2 reviews the fundamental aspects of fractional calculus. Section 3 gives some concepts of fractional-order control and proposes a fractional-order transfer function that will be used as reference system for PID controller tuning. Motivated by these results, Sections 4 and 5 establish a methodology for tuning PID controllers and present various simulation examples testing the effectiveness of the proposed method, respectively. Finally, Section 6 draws the main conclusions and addresses perspectives towards future developments.

2. Basics of fractional calculus

The area of fractional calculus has his birth at the same time than the classical theory of differential calculus. Since its foundation the generalization of a derivative and integral to a fractional order (more precisely, noninteger order of rational, irrational or even complex value) has been the subject of several approaches. Due to this reason there are various alternative definitions of fractional-order derivatives and integrals [1] - [3]. Nevertheless, from the control point of view some definitions seem more attractive, namely when thinking in a discrete-time implementation. Oldham and Spanier [1] define the Grünwald-Letnikov approximation to fractional-order derivatives and integrals as the most fundamental definition among all the others, since it unifies on a single operator the notions of derivative and integral and puts the fewest restrictions on the functions to which it applies. It is given by the following expression [1], [3]:

$$D^{\gamma} f(t) = \lim_{h \to 0} \left\{ \frac{1}{h^{\gamma}} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\gamma+1)}{\Gamma(k+1)\Gamma(\gamma-k+1)} f(t-kh) \right\}, \quad \gamma \in \mathrm{R}, \tag{1}$$

where $\Gamma(z)$ is the Gamma function, h is the time increment and $f(t)$ is the applied function. This definition reveals that while integer-order derivatives imply a finite series, the fractional-order derivatives require an infinite number of terms. This means that integer-order derivatives are "local" operators in opposition with the fractional-order derivatives that are "global" operators having a memory of all past function values.

In the analysis and synthesis of automatic control systems is usual the application of the Laplace-based analysis methods. Fortunately, the adaptation of these methods to a fractional-order is straightforward. In fact, the Laplace definition of a derivative and integral of fractional order γ of the signal $f(t)$, $D^{\gamma} f(t)$, under null initial conditions, is giving by the generalisation of the classical integer-order scheme, yielding:

$$L\{D^{\gamma} f(t)\} = s^{\gamma} L\{f(t)\}, \quad \gamma \in \mathrm{R}. \tag{2}$$

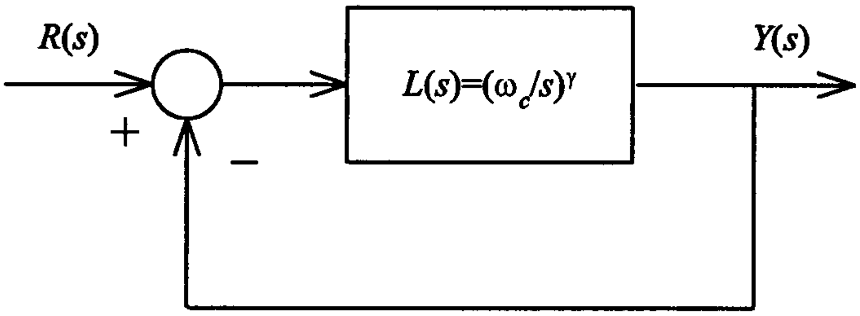

Figure 1: Fractional-order control system with open-loop transfer function $L(s)$.

This expression implies that the classical frequency-based methods have a direct adaptation to fractional-order control systems.

Although the area of fractional calculus goes back to the beginning of the theory of differential calculus its inherent complexity postponed the application of the associated concepts. In the last two decades, this area has been an active field of research with application in many areas of science and engineering [1] - [4]. However, this work is still giving its first steps and, consequently, many aspects remain to be investigated.

3. Concepts of fractional-order control

In this section we present the fundamental characteristics of the fractional-order system that will be used as reference model for the tuning of PID controllers. The model is a unit feedback control system with open-loop transfer function $L(s)$ given by a fractional-order integrator (Figure 1).

The open-loop transfer function $L(s)$ is defined as:

$$L(s) = \left(\frac{\omega_c}{s}\right)^{\gamma}, \quad \gamma \in \mathrm{R}^{+}, \tag{3}$$

where ω_c is the gain crossover frequency, that is, $|L(j\omega_c)| = 1$. The parameter γ is the slope of magnitude curve on log-log scale, and may assume integer as well noninteger values. In this study we consider $1 < \gamma < 2$. This transfer function is also known as the Bode's ideal loop transfer function since Bode study's on design of feedback amplifiers in the 1940's [14].

The Bode diagrams of amplitude and phase of $L(s)$ are illustrated in Figure 2. The amplitude curve is a straight line of constant slope -20γ

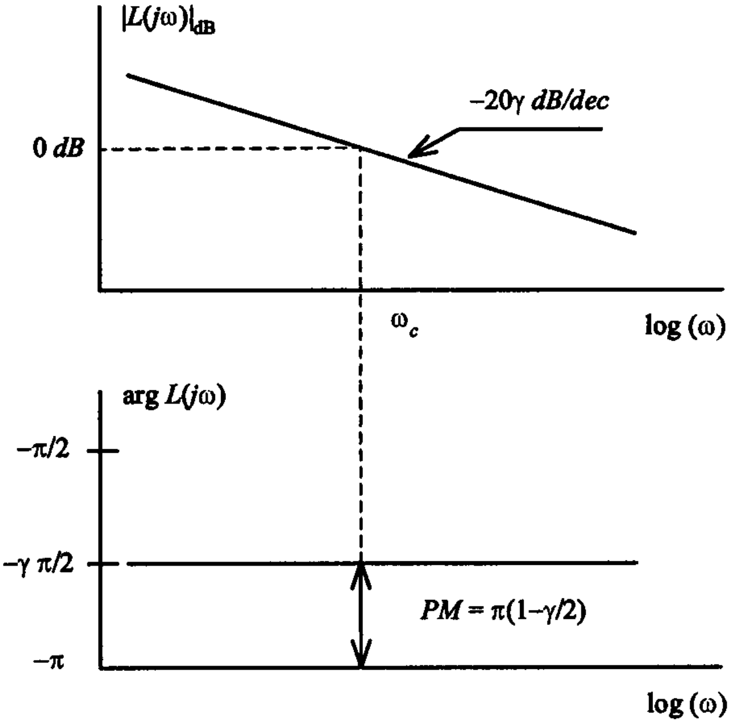

Figure 2: Bode diagrams of amplitude and phase of $L(j\omega)$ for $1 < \gamma < 2$.

dB/dec, and the phase curve is a horizontal line at $-\gamma\pi/2$ rad. The Nyquist curve is simply the straight line through the origin, arg $L(j\omega) = -\gamma\pi/2$ rad.

This choice of $L(s)$ gives a closed-loop system with the desirable property of being insensitive to gain changes. If the gain changes the crossover frequency ω_c will change but the phase margin of the system remains PM $= \pi(1 - \gamma/2)$ rad, independently of the value of the gain. This can be seen from the curves of amplitude and phase of Figure 2.

In the sequel we consider the fundamental characteristics of the time and frequency responses of the fractional-order control system represented in Figure 1. The closed-loop transfer function, $T(s) = Y(s)/R(s)$, is given by:

$$T(s) = \frac{L(s)}{1 + L(s)} = \frac{1}{\left({}^{s}\!/_{\omega_c}\right)^{\gamma} + 1}, \quad \gamma \in \mathrm{R}^{+}. \tag{4}$$

The Bode diagrams of amplitude and phase of fractional-order transfer function (4), $|T(j\omega)|_{dB}$ and arg $[T(j\omega)]$), are given by:

$$|T(j\omega)|_{dB} = 20\log_{10}\frac{1}{\sqrt{\left({}^{\omega}\!/_{\omega_c}\right)^{2\gamma} + 2\left({}^{\omega}\!/_{\omega_c}\right)^{\gamma}\cos(\gamma\pi/2) + 1}}, \tag{5}$$

$$\arg\left[T(j\omega)\right] = -\arctan\left[\frac{\sin(\gamma\pi/2)}{\cos(\gamma\pi/2) + (^{\omega_c}/_{\omega})^{\gamma}}\right]. \tag{6}$$

Considering the asymptotic behaviour, as $\omega \to +\infty$, of expressions (5) and (6), we have ($\gamma > 0$):

$$|T(j\omega)|_{dB} \approx -20\gamma \log_{10}\left(\frac{\omega}{\omega_c}\right), \quad \arg\left[T(j\omega)\right] \approx -\gamma\pi/2. \tag{7}$$

Hence, at high frequencies the asymptotes of magnitude and phase are given by straight lines of -20γ dB/dec and $-\gamma\pi/2$ rad, respectively. The low frequency behaviour approaches asymptotically the horizontal straight lines 0 dB and 0 rad, correspondingly to the magnitude and phase shift.

The resonance peak M_r and the frequency ω_r at which it occurs are given by the formulae:

$$M_r = \frac{1}{\sin(\gamma\pi/2)}, \quad \omega_r = \omega_c \left|\cos(\gamma\pi/2)\right|^{1/\gamma}. \tag{8}$$

The unit step response of $T(s)$ is given by the expression:

$$y(t) = L^{-1}\left\{\frac{\omega_c^{\gamma}}{s(s^{\gamma} + \omega_c^{\gamma})}\right\} = 1 - \sum_{n=0}^{\infty} \frac{[-(\omega_c t)^{\gamma}]^n}{\Gamma(1+\gamma n)} = 1 - E_{\gamma}\left[-(\omega_c t)^{\gamma}\right], \tag{9}$$

where $E_{\gamma}(x)$ is the one-parameter Mittag-Leffler function [2, 15]. This function is a generalization of the common exponential function since for $\gamma = 1$ we have $E_1(x) = e^x$.

Considering the asymptotic behaviour of $E_{\gamma}[-(\omega_c t)^{\gamma}]$, when $t \to +\infty$ and $t \to 0^+$, as [15]:

$$E_{\gamma}\left[-(\omega_c t)^{\gamma}\right] \approx \begin{cases} 1 - \frac{(\omega_c t)^{\gamma}}{\Gamma(\gamma+1)} & , \ \omega_c t \to 0^+ \\ \frac{(\omega_c t)^{-\gamma}}{\Gamma(1-\gamma)} & , \ \omega_c t \to +\infty \end{cases}, \tag{10}$$

we arrive to the final and initial values of the step response, $y(t \to +\infty)$ and $y(t \to 0^+)$, respectively:

$$y(\infty) = \lim_{t \to +\infty} y(t) = 1, \quad y(0^+) = \lim_{t \to 0^+} y(t) = 0. \tag{11}$$

Specifications for a control system design often involve certain requirements associated with the system time response. In the sequel, we derive some useful formulae to characterize the time response of the fractional-order transfer function $T(s)$. Like in the case of second-order systems we develop expressions for the overshoot M_p, peak time T_p, rise time T_r, time constant T_c and settling time T_s. Figure 3 shows the normalized step responses of $T(s)$ for $1 < \gamma < 2$. These variables can be approximated numerically leading to the following expressions:

- The overshoot M_p:

$$M_p = \frac{y_{\max} - y(\infty)}{y(\infty)}, \quad M_p \approx 0.8(\gamma - 1)(\gamma - 0.75), \quad 1 < \gamma < 2. \tag{12}$$

- The peak time T_p is the time at which the overshoot occurs (with an error smaller than 1%):

$$T_p \approx \frac{1.106(\gamma - 0.255)^2}{(\gamma - 0.921)\omega_c}, \quad 1 < \gamma < 2. \tag{13}$$

- The rise time T_r is the time for the response to evolve from 0.1 up to 0.9 of its final value (with an error smaller than 1%):

$$T_r \approx \frac{0.131(\gamma + 1.157)^2}{(\gamma - 0.724)\omega_c}, \quad 1 < \gamma < 2. \tag{14}$$

- The time constant T_c is the time required for the response to rise up to 63% of its final value (with an error smaller than 2%):

$$T_c \approx \frac{0.2(\gamma - 1)^2 + 1}{\omega_c}, \quad 1 < \gamma < 2. \tag{15}$$

- The settling time T_s is the time required for the response to settle within a small fraction of its steady state value and to stay there (for 2% and 5% criteria):

$$T_s(2\%) \approx \frac{4}{\cos(\pi - \pi/\gamma)\omega_c} = \frac{4}{\zeta\omega_c}, \quad 1.39 < \gamma < 2. \tag{16}$$

$$T_s(5\%) \approx \frac{3}{\cos(\pi - \pi/\gamma)\omega_c} = \frac{3}{\zeta\omega_c}, \quad 1.44 < \gamma < 2, \tag{17}$$

where $\zeta = \cos(\pi - \pi/\gamma)$ is the damping ratio of the fractional closed-loop system. Note the similarities of these formulae with those obtained for the underdamped second-order system ($0 < \zeta < 1$). In fact, some attempts have been made to establish relations between the two systems [13].

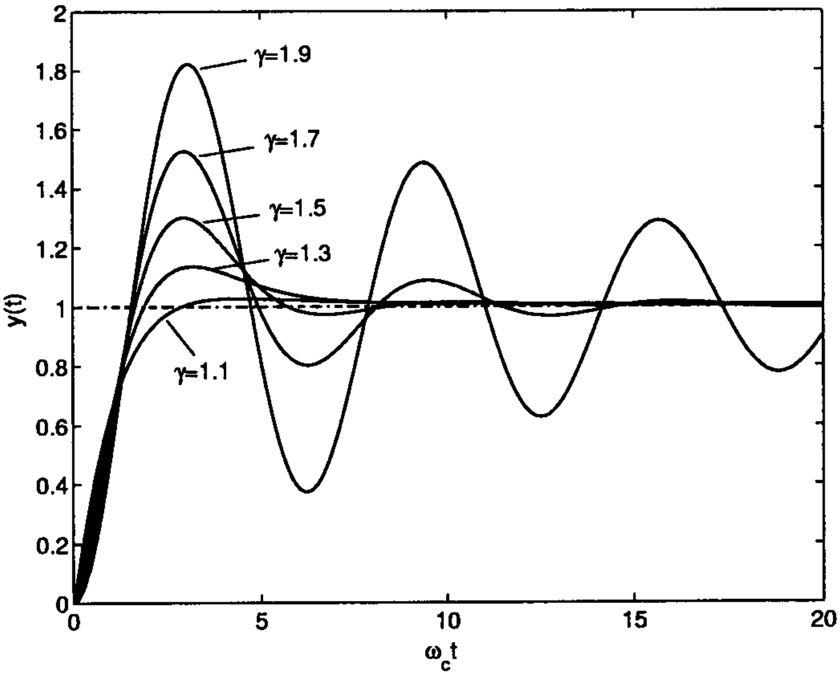

Figure 3: Unit step responses of $T(s)$ for several values of $1 < \gamma < 2$.

4. Tuning of PID controllers

In this section we address the closed-loop system with the fractional-order integrator $L(s) = (\omega_c/s)^\gamma$ in the forward path (Figure 1) as the reference system for PID tuning [16]. With the order γ and the gain crossover frequency ω_c we can establish the overshoot and the speed of the output response, respectively. For that purpose we consider the closed-loop system shown in Figure 4, where $G_c(s)$ and $G_p(s)$ are the PID controller and the plant transfer functions, respectively. The system may be subjected both for setpoint and disturbance signals, correspondingly $r(t)$ and $p(t)$.

The ideal time-domain equation of a PID controller has the following form:

$$u(t) = K\left(e(t) + \frac{1}{T_i}\int_0^t e(\tau)d\tau + T_d\frac{d}{dt}e(t)\right), \tag{18}$$

where $u(t)$ and $e(t)$ denote the control and the error signals, respectively. The set of variables (K, T_i, T_d) are the parameters to be tuned and are correspondingly the proportional gain, the integral time constant and the derivative time constant of the PID controller.

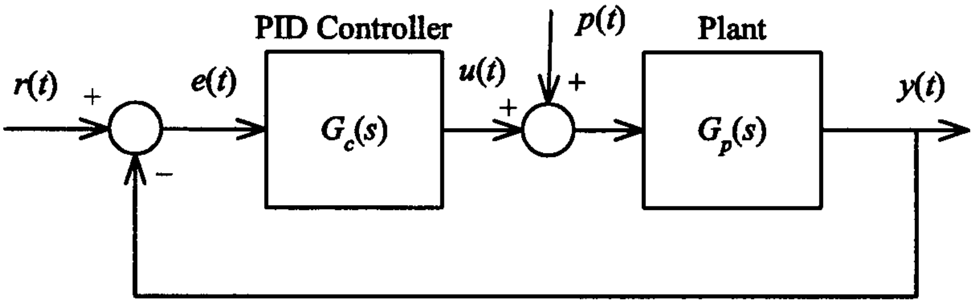

Figure 4: Closed-loop system with PID controller $G_c(s)$.

The corresponding transfer function is given by:

$$G_c(s) = \frac{U(s)}{E(s)} = K\left(1 + \frac{1}{T_i s} + T_d s\right). \tag{19}$$

In order to reduce the control effort or any existent high frequency measurement noise, a practical PID controller should implement the derivative term sT_d by a band-limited differentiator $sT_d/(1+sT_d/N)$, where usually $3 \le N \le 20$ [12].

The design of the PID controller will consist on the determination of the optimum PID set gains (K, T_i, T_d) that minimizes some performance criterion $J(K, T_i, T_d)$. In this study it was adopted the minimization of the integral of the square error (ISE), which is defined as:

$$J(K, T_i, T_d) = \int_0^\infty [y(t) - y_d(t)]^2 \, dt, \tag{20}$$

where $y(t)$ is the step response of the closed-loop system with the PID controller (Figure 4) and $y_d(t)$ is the desired step response of the fractional-order transfer function (4) given by expression (9). For the case under study, the order γ of equation (4) may assume real noninteger values such that $1 < \gamma < 2$.

5. Simulation examples

We apply the proposed methodology to the following plant transfer functions:

$$G_{pn}(s) = \frac{1}{(s+1)^n}, \quad n = 2, 3, 4. \tag{21}$$

Plant	K	T_i	T_d
$n=2$	0.4519	0.4452	1.0510
$n=3$	1.1562	1.0216	1.1034
$n=4$	1.6934	1.4227	1.5594

Table 1: PID parameters for the plants $G_{pn}(s) = 1/(s+1)^n$, $n = 2, 3, 4$. Proposed tuning methodology with $\gamma = 1.5$, $\omega_c = 0.6$ rad/s.

Figure 5 shows the time responses of the closed-loop system with the PID controller. The system was tuned for $\gamma = 1.5$ (PM = 45^o) and $\omega_c = 0.6$ rad/s and the corresponding PID parameters are shown on Table 1.

We verify that the step setpoint responses $y_r(t)$ are very similar, even for the highest order plant with $n = 4$. More precisely, the responses maintain an almost constant overshoot. On the other hand, and as expected, the step disturbance responses $y_p(t)$ differ from each other. This occurs because the PID tuning is established for setpoint changes. The resulting approximation can be viewed in the frequency domain through the Bode diagrams as illustrated in Figure 6. The graphs reveal that the phase curve is nearly flat around the gain crossover frequency $\omega_c = 0.6$ rad/s and that the system has a phase margin of approximately 45^o. Nevertheless, the almost constant phase margin is only verified in a limited frequency interval around the gain crossover frequency ω_c. Therefore, we can expect that the closed-loop system with the PID controller, tuned by the proposed method, is more robust to gain variations, exhibiting step responses with an almost constant overshoot (that is, with an iso-damping property) around ω_c. We will show this property in the next figures.

We must point out that the desirable iso-damping property will depend on the type of the plant and on the required specifications, namely the desired phase margin PM and in the gain crossover frequency ω_c at which it occurs.

To further illustrate the effectiveness of the proposed methodology we tune the PID controller for different specifications of (γ, ω_c). We plot the curves for gain variations around the nominal gain ($K_p = 1$) corresponding to $K_p = \{0.6, 0.8, 1.0, 1.2, 1.4\}$, that is, for a variation up to $\pm 40\%$ of its nominal value. In Figures 7 and 8 it is shown the step responses and the Bode diagrams of phase of $G_{p3}(s) = 1/(s+1)^3$, for $\gamma = 3/2$ (PM = 45^o), $\omega_c = 0.8$ rad/s, and $G_{p4}(s) = 1/(s+1)^4$, for $\gamma = 4/3$ (PM = 60^o), $\omega_c = 0.5$

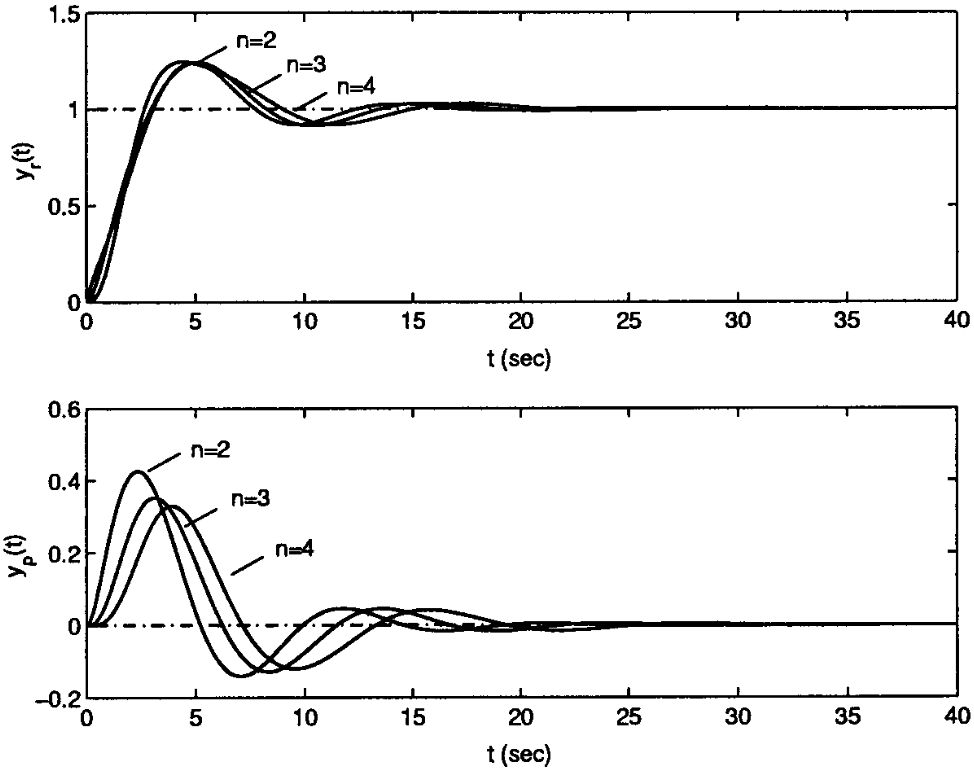

Figure 5: Step setpoint response $y_r(t)$ and step disturbance response $y_p(t)$ for the closed-loop system with a PID controller (tuned by the proposed methodology) and $G_{pn}(s) = 1/(s+1)^n$, $n = 2, 3, 4$. The desired specifications are $\gamma = 1.5$ (PM = 45^o) and $\omega_c = 0.6$ rad/s.

rad/s, respectively. In both cases, we get the same desired iso-damping property corresponding to the prescribed (γ, ω_c)-values. In fact, we observe that the step responses have an almost constant overshoot independently of the variation of the plant gain around the gain crossover frequency ω_c.

In conclusion, with the proposed algorithm we are capable of producing closed-loop systems robust to gain variations and step responses exhibiting an iso-damping property. However, it must be noted that this feature is limited to a frequency interval around ω_c and that the phase margin (PM) of the resulting closed-loop system is not exactly identical to the prescribed value defined by γ. This is due to a non perfect match between the desired and the obtained step responses (in fact, they are approximated in a least squares sense).

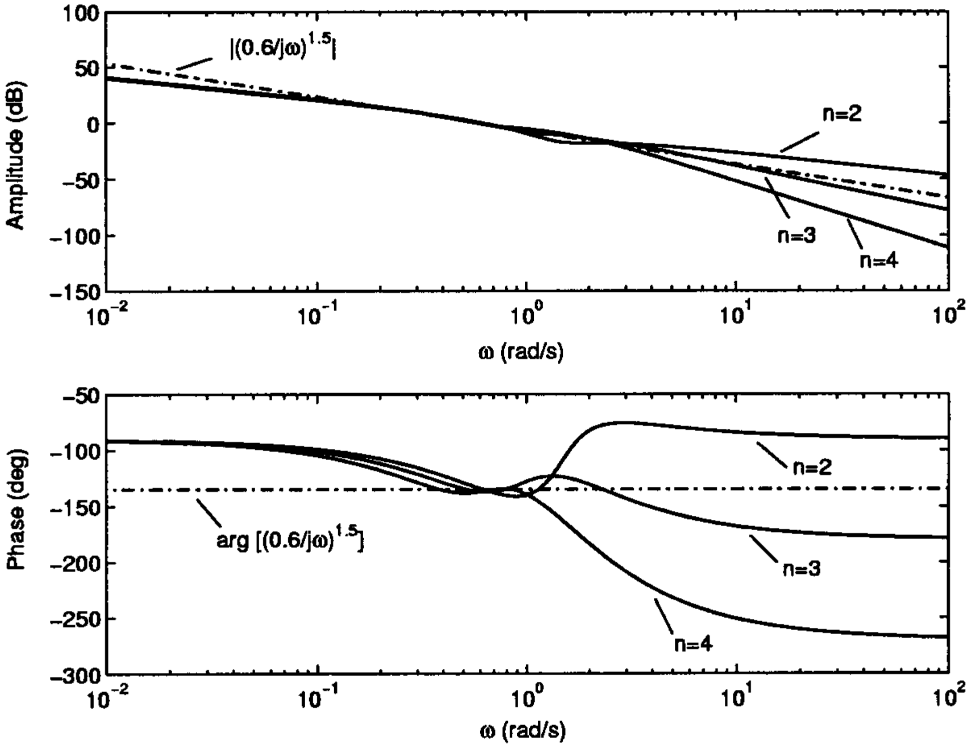

Figure 6: Bode diagrams of open-loop system with a PID controller (tuned by the proposed methodology) for $G_{pn}(s) = 1/(s+1)^n$, $n = 2, 3, 4$. The desired specifications are $\gamma = 1.5$ (PM = 45°) and $\omega_c = 0.6$ rad/s. For comparison purposes we also plot the corresponding ideal fractional-order integrator $(0.6/j\omega)^{1.5}$.

6. Conclusions

In this article we have presented a new approach for the tuning of PID controllers. The proposed method is based on the application of the fundamental concepts associated with fractional calculus. In fact, given the desired gain crossover frequency and the slope at that frequency (which is equivalent to prescribing a specific phase margin), and by minimizing a performance criterion like the integral of square error (ISE), we can ensure that the phase around the gain crossover frequency is nearly flat. Assuring this feature, we obtain closed-loop systems more robust to gain variations and step responses exhibiting an almost iso-damping property.

We have presented the fundamental time and frequency characteristics of the fractional-order system used as reference system for PID tuning. In this line of thought, were derived several formulae which can be used to

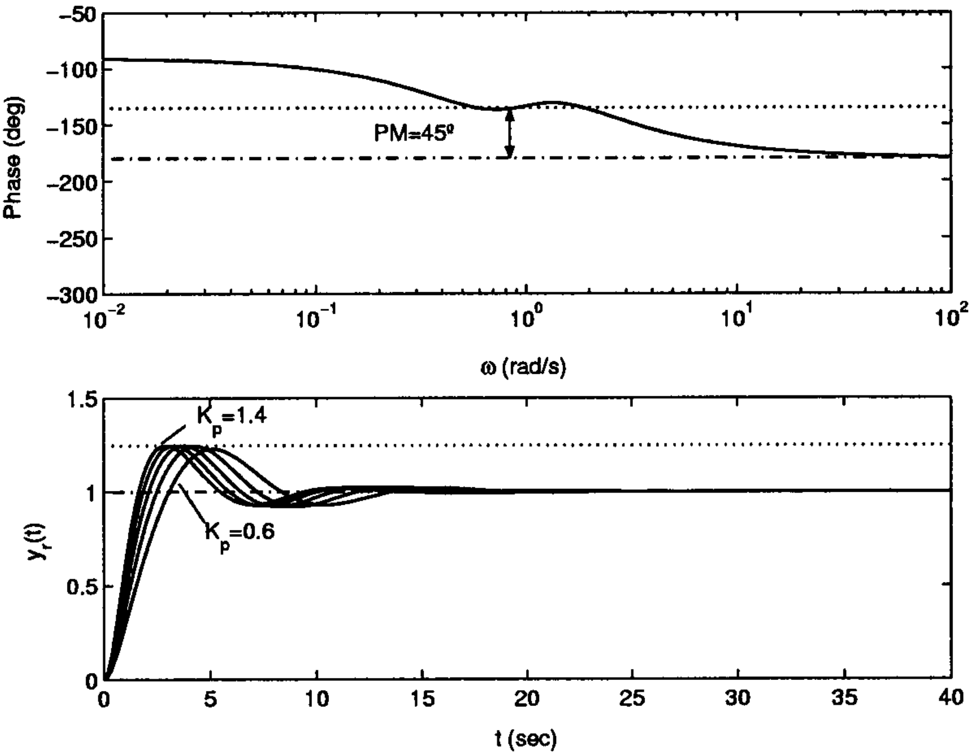

Figure 7: Bode phase diagram and setpoint step response $y_r(t)$ for the closed-loop system with a PID controller (tuned by the proposed methodology) for $G_{p3}(s) = 1/(s+1)^3$ and $\gamma = 3/2$ (PM $= 45^o$), $\omega_c = 0.8$ rad/s. The PID parameters are $K = 1.9158$, $T_i = 1.1407$ and $T_d = 0.9040$.

characterize the time response of the fractional-order system. We apply the methodology on several case studies that illustrates its effectiveness and applicability. It is interesting to note that although the resulting closed-loop system, with the PID controller, is an integer-order system it can be worked to yield fractional dynamics (over a limited range). This will allow a new point of view over the tuning of PID controllers as well the development of more robust tuning rules for the PID scheme.

Further research on this topic may include the development of a more systematic approach, leading to the establishment of expressions for the PID gains (K, T_i, T_d) as functions of the fractional parameters (γ, ω_c) and the definition of their range of applicability. Extension of the proposed methodology to other types of plants, including nonminimum phase systems, will be also a matter of interest.

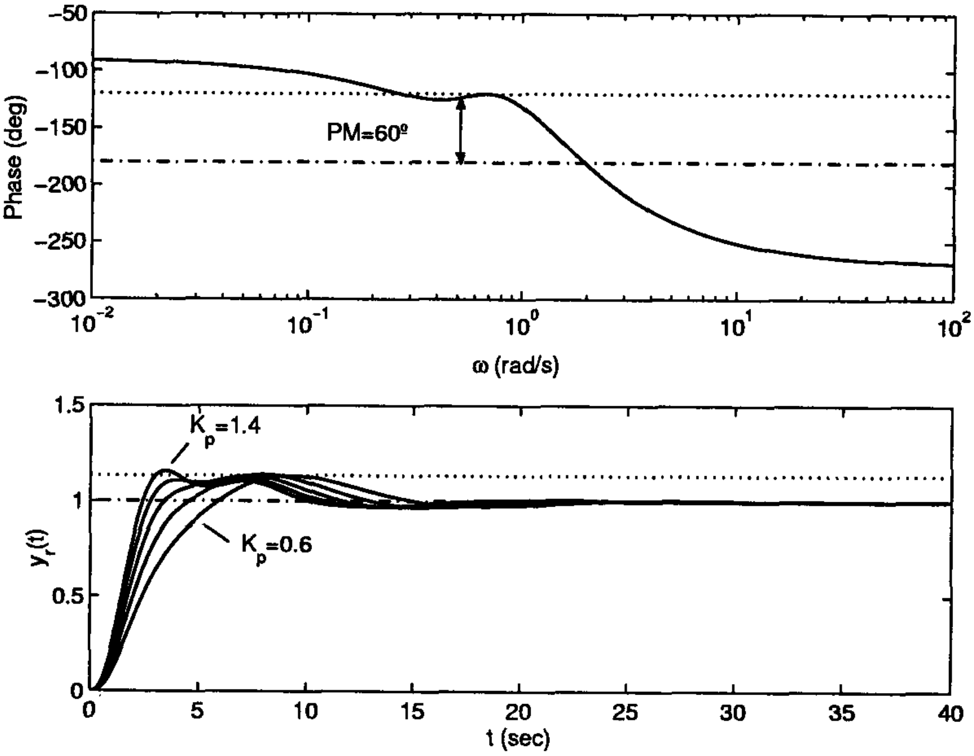

Figure 8: Bode phase diagram and setpoint step response $y_r(t)$ for the closed-loop system with a PID controller (tuned by the proposed methodology) for $G_{p4}(s) = 1/(s+1)^4$ and $\gamma = 4/3$ (PM $= 60^o$), $\omega_c = 0.5$ rad/s. The PID parameters are $K = 1.3774$, $T_i = 1.7030$ and $T_d = 1.7187$.

References

[1] K. B. Oldham and J. Spanier, *The Fractional Calculus: Theory and Application of Differentiation and Integration to Arbitrary Order.* Academic Press, New York (1974).

[2] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations.* Wiley & Sons, New York (1993).

[3] I. Podlubny, *Fractional Differential Equations.* Academic Press, San Diego (1999).

[4] R. Hilfer, *Applications of Fractional Calculus in Physics.* World Scientific, Singapore (2000).

[5] A. Oustaloup, *La Commande CRONE: Commande Robuste d'Ordre Non Entier.* Editions Hermès, Paris (1991).

[6] A. Oustaloup, *La Dérivation Non Entière: Théorie, Synthèse et Applications.* Editions Hermès, Paris (1995).

[7] J. A. Tenreiro Machado, Analysis and design of fractional-order digital control systems. *SAMS Journal Systems Analysis, Modelling, Simulation* **27** (1997), 107-122.

[8] J. A. Tenreiro Machado, Discrete-time fractional-order controllers. *Fractional Calculus & Applied Analysis* **4**, No 1 (2001), 47-66.

[9] B. M. Vinagre, I. Podlubny, A. Hernández, and V. Feliu, Some approximations of fractional order operators used in control theory and applications. *Fractional Calculus & Applied Analysis* **3**, No 3 (2000), 231-248.

[10] I. Podlubny, Fractional-order systems and $PI^{\lambda}D^{\mu}$-controllers. *IEEE Transactions on Automatic Control* **44**, No 1 (1999), 208-213.

[11] Ch. Lubich, Discretized fractional calculus. *SIAM Journal Math. Analysis* **17**, No 3 (1986), 704-719.

[12] K. Åström and T. Hägglund, *PID Controllers: Theory, Design, and Tuning.* Instrument Society of America, North Carolina (1995).

[13] S. Manabe, The non-integer integral and its aplication to control systems. *ETJ of Japan* **6**, No 3/4 (1961), 83-87.

[14] H. W. Bode, *Network Analysis and Feedback Amplifier Design.* Van Nostrand, New York (1945).

[15] R. Gorenflo and F. Mainardi, Fractional oscillations and Mittag-Leffler functions. *Pre-print A-14/96* , Free University of Berlin, Berlin (1996), 1-22.

[16] Ramiro S. Barbosa, J. A. Tenreiro Machado, and Isabel M. Ferreira, A fractional calculus perspective of PID tuning. In: *Proc. of the ASME International 19th Biennial Conference on Mechanical Vibration and Noise*, Chicago, Illinois-USA (2003). In CD-ROM.

[17] M. Zhuang and D. P. Atherton, Automatic tuning of optimum PID controllers. *IEE Proceedings-Part D: Control Theory and Applications* **140**, No 3 (1993), 216-224.

*) *Institute of Engineering of Porto*
Dept. of Electrotechnical Engineering
Rua Dr. António Bernardino de Almeida
4200-072 Porto, PORTUGAL
e-mails: rbarbosa@dee.isep.ipp.pt , jtm@dee.isep.ipp.pt

Received: July 15, 2003
Revised: February 23, 2004

**) *Faculty of Engineering of Porto*
Dept. of Electrotechnical Engineering
Rua Dr. Roberto Frias
4200-465 Porto, PORTUGAL
e-mail: imf@fe.up.pt