Fractional derivatives: Probability interpretation and frequency response of rational approximations

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ABSTRACT

The theory of fractional calculus (FC) is a useful mathematical tool in many applied sciences. Nevertheless, only in the last decades researchers were motivated for the adoption of the FC concepts. There are several reasons for this state of affairs, namely the co-existence of differ- ent definitions and interpretations, and the necessity of approximation methods for the real time calculation of fractional derivatives (FDs). In a first part, this paper introduces a proba- bilistic interpretation of the fractional derivative based on the Grünwald-Letnikov definition. In a second part, the calculation of fractional derivatives through Padé fraction approxima- tions is analyzed. It is observed that the probabilistic interpretation and the frequency response of fraction approximations of FDs reveal a clear correlation between both concepts.

Keywords:
Fractional derivative, Fractional calculus, Probabilistic interpretation, Frequency response, Fraction approximations

1. Introduction

Fractional calculus (FC) goes back to the beginning of the theory of differential calculus and deals with the generalization of standard integrals and derivatives to a non-integer, or even complex order [1–3]. A wide range of potential fields of application are possible by bringing to a broader paradigm concepts of physics, chemistry and engineering [4–15]. Nevertheless, until recently, FC was considered an “exotic” mathematical tool, being present day interest due to the important developments in the area of nonlinear dynamics [16–18]. Several researchers [19–32] proposed different approaches for the interpretation and the real time calculation of fractional derivatives (FDs), but the fact is that a final paradigm is not yet well established.

Bearing these ideas in mind, this paper is organized as follows. Section 2 interprets the Grünwald-Letnikov definition of FDs in the point of view of probability theory. The proposed viewpoint reduces to the standard interpretation for the case of a derivative of integer order. Section 3 analyzes the frequency response of rational fraction approximations of FDs. Section 4 compares the previous results and demonstrates that there is a clear relationship between them. Finally, Section 5 outlines the main conclusions.

2. A probabilistic perspective of the fractional-order derivative

In this paper it is addressed the Grünwald-Letnikov definition of a FD of order $0 \leqslant \alpha \leqslant 1$ of the signal $x(t)$, $D^{\alpha}[x(t)]$, given by the expression

$$D^{\alpha}[x(t)] = \lim_{h\to 0}\left[\frac{1}{h^{\alpha}}\sum_{k=0}^{\infty}\gamma(\alpha,k)x(t-kh)\right], \tag{1a}$$

$$\gamma(\alpha,k) = (-1)^{k}\frac{\Gamma(\alpha+1)}{k!\,\Gamma(\alpha-k+1)}, \tag{1b}$$

where Γ represents the gamma function and h the time increment.

Analyzing (1b) we have

$$\gamma(\alpha,0) = 1, \tag{2a}$$

$$-\sum_{k=1}^{\infty}\gamma(\alpha,k) = 1. \tag{2b}$$

From the point of view of probability theory, these results lead directly to the following conclusions [31]:

– According to expression (2a) the “present” (P), constituted by $x(t)$, is seen in expression (1a) with probability 1.
– Due to Eq. (2b), the totality of the “past/future” (PF), constituted by the samples $x(t), x(t-h), x(t-2h), \ldots$, is also captured with probability 1; however, each sample of $x(t)$ is weighted with a given probability $-\gamma(\alpha,k)$, that is higher the closer we get to the P.

Expression $-\sum_{k=1}^{\infty}\gamma(\alpha,k)x(t-kh)$ can be viewed as the expected value of the random variable, such that $P[x(t-kh)] = -\gamma(\alpha,k)$, $k = 1,2,\ldots$. Fig. 1 represents the geometric interpretation of (1) in the perspective of probability theory.

The Grünwald-Letnikov definition (1) gets the slope h of a triangle with upper corners $x(t)$ and $-\sum_{k=1}^{\infty}\gamma(\alpha,k)x(t-kh)$ that is, having vertical width the difference between the P sample of the signal x and the arithmetic average of its PF, and having horizontal width h^{α}. In the limit, as $h \to 0$ the slope yields $h \to D^{\alpha}[x(t)]$. Therefore, for the particular case of $\alpha = 1$ corresponds to the slope of a tangent line, because the samples $x(t)$ and $x(t-h)$ have probability one, while the rest of the PF has probability zero. In other words, the first-order derivative corresponds to a deterministic perspective and is just a limit situation of the more general case of a fractional value of α.

It is also important to analyse the amplitude of the probability distribution that captures and weights the PF of $x(t)$ for getting the expected value. Fig. 2 depicts $|\gamma(\alpha,k)|$ *versus* k for several values of α. As $k \to +\infty$ we get the asymptotic approximation $|\gamma(\alpha,k)| \approx \frac{\Gamma(\alpha+1)}{\pi}|\sin(\alpha\pi)|k^{-\alpha-1}$ [17] revealing a logarithmic memory and the importance that the FD gives to the PF sample values of $x(t)$, in opposition with the integer order case. On the other hand, the factor h^{α} in the denominator of expression (1a) means that for large values of h (i.e., very far in the PF), we have a slow variation, while for small values of h (i.e., near to the P) we have a fast variation, being this effect the stronger the closer to zero is the value of α. Therefore, we only have a uniform velocity in the calculation of definition (1a) in the case of $\alpha = 1$.

3. Rational approximations of fractional derivatives

The Grünwald-Letnikov definition of a FD (1) inspires a discrete-time calculation algorithm, based on the approximation of the time increment h through the sampling period T, yielding the equation in the z domain:

$$Z\{D^{\alpha}[x(t)]\} \approx \left[\frac{1}{T^{\alpha}}\sum_{k=0}^{\infty}\frac{(-1)^{k}\Gamma(\alpha+1)}{k!\,\Gamma(\alpha-k+1)}z^{-k}\right]X(z), \tag{3}$$

where $X(z) = Z\{x(t)\}$.

The implementation of expression (3) corresponds to an r-term truncated series given by

$$Z\{D^{\alpha}[x(t)]\} \approx \left[\frac{1}{T^{\alpha}}\sum_{k=0}^{r}\frac{(-1)^{k}\Gamma(\alpha+1)}{k!\,\Gamma(\alpha-k+1)}z^{-k}\right]X(z). \tag{4}$$

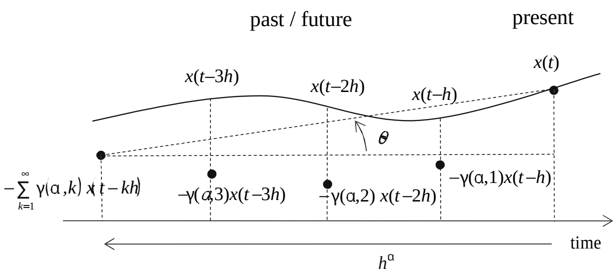

Fig. 1. Probabilistic interpretation of the Grünwald-Letnikov definition of the FD of order α of the signal $x(t)$.

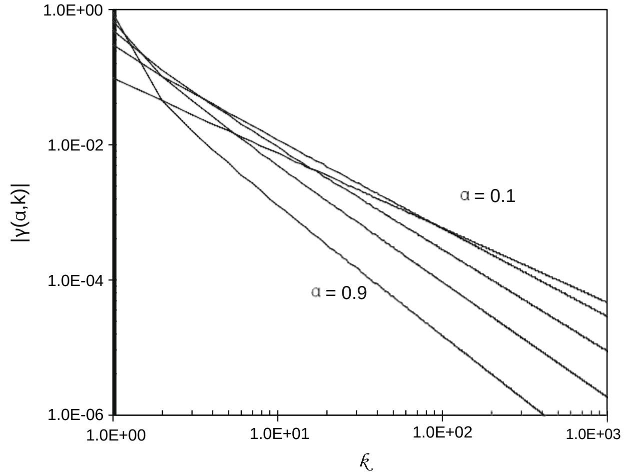

Fig. 2. Amplitude of the probability distribution $|\gamma(\alpha,k)|$ versus k for $\alpha = \{0.1, 0.2, \ldots, 0.8, 0.9\}$.

Expression (3) represents the Euler (or first backward difference) approximation, in the so-called $s \to z$ conversion scheme, where s and z represent the variables in the Laplace and Z domains. Another possibility, often adopted in control system design, consists in the Tustin (or bilinear) rule. The Euler and Tustin rational expressions, $H_0(z^{-1}) = \frac{1}{T}(1 - z^{-1})$ and $H_1(z^{-1}) = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$, are often called generating approximants of zero and first order, respectively. Therefore, the generalization of these conversion methods leads to the irrational expressions:

$$s^\alpha \approx \left[\frac{1}{T}(1 - z^{-1})\right]^\alpha = H_0^\alpha(z^{-1}), \tag{5a}$$

$$s^\alpha \approx \left(\frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}}\right)^\alpha = H_1^\alpha(z^{-1}). \tag{5b}$$

We can obtain a family of fractional differentiators generated by $H_0^\alpha(z^{-1})$ and $H_1^\alpha(z^{-1})$ that are weighted by the factors p and $1 - p$ [33], yielding

$$H_{av}(z^{-1}; p) = pH_0^\alpha(z^{-1}) + (1 - p)H_1^\alpha(z^{-1}). \tag{6}$$

For example, the Al-Alaoui operator corresponds to an interpolation of the Euler and Tustin rules with weighting factor $p = 3/4$ [34–36].

In order to get a rational expression, the final approximation corresponds to a truncated Taylor series or a rational fraction expansion. Due to its superior performance often it is used a Padé fraction expansion of order $r \in \mathbb{N}$:

$$H_r(z^{-1}) = \sum_{i=0}^{r} a_i z^{-i} / b_i z^{-i}, \quad a_i, b_i \in \Re. \tag{7}$$

Moreover, since one parameter is linearly dependent, it is established $b_0 = 1.0$.

In the sequel, for simplicity, we consider only $H_0^\alpha(z^{-1})$ (i.e., $p = 1$ in (6)), the cases of $r = \{1, 2, 3, 4\}$ in expression (7), and $T = 1$. For characterizing the performance of the r-order Padé approximations, we analyze their frequency response based on the transformation $z^{-1} = e^{-j\Omega}$, $\Omega = \omega T$, $j = \sqrt{-1}$. For example, Fig. 3 compares the frequency response of the ideal and

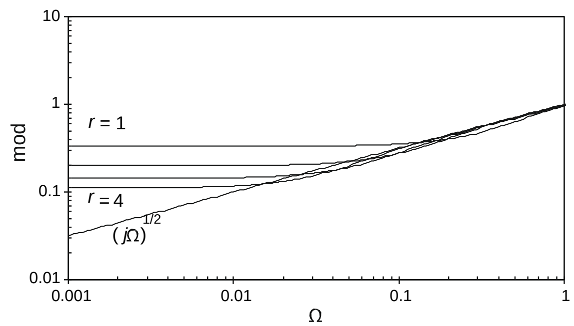

Fig. 3. Amplitude Bode diagram of the ideal and Padé approximations with $r = \{1, 2, 3, 4\}$, for a FD of order $\alpha = 0.5$.

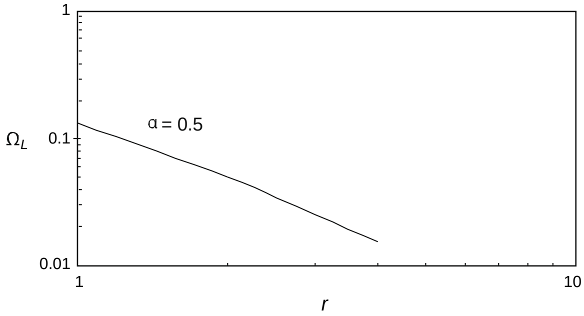

Fig. 4. Low frequency limit X_L of the Padé approximations versus $r = \{1, 2, 3, 4\}$ for a FD of order $a = 0.5$.

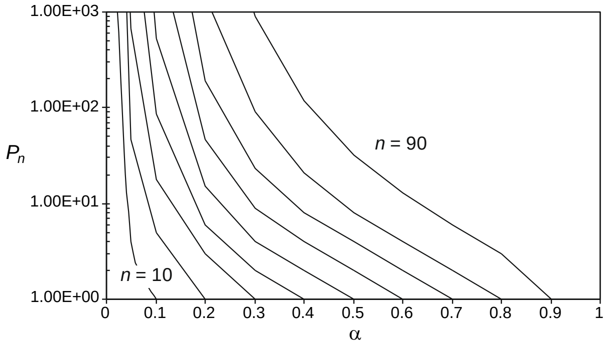

Fig. 5. Chart of P_n versus a for $n = \{10, 20, \ldots, 90\}$.

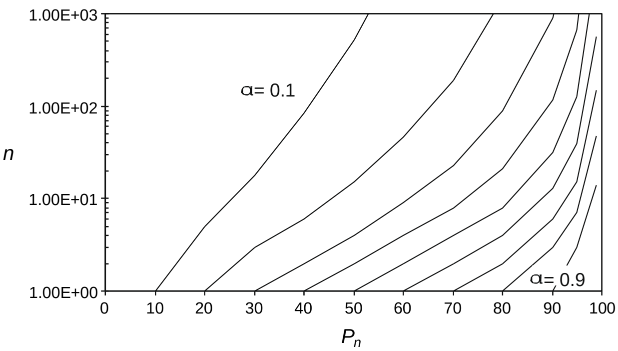

Fig. 6. Chart of n versus P_n for $a = \{0.1, 0.2, \ldots, 0.9\}$.

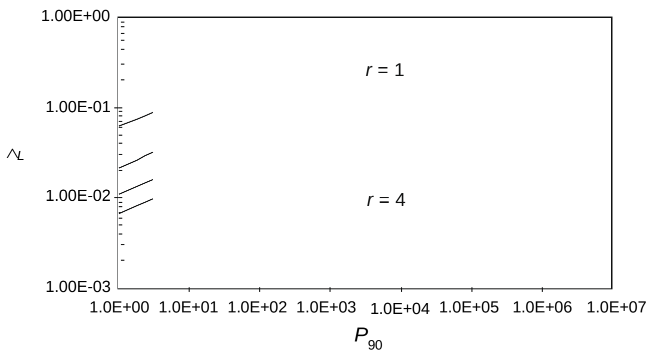

Fig. 7. Chart of X_L versus P_{90} for $r = \{1, 2, 3, 4\}$.

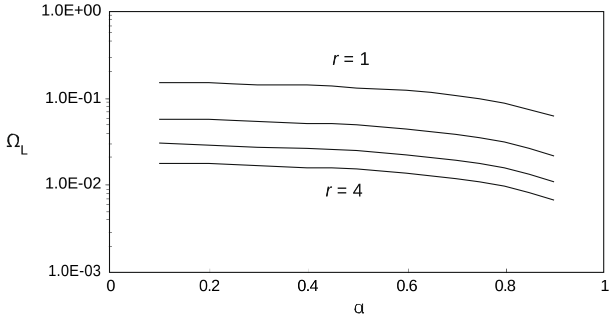

Fig. 8. Chart of X_L versus a for $r = \{1, 2, 3, 4\}$.

approximate expressions for the FD of order $a = 0.5$. The chart shows the well-known good behaviour at high frequencies in opposition with the poor curve fitting at low frequencies, being the approximation the better the higher the value of r. For measuring the bandwidth of the approximation, namely the limit at low frequencies, the frequency X_L of intersection between the ideal and the Padé fraction is considered. Fig. 4 depicts X_L versus $r = \{1, 2, 3, 4\}$ for a FD of order $a = 0.5$ demonstrating a clear correlation between both variables.

4. The probabilistic interpretation and the frequency response

This section compares the previous results in order to investigate possible relationships between them. We start by defining a suitable index for measuring the statistical properties of the probability distribution $-c(a, k)$. Unfortunately,

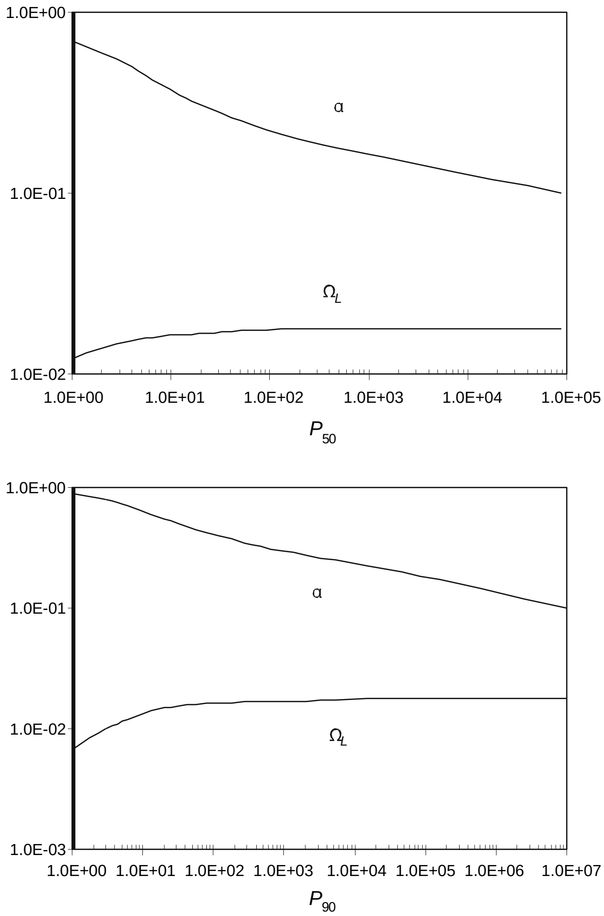

Fig. 9. Charts of a and X_L versus P_n, $n = \{50, 90\}$, for $r = 4$.

the calculation of the arithmetic mean and the standard deviation diverges and therefore, alternative indices are required. In this line of thought, for measuring the dispersion of $-c(a,k)$, the nth percentiles P_n, $n = \{10, 20, \ldots, 90\}$, are adopted.

Figs. 5 and 6 analyze P_n versus a and P_n versus n, respectively. As expected, we verify that FDs:

– take a considerable number of samples in the PF to accomplish a significant percentile value;
– weight more/less the recent/far PF the closer the value of a is to one/zero.

Figs. 7 and 8 show the relationship between the low frequency limit of the Padé approximation X_L versus P_{90} and X_L versus a, respectively, for $r = \{1, 2, 3, 4\}$.

We observe in Fig. 9 that the variation of X_L with P_n, $n = \{50, 90\}$, is strongly correlated with the variation of a with P_n, $n = \{50, 90\}$. Therefore, we verify that the probabilistic interpretation of the FD is compatible with the analysis in the frequency domain.

5. Conclusions

In the last years, the progress in the scientific knowledge motivated the adoption of the theory of FC as a useful mathematical tool to handle applications in the areas of physics, chemistry and engineering sciences. The work carried out so far is still preliminary but reveals interesting and promising aspects for future research and developments. Nevertheless, the lack of a simple interpretation for the base concept of the FD poses problems and, consequently, such limitations must be overcome.

In this line of thought, this paper presented a novel approach based on the probability theory and the Grünwald-Letnikov of a FD. The concepts are simple and lead to a clear geometric interpretation that is compatible with the standard case of integer order. The frequency response of rational Padé approximations was also investigated, namely the low frequency limit of the bandwidth. The comparison of both perspectives leads to similar conclusions about the memory effect of FDs.

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