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Nonmonotone Hybrid Tabu Search for Inequalities and Equalities: an Experimental Study

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Abstract

The main goal of this paper is to analyze the behavior of nonmonotone hybrid tabu search approaches when solving systems of nonlinear inequalities and equalities through the global optimization of an appropriate merit function. The algorithm combines global and local searches and uses a nonmonotone reduction of the merit function to choose the local search. Relaxing the condition aims to call the local search more often and reduces the overall computational effort. Two variants of a perturbed pattern search method are implemented as local search. An experimental study involving a variety of problems available in the literature is presented.

Mathematics Subject Classification: 90C15; 90C56; 90C30

Keywords: inequalities and equalities, tabu search, pattern search, merit function, nonmonotone condition

1 Introduction

In this paper, we aim at providing an experimental study with nonmonotone hybrid tabu search (TS) approaches for solving systems of p nonlinear inequalities and equalities with n variables and $p \geq n$. The system is considered to be in the form:

$$\begin{cases} f_i(x) \leq 0, & i = 1, \ldots, m \\ f_i(x) = 0, & i = m+1, \ldots, p \end{cases} \tag{1}$$

where each $f_i : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ and Ω is a closed convex set, herein defined as the interval $[l, u] = \{x : -\infty < l_i \leq x_i \leq u_i < \infty, i = 1, \ldots, n\}$. We assume that all functions $f_i(x)$, $i = 1, \ldots, p$ are continuously differentiable.

The TS is commonly known as a metaheuristic for global optimization and aims at exploring all the search space for global optimal solutions. It has the ability to guide the search out of local optima and explore new regions. It is an iterative procedure that maintains a list of the movements most recently made, avoiding in subsequent iterations the execution of movements that lead to solutions already known to have been visited. The term metaheuristic is used to describe any approximate method that combines basic heuristic methods and high level frameworks to explore the search space in an efficient and effective manner. The hybrid word is used to describe the combination of the TS with a local search procedure which may be a deterministic search or a random one, to enhance the exploitation ability of the TS. In the presented algorithm, at each iteration, the algorithm chooses between the global search and the local search, i.e., it chooses the search that is most appropriate depending on the properties of the involved functions in that region of the search space. Finally, the nonmonotone property of the algorithm is directly related with the condition that is used to choose the type of search, at each iteration. The choice depends on the progress of the algorithm and it does not require a sufficient decrease of merit function values. In the problem (1), a merit function aims to assess the fitness of the iterates, giving a measure of the progress towards the solution. When a nonmonotone reduction on the merit function is used to select a local search, it means that the classical simple decrease is relaxed and consequently the local search is selected more often. Thus, the exploitation phase in the vicinity of an iterate is privileged in detriment of the global search. We further remark that the presented algorithm is a derivative-free framework since neither analytical nor numerical derivatives are required in both exploration and exploitation phases.

The motivation of this work comes mainly from the detection of feasibility in nonlinear optimization problems. Systems of inequalities have been extensively studied because of various applications in data analysis, set separation problems, computer-aided design problems and image reconstructions. This type of system appears frequently in bound constrained variational inequalities and linear or nonlinear complementarity problems [30]. Classical methods for solving problem (1) use Newton-type methods [5]. In [25], the authors propose a method that combines the use of a modified Newton step and a conventional first-order step, and in [39], a solution of (1) is obtained by applying successively a Levenberg-Marquardt algorithm to solve smoothed versions of the problem. Smooth reformulation of (1) have also been proposed in [19, 45]. Instead of solving the system (1) of m inequalities and $p - m$ equalities, the authors construct a smooth and equivalent system of increased

dimension of $2m+(p-m)+1$ equalities and $p+m+1$ variables. A Newton-type method is then used with monotone and nonmonotone line search strategies to guarantee convergence, see [19] and [45] respectively. Systems of inequalities and equalities emerging from nonlinear complementarity problems have been reformulated as inequality constrained optimization problems and solved by filter-type methods. The derivative-free filter method proposed in [30] relies on the generalized pattern search method and on the filter methodology [7].

However, the simplest way to solve (1) is based on the reformulation of the inequalities into equalities using the following equivalence:

$$\begin{cases} f_i(x) \leq 0 & i = 1, \ldots, m \\ f_i(x) = 0, & i = m+1, \ldots, p \end{cases} \Leftrightarrow \begin{cases} \max\{0, f_i(x)\} = 0, & i = 1, \ldots, m \\ f_i(x) = 0, & i = m+1, \ldots, p \end{cases} . \tag{2}$$

Since some functions in the equivalent system (2) are nonsmooth, Newton's method cannot be directly applied to solve it. In this paper, we present a derivative-free metaheuristic strategy for solving systems of inequalities and equalities, with $p \geq n$, by solving the equivalent system of equations alone (2). The most famous techniques to solve nonlinear equations are based on the Newton's method [6, 11, 32]. They are computationally expensive since the Jacobian matrix with analytical first derivatives and the solution of a system of linear equations may be required at each iteration. Quasi-Newton methods have less expensive iterations than Newton since they avoid either the necessity of computing derivatives, or the necessity of solving a full linear system per iteration or both tasks [8, 24]. A common strategy to solve problem (2) is presented in [28, 38]. The authors use a scheme that transfers the system of nonlinear equations into an equality constrained optimization problem. A set of equations from the system is selected to define the objective function and the remaining ones define the equality constraints of the problem. The methodology applied to handle the constraints in [31] uses a line search filter approach. Recently, in [20], global convergence is proved with an algorithm that uses a nonmonotone line search filter approach, and in [38], a nonmonotone filter trust region algorithm is implemented. When solving structured variational inequalities by alternating direction-type methods, two complementarity problems are required to be solved at each iteration. Further developments in [23] held an algorithm that solves an equivalent system of nonlinear equations. When the derivatives are difficult to calculate or there exist no derivatives for some equations, methods that do not require derivatives are the most adequate. In [29], the generalized pattern search approach is extended to handle a system of nonlinear equations. An approach that transforms the system of nonlinear equations into a multiobjective optimization problem, whose number of objectives is equal to the number of equations, is presented in [14]. The therein implemented evolutionary approach to solve the multiobjective problem uses the sum of the absolute values of the objectives to measure fitness

and compare solutions.

The problem of solving a nonlinear system of equations can be naturally formulated as a global optimization problem. Problem (2) is equivalent to

$$\min_{x\in\Omega\subset\mathbb{R}^n} M(x) \equiv \sum_{i=1}^{m}(\max\{0, f_i(x)\})^2 + \sum_{i=m+1}^{p} f_i(x)^2, \tag{3}$$

in the sense that they have the same solutions. These required solutions are the global minima, and not just the local minima, of the function $M(x)$, known as merit function, in the set Ω. Problem (3) is similar to the usual least squares problem for which many iterative methods have been proposed. They basically assume that the objective function is twice continuously differentiable. However, the objective M in (3) is only once differentiable even if all $f_i, i = 1, \ldots, p$ are twice continuously differentiable. Thus, methods for solving the least squares problem cannot be directly applied to solve (3).

Besides, when a global solution is required, classical local search methods, like Newton-type methods, have some disadvantages, compared with global search methods. The final solution is heavily dependent on the initial approximation of the iterative process and they can be trapped in a local minimum. Furthermore, most of the methods require differentiable properties of all the equations in the nonlinear system, such as, for example, the trust-region based quasi-Newton method presented in [42]. A trust-region Gauss-Newton method for bound constrained nonlinear least-squares problem has been recently presented in [27]. The therein developed solver *TRESNEI* is prepared to solve systems of nonlinear inequalities and equalities without restrictions on their dimensions. However, the algorithm requires derivatives of the involved functions. Local optimization techniques guarantee globally only under certain convexity assumptions. Preventing premature convergence to a local solution, while trying to compute a global optimum, is to be required when solving problem (3) [33, 34, 35, 36].

The nonmonotone concept was firstly introduced in 1986 in the Newton's method for unconstrained optimization [12]. The concept is related with relaxing the usual condition that forces a sufficient decrease of the objective function, so that convergence to the solution is guaranteed, whatever the initial approximation. Grippo, Lampariello and Lucidi [13] developed a class of nonmonotone objective/merit function reduction strategies for unconstrained optimization. They showed that the usual sufficient reduction required by the traditional Armijo condition can slow the rate of convergence in the intermediate stages of the minimization process, especially when the merit function has narrow curved valley. They introduced a nonmonotone Armijo condition that chooses the step size $\alpha \in (0, 1]$ such that

$$M(x^{k+1}) \equiv M(x^k + \alpha d^k) \leq \max_{0\leq j\leq s^k} M(x^{k-j}) + \mu\alpha\nabla M(x^k)^T d^k, \tag{4}$$

where $\mu \in (0,1)$ is a constant, $\nabla M(x^k)$ represents the gradient of M computed at x^k, d^k is the search direction, and

$$s^0 = 0, \quad s^k = \min\{s^{k-1} + 1, s_{\max}\},\ k \geq 1 \ \text{ and } \ s_{\max} \ \text{ is a nonnegative integer.} \tag{5}$$

If $s_{max} = 0$, the above nonmonotone rule is just the condition for sufficient decrease. Slightly different strategies have been proposed to overcome some difficulties that the classical nonmonotone strategy has been shown. Instead of the maximum function value reached over (at most) the last $s_{\max}+1$ iterations, the alternative is to use the average of the successive merit function values [44]. Although this choice of nonmonotone technique looks particularly efficient, the calculations seem a little cumbersome. In [15], the line search technique uses a nonmonotone term which is a convex combination of the previous nonmonotone term and the current objective function value

$$\begin{aligned} M(x^k + \alpha d^k) \leq \ & D^k + \mu\alpha \nabla M(x^k)^T d^k, \\ \text{for } D^k = \ & \begin{cases} M(x^k), & \text{if } k = 1 \\ \eta D^{k-1} + (1-\eta) M(x^k), & \text{if } k \geq 2 \end{cases} \end{aligned} \tag{6}$$

where $\eta \in (0,1)$. A nonmonotone strategy can be applied within a line search approach [43] or in a traditional trust region method [15]. Its use is nowadays generalized. Under mild conditions, global convergence has been guaranteed. These nonmonotone strategies have been applied in different contexts, for example, coupled with Quasi-Newton methods for systems of equations in [8], for solving complementarity problems [18], integrated into primal-dual interior point and barrier methods [2, 3] and in a Newton-type method when solving a smooth reformulation of the system (1) [45].

In this paper, we aim at studying the behavior of a nonmonotone combined hybrid TS method when computing a solution of the system (1) by converging to a global minimizer of the merit function (3). The herein presented algorithm combines global and local searches. With both searches we are prepared to explore the search space for promising regions and refine the search in the vicinity of a global solution.

In the herein presented combined algorithm, the global search is based on the metaheuristic TS. Although TS is computationally successful in solving combinatorial problems [10] as well as continuous global optimization problems [4], it may fail to detect promising search directions in the neighborhood of a global minimum. TS is an iterative procedure that maintains a list of the movements most recently made – the tabu list – avoiding in subsequent iterations the execution of movements that lead to solutions already known to have been visited. Usually, the slow convergence of TS is overcome by incorporating a classical local search strategy into the main algorithm. We will use the Directed Tabu Search (DTS) method of Hedar and Fukushima [16], a

variant of the tabu search that implements a local search in the final stage of the process, therein denoted by the intensification search.

The local search phase of the herein proposed combined algorithm relies on a perturbed pattern search procedure. Some preliminary experiments have been conducted with the classical coordinate search [21] in an alike nonmonotone combined method [35]. The well-known Hooke and Jeeves (HJ) method [17] has recently been used to hybridized the TS method [36].

However, to increase the exploration feature of the local search procedure, avoiding to define new iterates solely along the coordinate directions, randomly perturbed directions are herein added to the chosen set of directions to locate a point that reduces the merit function value with respect to the current point. This idea is borrowed from the paper of Ali and Gabere [1], where these perturbed directions are integrated into the basic opportunist coordinate search. Further, we address the issue related with the choice between implementing the global search or the local search, by means of a nonmonotone F-rule reduction of the merit function similar to [37]. Here, the main goal is to reduce the computational effort required to reach a solution with a pre-specified accuracy.

The remainder of the paper proceeds as follows. Section 2 describes the combined global and local search algorithm and the main idea behind the nonmonotone condition for the selection between global and local phases. The experimental study is presented in Section 3 and we conclude the paper in Section 4.

2 Nonmonotone combined searches

In the presented nonmonotone combined global search and local search (nm–CGSLS), we aim to combine two alternative phases: the exploration phase, where a global search is carried out, and the exploitation one, where the search is refined in the vicinity of a promising region. The idea of combining two searches has been used in the past to solve nonlinear systems of equations, where a classical gradient-based Quasi-Newton nonmonotone strategy is used [8], and to address systems of nonlinear inequalities and equalities [33, 35, 36].

It seems acceptable that if a reduction in the merit function has been observed, a downhill progress is detected and the search is to be refined using a local search. On the other hand, if the merit function has not been reduced, a global search is appropriate since the search space is to be explored for a promising region [33, 34]. However, requiring a reduction in the merit function may prevent the local search phase of being implemented more often. Consequently, the global search steps dominate and the iterative process is computationally demanding in terms of function evaluations. Thus, we address this issue by relaxing the condition on the merit function to choose a

local search step. Borrowing the ideas present in nonlinear optimization, a nonmonotone merit function reduction strategy will be used.

2.1 The nonmonotone F-rule

When analytical or numerical derivatives are not available the second term on the right of the nonmonotone Armijo condition (4) cannot be used. Since it has been noticed that the term, that depends on the search direction, decreases as the corresponding algorithm converges to the solution, a commonly named forcing function (F-function) has been used instead [37].

Definition 2.1 *The function* $\sigma : [0, +\infty) \to [0, +\infty)$ *is an F-function if for any sequence* $\{t^k\}$, $\lim_{k\to\infty} \sigma(t^k) = 0$ *implies* $\lim_{k\to\infty} t^k = 0$.

Thus, based on the previous definition, the following nonmonotone F-rule is implemented in the nm–CGSLS. Let $s_{\max}$ be a nonnegative integer. For each k, let s^k be defined as (5). Then, if the nonmonotone decrease

$$M(x^{k+1}) \leq \max_{0\leq j\leq s^k} M(x^{k-j}) - \bar{\eta}^k \tag{7}$$

is verified, where $\bar{\eta}^k \to 0$ as $k \to \infty$, the region is considered a promising region and a local search procedure is required to refine the search. In our algorithm, the F-function σ is replaced by a monotone decreasing sequence of positive values $\bar{\eta}^{k+1} = \gamma\bar{\eta}^k$, as $k \to \infty$, where $\bar{\eta}^0 > 0$ and $0 < \gamma < 1$. We remark that condition (7) is more demanding at the beginning of the iterative process than at the end of it. This is important since, far from the solution, the likelihood of detecting a global solution instead of a local one is high when the merit function has a sufficient decrease, relative to the maximum merit values of previous iterations, of more than a certain quantity, $\bar{\eta}^k$.

Similar nonmonotone F-rules for line search with guaranteed convergence in optimization context are available in the literature [37, 41]. In [40], a nonmonotone trust region procedure using a filter methodology to handle the equality constraints of an optimization problem is presented. The goal of the nonmonotone F-rule (7) is to relax the condition that is usually used to control the decreasing behavior of merit function values along the iterative process, i.e., the progress towards a minimum of the merit function. Relaxing the descent condition on the merit function allows that trial solutions may be selected even if they do not provide an improvement on the merit function. As a consequence, trial solutions are accepted more often and a reduction in the overall computational effort, for instance in the number of function evaluations, is verified. This is the type of behavior that our algorithm aims to achieve. Relaxing the descent condition on the merit function, the local search procedure is selected more often and the number of function

evaluations is reduced. Indeed our previous work has shown that we gain in efficiency if a nonmonotone merit reduction condition is imposed to select the local search step [35, 36]. However, the therein used nonmonotone merit reduction $M(x^{k+1}) \leq \gamma_M \max_{0\leq j\leq s^k} M(x^{k-j})$, where $0 < \gamma_M < 1$, is invariant to the algorithm progress. The Algorithm 1 contains the pseudo-code of the herein presented nm–CGSLS method.

Algorithm 1 nm–CGSLS algorithm

Require: x^0, $\eta^* > 0$, $\bar{\eta}^0 = \eta^0 > \eta^*$, $0 < \gamma < 1$, $k_{\max} > 0$, $s_{\max} \geq 0$
Set $k = 0$, $s^0 = 0$
while $M(x^k) > \eta^*$ AND $k \leq k_{\max}$ **do**
  if $\left(k = 0 \text{ AND } M(x^k) \geq \eta^k\right)$ OR $M(x^k) > \max\limits_{1\leq j\leq s^k} M(x^{k-j}) - \bar{\eta}^k$ **then**
    Find an η^k–minimizer x^{k+1} of M, using a *global search procedure*
  else
    Find an η^k–minimizer x^{k+1} of M, using a *local search procedure*
  end if
  Set $\bar{\eta}^{k+1} = \gamma\bar{\eta}^k$, $\eta^{k+1} = \max\{\eta^*, \bar{\eta}^{k+1}\}$, $s^{k+1} = \min\{s^k + 1, s_{\max}\}$
  Set $k = k + 1$
end while

We remark that the solution x^{k+1}, obtained by either the *local search procedure* or the *global search procedure*, is required to be an η^k-approximation, where the sequence of η^k values decreases towards $\eta^* > 0$: $\eta^{k+1} = \max\{\eta^*, \gamma\eta^k\}$ for $0 < \gamma < 1$. This means that at the beginning of the iterative process, far from the solution of the problem (3), the computational effort to reach a solution could be lower than the effort to reach a solution during the final stage of the iterative process.

We further note that the algorithm starts with an exploration phase, by calling the *global search procedure*, unless the merit function value of the provided initial approximation, that could be a randomly generated point, falls below η^k. Finally, if an approximation is found with a merit function value less or equal to η^* the algorithm stops. However, if this last condition fails to be verified, we choose to stop the algorithm when the number of iterations exceeds $k_{\max}$.

2.2 Global search procedure

The *global search procedure* of the Algorithm 1 is based on the DTS heuristic of Hedar and Fukushima [16]. The DTS method is composed of three main search procedures: exploration, diversification and intensification. The main loop of the DTS method consists of the exploration and diversification search procedures. The exploration search aims to explore the search space Ω and

uses a direct search method to be able to stabilize the search, in particular in the vicinity of a local minimum. Cycling is prevented by the standard tabu list, as well as by other four TS memory elements: the multi-ranked tabu list, the tabu region, the semi-tabu region and the visited region list (VRL). The reader is refereed to [16] for details.

The diversification procedure aims to generate new trial points outside the previously visited regions. The VRL works as a diversification tool and is used to direct the search towards regions that have not been visited in the search space. When one of the best obtained trial solutions is within an η^k-neighborhood of the global minimum of M, at iteration k, or the number of iterations exceeds a target value – $5n$ in our implemented algorithm – the DTS algorithm leaves the exploration and diversification search procedures and enters the intensification procedure. Here, the DTS algorithm aims to compute a solution still closer to the global minimum of the merit function by implementing a local search. In [16], a modified Nelder-Mead method is used in this final stage of the process. However, since pattern search methods have better convergence properties [21], the herein implemented variant of the DTS algorithm allows a maximum of $2n$ iterations of the below described *local search procedure* in its intensification phase.

2.3 Local search procedure

The *local search procedure* of the Algorithm 1 is a hybrid pattern search method. It uses a classical generating set method and defines perturbed directions in the sense of [1]. It will be denoted by perturbed generating set (PGS) method. We remark that the set of coordinate directions is one example of a generating set [21]. At each iteration, a pattern search method uses a pattern of points to search for a minimizer. At least $n+1$ trial points are provided by the pattern, where n is the number of variables. Based on the current point, x^j, where j represents the iteration counter in this inner iterative process, each trial point, denoted by y^i, is generated along a search direction (starting from the current point) with a certain step size $\Delta^j > 0$:

$$y^i = x^j + \Delta^j d^i$$

where d^i is a search direction chosen from a finite generating set $\mathcal{G}$ for $\mathbb{R}^n$.

Definition 2.2 *([21]) Let $\mathcal{G} = \{d^1, \ldots, d^r\}$ be a set of $r \geq n+1$ vectors in $\mathbb{R}^n$. Then the set $\mathcal{G}$ generates (or positively spans) $\mathbb{R}^n$ if for any vector $v \in \mathbb{R}^n$, there exist $\lambda^1, \ldots, \lambda^r \geq 0$ such that*

$$v = \sum_{i=1}^{r} \lambda^i d^i.$$

A generating set must contain a minimum of $n+1$ vectors. A set with $r = n+1$ is called a minimal generating set. For $n = 2$, $\mathcal{G} = \{(1,0),(0,1),(-1,-1)\}$ and $\mathcal{G} = \{(1,0),(-1,-1),(-1,1)\}$ are two examples of minimal generating sets [21]. The most used set with the coordinate directions, defined as the positive and negative unit coordinate vectors $\mathcal{G} = \{e_1,\ldots,e_n,-e_1,\ldots,-e_n\}$ form a generating set with $2n$ vectors. The most important property is that at least one of the directions is a descent direction for M, so long as the current point is not a stationary point of M.

To avoid the searches solely along the chosen directions of the generating set, a scheme that adds randomly generated perturbations to the generating set directions, has been tested in [1]. Using this idea in our local search procedure, each trial point is defined by

$$y^i = x^j + \Delta^j d^i + \varsigma \Delta^j u\,,\ i = 1,\ldots,r$$

where u is a unit vector with randomly generated components from a uniform distribution in $[-1,1]$ and ς is a positive parameter. We use the classical approach to produce feasible iterates. A trial point that is generated outside Ω will have a merit function value of $+\infty$ so that it will never be selected.

In this PGS method, two variants are provided: a minimal generating set yielding $n+1$ trial points $y^i, i = 1,\ldots,n+1$ and the generating set defined by the $2n$ coordinate directions. In both variants, at each iteration j, the trial point with the smallest merit function value is chosen, y^{best}, and compared with the merit function at the current x^j. If this search fails to generate a trial point that is better than the current point, the iteration is called *unsuccessful*, the step size Δ^j is reduced by a factor $0 < \xi < 1$, in order to refine the search hereafter, and $x^{j+1} \leftarrow x^j$. The step size is then compared to a specified stopping tolerance. Since an η^k-approximate solution is required, when the merit function value falls below η^k, or the step size reaches a sufficiently small value, defined as $(\eta^k)^2$, the local search terminates and the current point is the required x^{k+1} (see Algorithm 1). If none of these conditions is satisfied, the iterative process runs for a maximum of $10n$ iterations. However, if at the end of each iteration, a simple decrease in M is verified, then the iteration is *successful*, Δ^j is not changed and $x^{j+1} \leftarrow y^{best}$. We note that the search will never stop after a successful iteration.

3 Experimental study

In this section, we use thirteen problems to test the performance of the nm–CGSLS algorithm. In four problems, the system has nonsmooth functions, and in five problems inequalities are also present in the system. There are two systems of equations that emerge from practical applications and the last

two problems are examples of nonlinear complementarity problems. Some examples are known in the literature as being difficult to be solved by Newton-type methods. For comparative purposes, we run some of the problems by `fsolve` from MATLAB^{TM}. The trust-region dogleg algorithm in MATLAB computes a step that is a convex combination of a Cauchy step and a Gauss-Newton step. The main computational effort with the dogleg is related with handling matrix-vector products and solving linear systems of equations. The results of these experiments are obtained in a personal computer with an AMD Turion 2.20 GHz processor and 3 GB of memory. Due to the stochastic nature of algorithms based on tabu search, each problem was run 30 times and the best of the 30 solutions is shown. We tested the nm–CGSLS algorithm with $s_{\max} = 3$ and $s_{\max} = 5$. The former value seems to give in general better performance, although the differences are not significant. The values set to the other parameters of the algorithm are: $k_{\max} = 15n$, $\eta^* = 10^{-6}$, and $\eta^0 = 1$ with $\gamma = 0.1$. Another set of values, $\eta^0 = 10$ with $\gamma = 0.01$, was tried. The results shown in this section that correspond to this last set are marked with '§'. We always report the best set of obtained results. Whenever PGS algorithm is called at the iteration k, we set $\Delta_0 = 1$, $\xi = 0.1$ and $\varsigma = 0.15$.

Problem 3.1 *Consider the problem of finding a solution of two nonlinear inequalities with two variables [45] in* $[-10, 10]^2$*:*

$$\begin{cases} x_1^2 + x_2^2 - 1 & \leq 0 \\ -x_1^2 - x_2^2 + (0.999)^2 & \leq 0. \end{cases}$$

This and the next two problems will be used to compare the results of the present study with those obtained by DTS method [16] and CGTSLS method in [33]. When DTS method is used separately, a solution within a η^* accuracy is also required. All methods are allowed to run for a maximum of $15n$ iterations. The results are reported in the first part of Table 1. In this table and in all the remaining ones, 'k' stands for the number of iterations, '$n.f.e.$' is the number of function evaluations and 'M' gives the merit function value at the best found solution. We also compare with the results available in [45] where a smoothing technique is applied to the system (2) so that it can be solved by a Newton-type method. In the hereinafter tables, '-' means not available. Even when the number of outer iterations (k in the table) is similar to that of the monotone CGTSLS algorithm, the overall number of function evaluations required by nm–CGSLS is smaller than that of CGTSLS. Besides the two variants of the PGS method in the *local search procedure* of the nm–CGSLS, the well-known Hooke and Jeeves (HJ) method [17] has also been used for comparative purposes. The previous experiments with the HJ local search available in [36] involve a limited set of five problems with $p = n$ [45]. We use the notation: 'PGS' for the generating set with the $2n$ coordinate directions

and 'PGSm' for the minimal generating set with $n+1$ directions. We also compare the nonmonotone F-rule (7) with the nonmonotone condition that comes from the convex combination of merit function values, similar to (6). In the following tables we will use the notation 'nm' to refer to 'nm–CGSLS'.

Table 1: Comparative results for Problems 3.1–3.3.

Algorithm	k	$n.f.e.$	M	Algorithm	k	$n.f.e.$	M
Problem 3.1							
DTS	30	341	0	CGTSLS	4	271	0
nm+PGS+(7)	3	26	0	in [45]	8	9	-
nm+PGSm+(7)	2	21	0	nm+HJ+(7)	3	32	0
nm+PGSm+(6)	7	55	0	nm+HJ+(6)	3	41	0
Problem 3.2							
DTS	75	1712	3.5e-7	CGTSLS	7	1860	3.5e-7
nm+PGS+(7)	$4^{\S}$	96	5.8e-8	in [45]	4	4	-
nm+PGSm+(7)	8	136	2.0e-8	nm+HJ+(7)	5	230	3.5e-8
nm+PGSm+(6)	7	176	9.2e-7	nm+HJ+(6)	6	393	3.5e-8
Problem 3.3							
DTS	45	815	3.8e-7	CGTSLS	7	889	5.3e-7
nm+PGS+(7)	$5^{\S}$	180	2.5e-8	in [45]	5	5	-
nm+PGSm+(7)	$5^{\S}$	78	3.1e-7	nm+HJ+(7)	5	151	6.1e-8
nm+PGSm+(6)	8	331	7.3e-7	nm+HJ+(6)	6	1068	5.4e-10

Problem 3.2 *Consider the problem of finding a solution of three nonlinear inequalities and two equalities with five variables [45] in* $[-10,10]^5$*:*

$$\begin{cases} x_1 + x_3 - 1.6 & \leq 0 \\ 1.333x_2 + x_4 - 3 & \leq 0 \\ -x_3 - x_4 + x_5 & \leq 0 \\ x_1^2 + x_3^2 - 1.25 & = 0 \\ x_2^{1.5} + 1.5x_4 - 3 & = 0. \end{cases}$$

The results are reported in the second part of Table 1. Both variants of the proposed nm–CGSLS, PGS and PGSm, outperform DTS and the monotone version CGTSLS. The nonmonotone F-rule (7) is much more effective in converging to a solution than the nonmonotone rule based on the condition (6).

Problem 3.3 *Another problem available in [45], with one nonlinear inequality, two equalities with three variables and* $\Omega = [-10,10]^3$*,*

$$\begin{cases} x_1 + x_2\exp(0.8x_3) + \exp(1.6) & \leq 0 \\ x_1^2 + x_2^2 + x_3^2 - 5.2675 & = 0 \\ x_1 + x_2 + x_3 - 0.2605 & = 0. \end{cases}$$

The results are listed in the third part of Table 1. The performance of the tested versions is similar to that of the previous problems.

Problem 3.4 *Consider the problem of finding a solution of the nonlinear system with two variables [20, 31, 38]*

$$\begin{cases} x_1 & = 0 \\ 10x_1/(x_1 + 0.1) + 2x_2^2 & = 0 \end{cases}$$

in $[-10, 10]^2$*, which has the unique solution* $(0, 0)$*.*

Although it has been noticed in the past that starting from $(3, 1)$ some iterative processes do not converge, recent techniques have been able to converge to the optimal solution whatever the initial value. Table 2 reports the results obtained by nm–CGSLS and the monotone algorithm developed in [34] for nonlinear equations alone, therein denoted by GTSLS3. For comparative purposes we list the results from other papers [20, 28, 31, 38]. These last cited papers use derivatives.

Table 2: Results for Problem 3.4.

Algorithm	k	$n.f.e.$	M	Algorithm	k	$n.f.e.$	M
$x^0 = (3, 1)^T$				$x^0 = (24, 8)^T$			
GTSLS3	5	312	1.5e-8	GTSLS3	5	283	6.1e-8
nm+PGS+(7)	$5^{\S}$	146	4.4e-7	nm+PGS+(7)	$5^{\S}$	174	9.6e-8
nm+PGSm+(7)	$3^{\S}$	131	7.3e-7	nm+PGSm+(7)	5	130	6.9e-7
nm+PGSm+(6)	8	261	5.3e-8	nm+PGSm+(6)	6	249	5.8e-7
nm+HJ+(7)	5	353	1.5e-8	nm+HJ+(7)	7	521	8.2e-11
in [20]	6	10	-‡				
in [28]	10	-	-†				
in [31]	6	8	-†	in [31]	13	16	-†
in [38]	6	6	-†	in [38]	13	13	-†

† stops with error tolerance $\|f\| \leq$ 1.0e-6; ‡ stops with error tolerance $\|f\| \leq$ 1.0e-5

Problem 3.5 *Another nonlinear system with two variables from [20, 31]*

$$\begin{cases} x_1 + 3x_2^2 & = 0 \\ (x_1 - 1.0)x_2 & = 0 \end{cases}$$

which has the unique solution $(0, 0)$ *in* $[-10, 10]^2$*.*

It has been reported that Newton iterates converge to a point on the line $\Gamma = \{(1, y) : y \in \mathbb{R}\}$ if the initial approximation is also a point of the line [20]. Two initial points $(1, 0)$ and $(1, 2)$ have been used to test convergence to the solution. The results are reported in Table 3.

Table 3: Results for Problem 3.5.

Algorithm	k	$n.f.e.$	M	Algorithm	k	$n.f.e.$	M
$x^0 = (1,0)^T$				$x^0 = (1,2)^T$			
GTSLS3	5	95	8.7e-8	GTSLS3	4	56	9.6e-7
nm+PGS+(7)	$5^{\S}$	174	7.9e-8	nm+PGS+(7)	$5^{\S}$	78	5.9e-8
nm+PGSm+(7)	8	99	4.5e-8	nm+PGSm+(7)	$5^{\S}$	66	8.5e-8
nm+HJ+(7)	5	109	2.0e-7	nm+HJ+(7)	5	91	3.8e-8
in [20]	2	5	-‡	in [20]	6	13	-‡
				in [31]	-	15	-†

† stops with error tolerance $\|f\| \leq$ 1.0e-6; ‡ stops with error tolerance $\|f\| \leq$ 1.0e-5

Problem 3.6 *Starting from (-2,1), the solution (0, 0.1196) of the system*

$$\begin{cases} \exp(\sin(x_1^3)) + x_1 - 1 & = 0 \\ 2x_1^2 + 3x_2^2 - 4x_1x_2 + 8(x_2 - x_1) - 1 & = 0 \end{cases}$$

in $[-10, 10]^2$ *was obtained after six iterations of the globalized centered Newton algorithm presented in [11].*

Solver `fsolve` was not able to converge to a solution of the system. The results obtained with nm–CGSLS+PGS+(7) can be summarized as follows: after $k = 5^{\S}$ iterations and 94 function evaluations a solution is reached with M=3.0e-7. The variant nm–CGSLS+PGSm+(7) (with the minimal generating set) needs $k = 5^{\S}$ and 93 function evaluations to converge to a solution. When using the HJ method in the local search procedure and condition (7), the value M=1.5e-12 is reached, after 16 iterations and 1339 function evaluations. When running the monotone version GTSLS3, six iterations and 405 function evaluations are needed to obtain a solution with a merit function value of 9.5e-10.

Problem 3.7 *Consider the system of ten nonlinear equations:*

$$\begin{cases} (3 - 5x_1)x_1 + 1 - 2x_2 & = 0, \\ (3 - 5x_i)x_i + 1 - x_{i-1} - 2x_{i+1} & = 0, \quad 2 \leq i \leq 9 \\ (3 - 5x_{10})x_{10} + 1 - x_9 & = 0 \end{cases}$$

presented and solved in [26] by a conjugate direction particle swarm optimization (CDPSO).

The algorithm in [26] was run 100 times and was regarded as successful if $\max f_i(x)$ <1e-6. The paper reports a successful rate (s.r.) of 95% for CDPSO, while the basic particle swarm optimization algorithm has a s.r. of 81%

and the Newton's method got trapped into a local minimum. We run nm–CGSLS+PGS+(7), using the initial point $(-1,-1,\ldots,-1)$ and $\Omega=[-2,2]^{10}$, and obtain the best solution with M=7.4e-7, for $k=5^{\S}$ and 1920 function evaluations. We report a s.r. of 57%. The variant nm–CGSLS+PGSm+(7) requires $k=6^{\S}$ and $n.f.e.=4265$. When running nm–CGSLS+PGS+(6) we obtain M=9.1e-7 with $k=8$ and $n.f.e.=6118$. When nm–CGSLS+HJ+(7) is used, a s.r. of 27% is registered and the value M=2.8e-8 is reached after six iterations and 6124 function evaluations.

Problem 3.8 *Consider the neurophysiology application which consists of six nonlinear equations of six variables [14]:*

$$\begin{cases} x_1^2+x_3^2-1 &= 0\\ x_2^2+x_4^2-1 &= 0\\ x_5x_3^3+x_6x_4^3-c_1 &= 0\\ x_5x_1^3+x_6x_2^3-c_2 &= 0\\ x_5x_1x_3^2+x_6x_4^2x_2-c_3 &= 0\\ x_5x_1^2x_3+x_6x_2^2x_4-c_4 &= 0 \end{cases}$$

where the constants $c_i=0$, $i=1,\ldots,4$, although they can be randomly generated, and $\Omega=[-10,10]^6$.

The evolutionary approach suggested in [14] obtains a large set of Pareto nondominated solutions, some of which have fitness larger than one, after 200 generations. Although this problem has been used to show some limitations of Newton's method since running time increases exponentially with the initial intervals, we run `fsolve` and a solution was obtained after seven iterations and 56 function evaluations, with a merit function value of around 1.0e-10. While nm–CGSLS+PGS+(7) reaches a merit function value of 9.1e-7 after $k=5^{\S}$ and 397 function evaluations, nm–CGSLS+PGSm+(7) needs $k=5^{\S}$ and 455 function evaluations to reach M=3.6e-7, nm–CGSLS+PGS+(6) needs $k=8$ and $n.f.e.=1144$, nm–CGSLS+PGSm+(6) needs $k=8$ and $n.f.e.=1553$ and nm–CGSLS+HJ+(7) requires $k=5$ and $n.f.e.=544$ to reach M=6.5e-8.

Problem 3.9 *The following problem is considered to be difficult and represents an economics modeling application which can be scaled up to an arbitrary dimension [14]:*

$$\begin{cases} \left(x_j+\displaystyle\sum_{i=1}^{n-j-1} x_i\,x_{i+j}\right)x_n-c_j &= 0, \quad 1\le j\le n-1\\ \displaystyle\sum_{l=1}^{n-1} x_l+1 &= 0 \end{cases}$$

where we set $c_i=0$, $i=1,\ldots,n-1$, although random values can be chosen, consider the case with $n=20$ and $\Omega=[-10,10]^{20}$.

The evolutionary approach used to solve this problem in [14] as a multiobjective optimization problem obtains a set of nondominated solutions, some of which have sum of objectives (in absolute value) greater than 1.5, after 300 generations. The solver `fsolve` was able to converge to a solution with merit function value of 1.0e-10 after four iterations of the trust-region dogleg algorithm, and 105 function evaluations. The nm–CGSLS+PGSm+(7) reaches a solution with a merit function value of 8.4e-7 after eight iterations and 1476 function evaluations. The nm–CGSLS based on PGS with coordinate directions uses $k = 5^{\S}$ and $n.f.e. = 1395$. When the condition (6) is considered, the statistics are: PGSm needs $k = 8$ and $n.f.e. = 9105$ and PGS uses $k = 8$ and $n.f.e. = 8870$. The HJ search requires $k = 6$ and $n.f.e. = 21683$ to converge to a solution with M=3.0e-10.

Problem 3.10 *This problem represents a system of nonsmooth nonlinear equations [29]:*

$$\begin{cases} |x_1| - 0.5x_1x_2 &= 0 \\ x_2^2 + 2x_1 - 2 &= 0 \end{cases}$$

which has two solutions.

The generalized pattern search method presented in [29] converges to $(0, \sqrt{2})$ in 16 iterations and 37 function evaluations, starting from (1,1); and converges to (2,-1) in five iterations and nine function evaluations when the initial point is (1,0). With $\Omega = [-10, 10]^2$, the solution $(0, \sqrt{2})$ is obtained with nm–CGSLS+PGS+(7) in eight iterations and 102 function evaluations, when starting from (1,1), and in eight iterations and 110 function evaluations, when starting from (1,0). On the other hand, nm–CGSLS+PGSm+(7) needs $k = 5^{\S}$ and $n.f.e. = 51$ with the first initial point, and $k = 5^{\S}$ and $n.f.e. = 172$ with the other point. Further, nm–CGSLS+HJ+(7) requires seven iterations and 341 function evaluations, when starting from (1,1), and six iterations and 424 function evaluations, when starting from (1,0).

Problem 3.11 *Consider another nonsmooth nonlinear system, with two variables [14]:*

$$\begin{cases} x_1^2 - x_2^2 &= 0 \\ 1 - |x_1 - x_2| &= 0 \end{cases}$$

which has more than one solution.

The evolutionary approach in [14] converges to a Pareto curve with a set of nine nondominated solutions after 300 generations. The least fitness (sum of the absolute values of the objectives) is around 0.01. We also run `fsolve` and after 20 iterations and 23 function evaluations, the point $(1, 1)$, with merit function value equals to one, is obtained. The solver stopped due to a very

small direction vector. The output parameter 'exitflag' is '-2'. After trying several initial points, the solution $(-0.5, 0.5)$ was finally reached (in three iterations and 12 function evaluations). With $\Omega = [-10, 10]^2$, our proposed nm–CGSLS+PGSm+(7) converges to the same solution with a merit function value of 5.6e-8, after $k = 5^{\S}$ and 59 function evaluations. With the variant based on the generating set of $2n$ coordinate directions, the solution is reached after $k = 5^{\S}$ and 97 function evaluations. The version that is based on the HJ search reaches the solution in four iterations and 68 function evaluations.

Problem 3.12 *Consider the following nonlinear complementarity problem (NCP) [30], which has the three solutions* $(0, 0.5)$, $(0, 3.5)$, $(\frac{\sqrt{2}}{2}, 0.5)$*: find* $(x_1, x_2) \geq 0$, $F(x_1, x_2) \geq 0$ *such that* $x^T F(x) = 0$, *where*

$$F_1(x_1, x_2) = \begin{cases} x_1^2 + |x_2 - 1| - 1 & \text{if } x_1 \neq 0 \\ x_2 & \text{if } x_1 = 0 \end{cases},$$
$$F_2(x_1, x_2) = \begin{cases} x_1 - 0.5 & \text{if } x_2 = 0 \\ |x_2 - 2| - 1.5 & \text{if } x_2 \neq 0 \end{cases}.$$

When running the solver `fsolve`, the trust-region dogleg algorithm cannot be applied and the Levenberg-Marquardt algorithm is used instead, since using the reformulation (2) the overall number of equations is larger than the number of variables. The algorithm solves a least-squares problem. The results obtained with two initial points are depicted in Table 4. In [30], a generalized pattern search filter method is used to solve an equivalent constrained optimization problem. The results therein reported are listed in the Table 4. Finally, the table also reports the best results obtained by the monotone version of the combined algorithm, GTSLS3 [34], as well as those of the proposed nm–CGSLS. Both variants of the PGS outperform the HJ.

Table 4: Results for Problem 3.12.

Algorithm	k	$n.f.e.$	M	Algorithm	k	$n.f.e.$	M
$x^0 = (1, 1)^T$				$x^0 = (2, 2)^T$			
GTSLS3	4	77	1.1e-14	GTSLS3	4	84	3.6e-11
nm+PGS+(7)	$4^{\S}$	53	9.2e-9	nm+PGS+(7)	$4^{\S}$	49	9.5e-7
nm+PGSm+(7)	$5^{\S}$	36	2.3e-7	nm+PGSm+(7)	$5^{\S}$	61	1.9e-8
nm+PGSm+(6)	8	52	5.1e-7	nm+PGSm+(6)	8	85	9.9e-7
nm+HJ+(7)	4	334	1.1e-14	nm+HJ+(7)	5	146	4.8e-11
in [30]	4	9	-	in [30]	12	33	-
`fsolve`	7	26	3.3e-12	`fsolve`	6	21	7.0e-8

Problem 3.13 *Another NCP problem is used to analyze the performance of the proposed algorithm. We aim to find* $x \in \mathbb{R}^4$ *such that* $x \geq 0$, $F(x) \geq 0$ *and* $x^T F(x) = 0$, *where*

$$\begin{aligned} F_1(x) &= 3x_1 + 2x_1x_2 + 2x_2^2 + |x_3 - 3| + 3x_4 - 3, \\ F_2(x) &= 2x_1^2 + x_1 + x_2^2 + 10x_3 + 2x_4 - 2, \\ F_3(x) &= 3x_1^2 + x_1x_2 + 2x_2^2 + 2x_3 + 9x_4 - 9, \\ F_4(x) &= x_1^2 + 3x_2^2 + 2x_3 + 3x_4 - 3, \end{aligned}$$

which has a solution at $(1, 0, 3, 0)$ *[30].*

Based on the initial approximation $(2, 2, 1, 1)$, the generalized pattern search filter method solves an equivalent constrained optimization problem in [30], using 14 iterations and 55 function evaluations. The Levenberg-Marquardt algorithm from MATLAB converges to (0,0,0,1) in 43 iterations and 259 function evaluations. The nm–CGSLS+PGS+(7) uses $k = 7^{\S}$ and 1575 function evaluations to reach a solution with M=5.4e-7. When testing nm–CGSLS+HJ+(7), a solution is obtained in 21 iterations and 9094 function evaluations, with M=3.2e-11.

The main advantage of the herein presented algorithm comes from the fact that no derivatives are required, which is useful when solving systems with nonsmooth functions, such as Problems 3.10 – 3.13, and overdetermined systems can be easily solved, like those that emerge from Problems 3.12 and 3.13. Furthermore, each iteration of the algorithm is not computationally expensive since matrices and the solution of systems of linear equations are not required.

4 Conclusions

The nm–CGSLS algorithm for solving systems of nonlinear inequalities and equalities has been presented. The nonmonotone perspective of the algorithm emerges from the condition that is used to choose between the global search and the local one. A global search aims at exploring the search space for a region where a global minimizer of the merit function is located. A local search procedure aims at refining the search around a located global minimizer and is to be implemented as soon as a promising region is detected. This way, computational requirements are reduced.

The proposed algorithm is a hybrid tabu search. The global search procedure is a variant of the DTS method developed in [16]. In its final intensification phase, the PGS procedure is implemented. In the local search procedure, the nm–CGSTS algorithm implements PGS. The PGS, a hybrid pattern search, gathers ideas from the classical generating set method, either

with a minimal set of $n+1$ directions or with the $2n$ coordinate directions [21], as well as from the perturbed direction strategy [1].

An experimental study has been carried out with a variety of problems collected from the literature. From this study, we may conclude that allowing a nonmonotone reduction of the merit function to choose a local search procedure, in detriment of a global one, reduces computational requirements. Further, the nonmonotone F-rule based on the maximum merit function value of previous iterations has shown to reduce significantly the function evaluations, when compared with the rule that uses a convex combination of previous merit values. The proposal based on the PGS method turns out to be more efficient than the version based on the classical HJ local search.

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