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Research Article

Systems of Navier-Stokes Equations on Cantor Sets

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We present systems of Navier-Stokes equations on Cantor sets, which are described by the local fractional vector calculus. It is shown that the results for Navier-Stokes equations in a fractal bounded domain are efficient and accurate for describing fluid flow in fractal media.

1. Introduction

The Navier-Stokes equations [1–3] are commonly used in describing motion of fluids in models relevant to weather, ocean currents, water flow in pipes, and so forth.

However, the turbulent flows may be of fractal character, and the smoothness required in the common problem modelled by the Navier-Stokes equations may be violated in fractal flows. Such distortions exist in both small-scale [4] and large-scale [5, 6] turbulent flows. In this context, Benzi et al. [7] investigated the scaling properties of turbulent flows while She and Leveque [8] stressed the attention on universal scaling laws in fully developed turbulence. Shraiman and Siggia also reported scalar turbulence [9], while the self-similarity in turbulent flows was analyzed by [10]. The fractal dimension of turbulent flows was pointed out in several studies [10–12], and some applications involving the Navier-Stokes equations were suggested [13–15] taking into account the multifractal nature of fully developed turbulence and chaotic systems [16]. In this context, the alpha model of turbulence [17, 18] and its applications to Navier Stokes have been discussed [19].

Based on fractional calculus theory [20–23] time-fractional Navier-Stokes equation has been recently proposed by El-Shahed and Salem [24]. Several analytical solutions have been developed by the Adomian decomposition method [25] and the homotopy perturbation method [26].

Recently, the element of fractal arc length squared in fractal time-space was discussed [27]. Based on it, the Schrödinger equation [28], the heat conduction equation [27, 29], the wave equation [27, 30], and the diffusion equation on Cantor time-space [31] were suggested based on local fractional calculus theory [27, 32, 33]. Carpinteri and Sapora had reported diffusion equation on Cantor space [34]. Kolwankar and Gangal had proposed the Fokker-Planck equation on Cantor space [35].

In this paper we employ the local fractional vector calculus [27], which is set up on fractals, to model systems of Navier-Stokes equations on Cantor sets. The paper is organized as follows. In Section 2, a short introduction to local fractional vector calculus theory is given. Fractal kinematics, continuity equations, and constitutive equations for Newtonian fluid on Cantor sets are discussed in Section 3. Cauchy's equations of motion and mechanical energy equations on Cantor sets are investigated in Section 4. Systems of Navier-Stokes equations on Cantor sets with Cantorian coordinates systems are presented in Section 5. Finally, in Section 6, the conclusions are presented.

2. Mathematical Tool

Local fractional vector form of function gives [27]

$$\mathbf{r}(x) = M(x)\mathbf{e}_1^{\alpha} + N(x)\mathbf{e}_2^{\alpha} \quad (1)$$

which is expressed by

$$\mathbf{r}(x) = (M(x), N(x)). \quad (2)$$

For a function of three variables, the vector form can be written in the form [27]

$$\mathbf{r}(x,y,z) = L(x,y,z)\mathbf{e}_1^{\alpha} + M(x,y,z)\mathbf{e}_2^{\alpha} + N(x,y,z)\mathbf{e}_3^{\alpha}, \quad (3)$$

or

$$\mathbf{r}(x,y) = (L(x,y,z), M(x,y,z), N(x,y,z)). \quad (4)$$

Let $\mathbf{u}(x,y,z) = u(x,y,z)\mathbf{e}_1^{\alpha} + u_2(x,y,z)\mathbf{e}_2^{\alpha} + u_3(x,y,z)\mathbf{e}_3^{\alpha}$ a local fractional vector field and $\varphi(x,y,z)$ be a local fractional scalar field, the computing rules of Hamilton operator are valid as follows [27].

(1) The local fractional gradient and Laplace operator of a local fractional scalar field are, respectively, defined as [27]

$$\nabla^{\alpha}\varphi = \frac{\partial^{\alpha}\varphi}{\partial x_1^{\alpha}}\mathbf{e}_1^{\alpha} + \frac{\partial^{\alpha}\varphi}{\partial x_2^{\alpha}}\mathbf{e}_2^{\alpha} + \frac{\partial^{\alpha}\varphi}{\partial x_3^{\alpha}}\mathbf{e}_3^{\alpha},$$
$$\nabla^{2\alpha}\varphi = \frac{\partial^{2\alpha}\varphi}{\partial x_1^{2\alpha}} + \frac{\partial^{2\alpha}\varphi}{\partial x_2^{2\alpha}} + \frac{\partial^{2\alpha}\varphi}{\partial x_3^{2\alpha}}, \quad (5)$$

where local fractional partial derivatives of high order are defined as [27, 31–33]

$$\frac{\partial^{k\alpha}}{\partial x^{k\alpha}}f(x) = \overbrace{\frac{\partial^{\alpha}}{\partial x^{\alpha}}\cdots\frac{\partial^{\alpha}}{\partial x^{\alpha}}}^{k\text{ times}} f(x). \quad (6)$$

(2) The local fractional divergence and curl of a local fractional vector field are written in the form [27, 31]

$$\mathbf{div}^{\alpha}\mathbf{u} = \nabla^{\alpha}\cdot\mathbf{u} = \frac{\partial^{\alpha}u_1}{\partial x_1^{\alpha}} + \frac{\partial^{\alpha}u_2}{\partial x_2^{\alpha}} + \frac{\partial^{\alpha}u_3}{\partial x_3^{\alpha}},$$
$$\mathbf{curl}^{\alpha}\mathbf{u} = \nabla^{\alpha}\times\mathbf{u} = \left(\frac{\partial^{\alpha}u_3}{\partial x_2^{\alpha}} - \frac{\partial^{\alpha}u_2}{\partial x_3^{\alpha}}\right)\mathbf{e}_1^{\alpha} + \left(\frac{\partial^{\alpha}u_1}{\partial x_3^{\alpha}} - \frac{\partial^{\alpha}u_3}{\partial x_1^{\alpha}}\right)\mathbf{e}_2^{\alpha} + \left(\frac{\partial^{\alpha}u_2}{\partial x_1^{\alpha}} - \frac{\partial^{\alpha}u_1}{\partial x_2^{\alpha}}\right)\mathbf{e}_3^{\alpha}. \quad (7)$$

(3) The following equations are valid only in Cantor coordinates [27]:

$$\nabla^{\alpha}(\mathbf{u}\mathbf{v}) = (\nabla^{\alpha}\mathbf{u})\mathbf{v} + \mathbf{u}\nabla^{\alpha}\mathbf{v},$$
$$\nabla^{2\alpha}\mathbf{A} = \nabla^{\alpha}(\nabla^{\alpha}\cdot\mathbf{A}) - \nabla^{\alpha}\times(\nabla^{\alpha}\times\mathbf{A}), \quad (8)$$
$$\mathbf{A}\cdot\nabla^{\alpha}\mathbf{A} = \nabla^{\alpha}\left(\frac{\mathbf{A}\cdot\mathbf{A}}{2}\right) - (\nabla^{\alpha}\times\mathbf{A})\times\mathbf{A},$$

where $\nabla^{2\alpha} = \nabla^{\alpha}\cdot\nabla^{\alpha}$ is local fractional Laplace operator [27].

The local fractional line integral of the function $\mathbf{u}(x_P, y_P, z_P)$ in the local fractional vector form

$$\mathbf{u}(x_P, y_P, z_P) = u_1(x_P, y_P, z_P)\mathbf{e}_1^{\alpha} + u_2(x_P, y_P, z_P)\mathbf{e}_2^{\alpha} + u_3(x_P, y_P, z_P)\mathbf{e}_3^{\alpha} \quad (9)$$

along a fractal line l^{α} is written as [27]

$$\int_{l^{(\alpha)}}\mathbf{u}(x_P, y_P, z_P)\cdot d\mathbf{l}^{\alpha} = \lim_{N\to\infty}\sum_{P=1}^{N}\mathbf{u}(x_P, y_P, z_P)\cdot\Delta\mathbf{l}_P^{\alpha}, \quad (10)$$

where for $P = 1, 2, \ldots, N$ and N elements of line $\Delta\mathbf{l}_P^{\alpha}$, it is required that all $|\Delta l_P^{\alpha}| \to 0$ as $N \to \infty$.

The local fractional surface integral of the given function (9) across a surface $S^{(\beta)}$ is defined as [27]

$$\iint_{S^{(\beta)}}\mathbf{u}(x_P, y_P, z_P)\cdot d\mathbf{S}^{(\beta)} = \lim_{N\to\infty}\sum_{P=1}^{N}\mathbf{u}(x_P, y_P, z_P)\cdot\mathbf{n}_P\Delta S_P^{(\beta)}, \quad (11)$$

where for $P = 1, 2, \ldots, N$ and N elements of area $\Delta S_P^{(\beta)}$ with a unit normal local fractional vector n_P, it is required that all $\Delta S_P^{(\beta)} \to 0$ as $N \to \infty$.

The local fractional volume integral of the given function (9) in a fractal region $V^{(\gamma)}$ is given by [27]

$$\iiint_{V^{(\gamma)}}\mathbf{u}(x_P, y_P, z_P)\,dV^{(\gamma)} = \lim_{N\to\infty}\sum_{P=1}^{N}\mathbf{u}(x_P, y_P, z_P)\,\Delta V_P^{(\gamma)}, \quad (12)$$

where for $P = 1, 2, \ldots, N$ and N elements of volume $\Delta V_P^{(\gamma)}$, it is required that all $\Delta V_P^{(\gamma)} \to 0$ as $N \to \infty$.

Let us consider a local fractional vector field $\mathbf{u} = u_1\mathbf{e}_1^{\alpha} + u_2\mathbf{e}_2^{\alpha} + u_3\mathbf{e}_3^{\alpha}$, the following results hold [27]:

(a) Divergence Theorem of local fractional field states that

$$\iiint_{V^{(\gamma)}}\nabla^{\alpha}\cdot\mathbf{u}\,dV^{(\gamma)} = \oiint_{S^{(\beta)}}\mathbf{u}\cdot d\mathbf{S}^{(\beta)}. \quad (13)$$

(b) Stokes' theorem of local fractional field states that

$$\oint_{l^{(\alpha)}}\mathbf{u}\cdot d\mathbf{l}^{\alpha} = \iint_{S^{(\beta)}}(\nabla^{\alpha}\times\mathbf{u})\cdot d\mathbf{S}^{(\beta)}. \quad (14)$$

(c) Green's first theorem in fractal domain states that

$$\oiint_{S^{(\beta)}}\phi\nabla^{\alpha}\varphi\cdot d\mathbf{S}^{(\beta)} = \iiint_{V^{(\gamma)}}\left(\phi\nabla^{2\alpha}\varphi + (\nabla^{\alpha}\phi)\cdot(\nabla^{\alpha}\varphi)\right)dV^{(\gamma)}. \quad (15)$$

(d) Green's second theorem in fractal domain states that

$$\begin{aligned}&\oiint_{S^{(\beta)}}\left(\phi\nabla^{\alpha}\varphi-\phi\nabla^{\alpha}\varphi\right)\cdot d\mathbf{S}^{(\beta)}\\&=\iiint_{V^{(\gamma)}}\left(\phi\nabla^{2\alpha}\varphi-\varphi\nabla^{2\alpha}\phi\right)dV^{(\gamma)},\end{aligned}\tag{16}$$

where the relationships of fractal dimensions of line, surface, and volume fields are

$$\beta=2\alpha,\qquad\gamma=3\alpha.\tag{17}$$

3. Fractal Kinematics, Continuity Equations, and Constitutive Equations for Newtonian Fluid on Cantor Sets

In this section, we investigate the fractal kinematics, balance of mass on Cantor sets, and constitutive equations for Newtonian fluid on Cantor sets. We first start with fractal kinematics.

3.1. Fractal Kinematics. For any scalar property ψ associated with an abstract Cantor body B expressed as

$$\psi=\psi\left(P,t\right)=\widehat{\psi}\left(X,t\right)=\widetilde{\psi}\left(x,t\right),\tag{18}$$

we define the following Lagrangian and Eulerian temporal and spatial derivatives with local fractional differential operator:

$$\begin{gathered}\dot{\psi}=\frac{D^{\alpha}\widehat{\psi}}{Dt^{\alpha}},\qquad\psi_{,t}^{\alpha}=\frac{\partial^{\alpha}\widetilde{\psi}}{\partial t^{\alpha}},\qquad\nabla_{X}^{\alpha}\psi=\frac{\partial^{\alpha}\widehat{\psi}}{\partial X^{\alpha}},\\\nabla_{x}^{\alpha}\psi=\nabla^{\alpha}\psi=\frac{\partial^{\alpha}\widetilde{\psi}}{\partial x^{\alpha}}.\end{gathered}\tag{19}$$

The velocity $\boldsymbol{v}$ and the acceleration $\mathbf{a}$ with local fractional operator are defined through

$$\boldsymbol{v}=\frac{D^{\alpha}\boldsymbol{\chi}}{Dt^{\alpha}},\qquad\mathbf{a}=\frac{D^{2\alpha}\boldsymbol{\chi}}{Dt^{2\alpha}}.\tag{20}$$

It immediately follows that the velocity gradient $\boldsymbol{\Theta}$ and its symmetric part $\boldsymbol{\Lambda}$ are expressed through

$$\boldsymbol{\Theta}=\nabla^{\alpha}\cdot\boldsymbol{v},\qquad\boldsymbol{\Theta}^{T}=\boldsymbol{v}\cdot\nabla^{\alpha},\qquad\boldsymbol{\Lambda}=\frac{1}{2}\left(\boldsymbol{\Theta}+\boldsymbol{\Theta}^{T}\right).\tag{21}$$

The deformation gradient $\boldsymbol{\Sigma}$ is given by

$$\boldsymbol{\Sigma}=\nabla^{\alpha}\cdot\boldsymbol{\chi},\tag{22}$$

which leads to

$$\boldsymbol{\Theta}=\frac{\partial^{\alpha}\boldsymbol{\Sigma}}{\partial t^{\alpha}}\boldsymbol{\Sigma}^{-1}.\tag{23}$$

The fractal material derivative of the fluid density ρ is defined as [27]

$$\frac{D^{\alpha}\rho}{Dt^{\alpha}}=\frac{\partial^{\alpha}\rho}{\partial t^{\alpha}}+\boldsymbol{v}\cdot\nabla^{\alpha}\rho.\tag{24}$$

3.2. Balance of Mass on Cantor Sets. The mass of fractal fluid in $V^{(\gamma)}$ is defined through [27]

$$M=\iiint_{V^{(\gamma)}}\rho dV^{(\gamma)},\tag{25}$$

which yields that the balance of mass on Cantor sets takes the form [27]

$$\frac{\partial^{\alpha}M}{\partial t^{\alpha}}=-\oiint_{S^{(\beta)}}\rho\boldsymbol{v}\cdot d\mathbf{S}^{(\beta)}.\tag{26}$$

Using Divergence Theorem of local fractional field, we deduce to

$$\begin{aligned}&\iiint_{V^{(\gamma)}}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\rho dV^{(\gamma)}+\oiint_{S^{(\beta)}}\rho\boldsymbol{v}\cdot d\mathbf{S}^{(\beta)}\\&=\iiint_{V^{(\gamma)}}\left[\frac{\partial^{\alpha}\rho}{\partial t^{\alpha}}+\nabla^{\alpha}\cdot\left(\rho\boldsymbol{v}\right)\right]dV^{(\gamma)}=0\end{aligned}\tag{27}$$

which implies that

$$\frac{\partial^{\alpha}\rho}{\partial t^{\alpha}}+\nabla^{\alpha}\cdot\left(\rho\boldsymbol{v}\right)=0.\tag{28}$$

This is called continuity equation on Cantor sets.

It becomes

$$\begin{aligned}\frac{\partial^{\alpha}\rho}{\partial t^{\alpha}}+\nabla^{\alpha}\cdot\left(\rho\boldsymbol{v}\right)&=\frac{\partial^{\alpha}\rho}{\partial t^{\alpha}}+\boldsymbol{v}\cdot\left(\nabla^{\alpha}\rho\right)+\rho\left(\nabla^{\alpha}\cdot\boldsymbol{v}\right)\\&=\frac{D^{\alpha}\rho}{Dt^{\alpha}}+\rho\left(\nabla^{\alpha}\cdot\boldsymbol{v}\right)=0,\end{aligned}\tag{29}$$

where the fractal material derivative of the fluid density ρ is noted by

$$\frac{D^{\alpha}\rho}{Dt^{\alpha}}=\frac{\partial^{\alpha}\rho}{\partial t^{\alpha}}+\boldsymbol{v}\cdot\left(\nabla^{\alpha}\rho\right).\tag{30}$$

If the fractal fluid is incompressible, we deduce that

$$\frac{\partial^{\alpha}\rho}{\partial t^{\alpha}}+\boldsymbol{v}\cdot\left(\nabla^{\alpha}\rho\right)=0\Longleftrightarrow\frac{D^{\alpha}\rho}{Dt^{\alpha}}=0\quad\text{or}\quad\nabla^{\alpha}\cdot\boldsymbol{v}=0.\tag{31}$$

It follows that Reynolds transport theorem on Cantor sets states that

$$\begin{aligned}\frac{D^{\alpha}}{Dt^{\alpha}}\iiint_{V^{(\gamma)}}F\left(x,t\right)dV^{(\gamma)}=&\iiint_{V^{(\gamma)}}\frac{\partial^{\alpha}}{\partial t^{\alpha}}F\left(x,t\right)dV^{(\gamma)}\\&+\oiint_{S^{(\beta)}}F\left(x,t\right)\boldsymbol{v}\cdot d\mathbf{S}^{(\beta)},\end{aligned}\tag{32}$$

where $\boldsymbol{v}$ is the fractal fluid velocity, $V^{(\gamma)}$ is the fractal material volume, and $\mathbf{S}^{(\beta)}$ is the fractal surfaces moving with the fractal fluid.

3.3. Constitutive Equations for Newtonian Fluid on Cantor Sets. We will assume a linear relation of the type of fractal Cauchy stress

$$\mathbf{J}=-p\mathbf{I}+\mathbf{K}:\boldsymbol{\Lambda},\tag{33}$$

where p is the thermodynamic pressure, $\mathbf{K}$ is a fourth-order fractal tensor, $\mathbf{\Lambda}$ is the fractal strain rate tensor, and $\mathbf{I}$ is unit vector in local fractional field.

The fractal velocity gradient tensor can be decomposed into symmetric and antisymmetric parts [27]

$$\nabla^{\alpha} \cdot \boldsymbol{v} = \frac{1}{2}\left(\mathbf{\Theta} + \mathbf{\Theta}^{T}\right) + \frac{1}{2}\left(\mathbf{\Theta} - \mathbf{\Theta}^{T}\right) = \mathbf{\Lambda} + \frac{1}{2}\left(\mathbf{\Theta} - \mathbf{\Theta}^{T}\right). \tag{34}$$

We can write

$$\mathbf{K} : \mathbf{\Lambda} = 2\mu\mathbf{\Lambda} + \lambda\left(\nabla^{\alpha} \cdot \boldsymbol{v}\right)\mathbf{I} \tag{35}$$

which leads to

$$\mathbf{J} = -p\mathbf{I} + 2\mu\mathbf{\Lambda} + \lambda\left(\nabla^{\alpha} \cdot \boldsymbol{v}\right)\mathbf{I}, \tag{36}$$

where $(\nabla^{\alpha} \cdot \boldsymbol{v})$ is the fractal volumetric strain rate and $\mathbf{\Lambda} = (1/2)(\nabla^{\alpha} \cdot \boldsymbol{v} + \boldsymbol{v} \cdot \nabla^{\alpha})$ [27], λ and μ are the bulk and shear moduli of viscosity.

The constitutive equation of homogeneous compressible Euler fluid on Cantor sets reads

$$\mathbf{J} = -p\left(\rho\right)\mathbf{I}. \tag{37}$$

The constitutive equation of homogeneous incompressible Euler fluid on Cantor sets is

$$\mathbf{J} = -p\mathbf{I}, \qquad \nabla^{\alpha} \cdot \boldsymbol{v} = 0. \tag{38}$$

The constitutive equation of homogeneous compressible Navier-Stokes fluid on Cantor sets is written in the form

$$\mathbf{J} = -p\left(\rho\right)\mathbf{I} + 2\mu\left(\rho\right)\mathbf{\Lambda} + \lambda\left(\rho\right)\left(\nabla^{\alpha} \cdot \boldsymbol{v}\right)I. \tag{39}$$

The constitutive equation of homogeneous incompressible Navier-Stokes fluid on Cantor sets is expressed as

$$\mathbf{J} = -p\mathbf{I} + 2\mu\mathbf{\Lambda}, \qquad \nabla^{\alpha} \cdot \boldsymbol{v} = 0. \tag{40}$$

By using Stokes' assumption $\lambda(\rho) = -2\mu(\rho)/3$, the constitutive equation of homogeneous compressible Navier-Stokes fluid on Cantor sets becomes

$$\mathbf{J} = -p\left(\rho\right)\mathbf{I} + 2\mu\left(\rho\right)\left(\nabla^{\alpha} \cdot \boldsymbol{v} + \boldsymbol{v} \cdot \nabla^{\alpha}\right) - \frac{2}{3}\mu\left(\rho\right)\left(\nabla^{\alpha} \cdot \boldsymbol{v}\right), \tag{41}$$

while the constitutive equation of homogeneous incompressible Navier-Stokes fluid on Cantor sets can obtain that

$$\mathbf{J} = -p\mathbf{I} + \mu\left(\nabla^{\alpha} \cdot \boldsymbol{v} + \boldsymbol{v} \cdot \nabla^{\alpha}\right), \qquad \nabla^{\alpha} \cdot \boldsymbol{v} = 0. \tag{42}$$

4. Cauchy's Equations of Motion and Mechanical Energy Equations on Cantor Sets

In this section, we consider balance of linear and angular momentums and energy on Cantor sets.

4.1. Balances of Linear and Angular Momentums on Cantor Sets. If the second law of Newton in fractal mechanics is valid, the balance of linear momentum for flows on Cantor sets takes the form [27]

$$\frac{D^{\alpha}}{Dt^{\alpha}}\iiint_{V^{(\gamma)}}\rho\boldsymbol{v}dV^{(\gamma)} = \iiint_{V^{(\gamma)}}\rho\mathbf{b}dV^{(\gamma)} + \oiint_{S^{(\beta)}}\mathbf{J} \cdot d\mathbf{S}^{(\beta)}, \tag{43}$$

where $\mathbf{J}$ denotes the fractal Cauchy stress tensor [27] and $\mathbf{b}$ denotes the specific fractal body force [27].

By using (32), (43) becomes

$$\begin{aligned}\frac{D^{\alpha}}{Dt^{\alpha}}\iiint_{V^{(\gamma)}}\rho\boldsymbol{v}dV^{(\gamma)} &= \iiint_{V^{(\gamma)}}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\left(\rho\boldsymbol{v}\right)dV^{(\gamma)} \\ &\quad + \oiint_{S^{(\beta)}}\rho\boldsymbol{v}\boldsymbol{v} \cdot d\mathbf{S}^{(\beta)} \\ &= \iiint_{V^{(\gamma)}}\rho\mathbf{b}dV^{(\gamma)} + \oiint_{S^{(\beta)}}\mathbf{J} \cdot d\mathbf{S}^{(\beta)}\end{aligned} \tag{44}$$

which is rewritten as

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}}\left(\rho\boldsymbol{v}\right) + \nabla^{\alpha} \cdot \left[\rho\boldsymbol{v}\boldsymbol{v}\right] = \rho\mathbf{b} + \nabla^{\alpha} \cdot \mathbf{J}. \tag{45}$$

In view of (7), (45) yields

$$\frac{\partial^{\alpha}\rho}{\partial t^{\alpha}}\boldsymbol{v} + \rho\frac{\partial^{\alpha}\boldsymbol{v}}{\partial t^{\alpha}} + \boldsymbol{v}\boldsymbol{v} \cdot \nabla^{\alpha}\rho + \rho\boldsymbol{v}\nabla^{\alpha} \cdot \boldsymbol{v} + \rho\boldsymbol{v} \cdot \nabla^{\alpha}\boldsymbol{v} = \rho\mathbf{b} + \nabla^{\alpha} \cdot \mathbf{J} \tag{46}$$

which implies that

$$\boldsymbol{v}\left(\frac{\partial^{\alpha}\rho}{\partial t^{\alpha}} + \nabla^{\alpha}\left(\boldsymbol{v} \cdot \rho\right)\right) + \rho\left(\frac{\partial^{\alpha}\boldsymbol{v}}{\partial t^{\alpha}} + \boldsymbol{v} \cdot \nabla^{\alpha}\boldsymbol{v}\right) = \rho\mathbf{b} + \nabla^{\alpha} \cdot \mathbf{J} \tag{47}$$

or

$$\begin{aligned}&\boldsymbol{v}\left(\frac{\partial^{\alpha}\rho}{\partial t^{\alpha}} + \boldsymbol{v} \cdot \nabla^{\alpha}\rho + \rho\nabla^{\alpha} \cdot \boldsymbol{v}\right) + \rho\left(\frac{\partial^{\alpha}\boldsymbol{v}}{\partial t^{\alpha}} + \boldsymbol{v} \cdot \nabla^{\alpha}\boldsymbol{v}\right) \\ &\quad = \rho\mathbf{b} + \nabla^{\alpha} \cdot \mathbf{J}.\end{aligned} \tag{48}$$

Therefore, (48) is reexpressed by

$$\boldsymbol{v}\left(\frac{\partial^{\alpha}\rho}{\partial t^{\alpha}} + \nabla^{\alpha} \cdot \left(\rho\boldsymbol{v}\right)\right) + \rho\left(\frac{\partial^{\alpha}\boldsymbol{v}}{\partial t^{\alpha}} + \boldsymbol{v} \cdot \nabla^{\alpha}\boldsymbol{v}\right) = \rho\mathbf{b} + \nabla^{\alpha} \cdot \mathbf{J}. \tag{49}$$

Taking continuity equation on Cantor sets and fractal material derivative yields

$$\rho\frac{D^{\alpha}\boldsymbol{v}}{Dt^{\alpha}} = \rho\mathbf{b} + \nabla^{\alpha} \cdot \mathbf{J}. \tag{50}$$

Hence, balance of linear momentum in its local Cantorian form (the Newton's law in its local Cantorian form) is rewritten as

$$\rho\frac{D^{\alpha}\boldsymbol{v}}{Dt^{\alpha}} = \nabla^{\alpha} \cdot \mathbf{J} + \rho\mathbf{b} \tag{51}$$

or

$$\rho \frac{\partial^{\alpha} \boldsymbol{v}}{\partial t^{\alpha}} = \nabla^{\alpha} \cdot \mathbf{J} + \rho \mathbf{b} - \rho \boldsymbol{v} \left(\nabla^{\alpha} \cdot \boldsymbol{v}\right). \tag{52}$$

It is called as the Cauchy's equation of motion of flows on Cantor sets.

In the absence of internal couples, the balance of angular momentum on Cantor sets implies that the fractal Cauchy stress is symmetric [27]; that is,

$$\rho \frac{D^{\alpha} \boldsymbol{v}}{D t^{\alpha}} = \nabla^{\alpha} \cdot \mathbf{J}^{T} + \rho \mathbf{b} \quad \text{or} \quad \mathbf{J} = \mathbf{J}^{T}. \tag{53}$$

Taking fractal material derivative, (45) is expressed by

$$\rho \frac{\partial^{\alpha} \boldsymbol{v}}{\partial t^{\alpha}} + \rho \boldsymbol{v} \left(\nabla^{\alpha} \cdot \boldsymbol{v}\right) = \nabla^{\alpha} \cdot \mathbf{J}^{T} + \rho \mathbf{b} \tag{54}$$

or

$$\mathbf{J} = \mathbf{J}^{T}. \tag{55}$$

which yields the Newton's law in local fractional integration form given by

$$\iiint_{V^{(\gamma)}} \nabla^{\alpha} \cdot \left(\rho \boldsymbol{v} \cdot \boldsymbol{v} - \mathbf{J}^{T}\right) dV^{(\gamma)} = \oiint_{S^{(\beta)}} \left[\rho \boldsymbol{v} \cdot \boldsymbol{v} - \mathbf{J}^{T}\right] d\mathbf{S}^{(\beta)} \tag{56}$$

if the compressible flow on Cantor sets is steady and body forces are absent. The results are different from [36–38] because of the fractional operators.

In local Eulerian form, Cauchy's equation of motion of flows on Cantor sets is given by using (37)

$$\frac{D^{\alpha}}{D t^{\alpha}} \iiint_{V^{(\gamma)}} \rho \boldsymbol{v} dV^{(\gamma)} = \iiint_{V^{(\gamma)}} \rho \mathbf{b} dV^{(\gamma)} - \oiint_{S^{(\beta)}} p \mathbf{I} \cdot d\mathbf{S}^{(\beta)} \tag{57}$$

which implies that

$$\rho \frac{D^{\alpha} \boldsymbol{v}}{D t^{\alpha}} = -\nabla^{\alpha} p\left(\rho\right) + \rho \mathbf{b}. \tag{58}$$

For compressible fluid, Cauchy's equation of motion of flows on Cantor sets is obtained by using (39)

$$\begin{aligned} \frac{D^{\alpha}}{D t^{\alpha}} & \iiint_{V^{(\gamma)}} \rho \boldsymbol{v} dV^{(\gamma)} \\ &= \iiint_{V^{(\gamma)}} \rho \mathbf{b} dV^{(\gamma)} \\ &\quad - \oiint_{S^{(\beta)}} \left(p\left(\rho\right) \mathbf{I} + \mu\left(\rho\right) \left(\frac{1}{3} \left(\nabla^{\alpha} \cdot \boldsymbol{v}\right) + \boldsymbol{v} \cdot \nabla^{\alpha}\right)\right) \cdot d\mathbf{S}^{(\beta)} \end{aligned} \tag{59}$$

which is

$$\begin{aligned} \rho \frac{D^{\alpha} \boldsymbol{v}}{D t^{\alpha}} = & -\nabla^{\alpha} p\left(\rho\right) + \frac{1}{3} \nabla^{\alpha} \cdot \left[\mu\left(\rho\right) \left(\nabla^{\alpha} \cdot \boldsymbol{v}\right)\right] \\ & + \nabla^{\alpha} \cdot \left(\mu\left(\rho\right) \boldsymbol{v} \cdot \nabla^{\alpha}\right) + \rho \mathbf{b}. \end{aligned} \tag{60}$$

For incompressible fluid, Cauchy's equation of motion of flows on Cantor sets is obtained by using (40)

$$\begin{aligned} \frac{D^{\alpha}}{D t^{\alpha}} & \iiint_{V^{(\gamma)}} \rho \boldsymbol{v} dV^{(\gamma)} \\ &= \iiint_{V^{(\gamma)}} \rho \mathbf{b} dV^{(\gamma)} \\ &\quad + \oiint_{S^{(\beta)}} \left(-p \mathbf{I} + \mu \left(\nabla^{\alpha} \cdot \boldsymbol{v} + \boldsymbol{v} \cdot \nabla^{\alpha}\right)\right) \cdot d\mathbf{S}^{(\beta)} \end{aligned} \tag{61}$$

which leads to

$$\rho \frac{D^{\alpha} \boldsymbol{v}}{D t^{\alpha}} = -\nabla^{\alpha} p + \mu \nabla^{\alpha} \left(\boldsymbol{v} \cdot \nabla^{\alpha}\right) + \rho \mathbf{b} \tag{62}$$

or

$$\rho \frac{D^{\alpha} \boldsymbol{v}}{D t^{\alpha}} = -\nabla^{\alpha} p + \mu \nabla^{\alpha} \left(\boldsymbol{v} \cdot \nabla^{\alpha}\right). \tag{63}$$

4.2. Balance of Energy on Cantor Sets. The mechanical energy equation with viscous dissipation on Cantor sets is repeated by

$$\begin{aligned} \frac{D^{\alpha}}{D t^{\alpha}} & \iiint_{V^{(\gamma)}} \left(\rho \varphi\right) dV^{(\gamma)} + \oiint_{S^{(\beta)}} \left(\rho \varphi \boldsymbol{v}\right) \cdot d\mathbf{S}^{(\beta)} \\ &= \iiint_{V^{(\gamma)}} \left(\rho \mathbf{b} \cdot \boldsymbol{v}\right) dV^{(\gamma)} + \oiint_{S^{(\beta)}} \boldsymbol{v} \cdot \mathbf{J} d\mathbf{S}^{(\beta)} \\ &\quad - \iiint_{V^{(\gamma)}} \nabla^{\alpha} \cdot \left(p \boldsymbol{v}\right) dV^{(\gamma)} - \iiint_{V^{(\gamma)}} \phi dV^{(\gamma)}, \end{aligned} \tag{64}$$

where the first term is rate of change of kinetic energy in fractal domain, the second term is rate of outflow across fractal boundary, the third term is rate of work by fractal body force, the fourth term is rate of work by surface force, the fifth term is rate of work by volume expansion, and the last term is rate of viscous dissipation.

Local fractional differential form of the mechanical energy equation with viscous dissipation on Cantor sets gives

$$\begin{aligned} \frac{D^{\alpha}}{D t^{\alpha}} \left(\rho \varphi\right) + \nabla^{\alpha} \cdot \left(\rho \varphi \boldsymbol{v}\right) = & \rho \mathbf{b} \cdot \boldsymbol{v} + \boldsymbol{v} \cdot \left(\nabla^{\alpha} \cdot \mathbf{J}\right) \\ & + \nabla^{\alpha} \cdot \left(p \boldsymbol{v}\right) - \phi, \end{aligned} \tag{65}$$

which implies for fractal material derivative of $\rho \varphi$

$$\begin{aligned} \frac{\partial^{\alpha}}{\partial t^{\alpha}} & \left(\rho \varphi\right) + \boldsymbol{v} \cdot \nabla^{\alpha} \left(\rho \varphi\right) + \nabla^{\alpha} \cdot \left(\rho \varphi \boldsymbol{v}\right) \\ &= \rho \mathbf{b} \cdot \boldsymbol{v} + \boldsymbol{v} \cdot \left(\nabla^{\alpha} \cdot \mathbf{J}\right) - \nabla^{\alpha} \cdot \left(p \boldsymbol{v}\right) - \phi, \end{aligned} \tag{66}$$

where the rate of viscous dissipation of kinetic energy per unit volume is ϕ.

The integration form of the balance of energy on Cantor sets is expressed by

$$\begin{aligned}\frac{D^{\alpha}}{Dt^{\alpha}}\iiint_{V^{(\gamma)}}\rho(\theta+\varphi)\,dV^{(\gamma)} &= \oiint_{S^{(\beta)}}\boldsymbol{v}\cdot\mathbf{J}\cdot d\mathbf{S}^{(\beta)}-\oiint_{S^{(\beta)}}\rho(\theta+\varphi)\,\boldsymbol{v}\cdot d\mathbf{S}^{(\beta)} \\ &\quad+\iiint_{V^{(\gamma)}}(\rho\mathbf{b}\cdot\boldsymbol{v})\,dV^{(\gamma)}+\oiint_{S^{(\beta)}}K^{2\alpha}\mathbf{q}\cdot d\mathbf{S}^{(\beta)} \\ &\quad-\iiint_{V^{(\gamma)}}\phi dV^{(\gamma)}\end{aligned} \tag{67}$$

which leads to

$$\begin{aligned}\frac{D^{\alpha}}{Dt^{\alpha}}[\rho(\theta+\varphi)] &= \boldsymbol{v}\cdot(\nabla^{\alpha}\cdot\mathbf{J})+\rho\mathbf{b}\cdot\boldsymbol{v}-\nabla^{\alpha}\cdot(p\boldsymbol{v}) \\ &\quad-\nabla^{\alpha}\cdot\left[\rho(\theta+\varphi)\,\boldsymbol{v}-K^{2\alpha}\cdot\mathbf{q}\right]-\phi,\end{aligned} \tag{68}$$

where $K^{2\alpha}$ is denoted as fractal heat conduction coefficient and $\mathbf{q}$ is fractal temperature flied.

The term

$$\oiint_{S^{(\beta)}}\rho(\theta+\varphi)\,\boldsymbol{v}\cdot d\mathbf{S}^{(\beta)}=\iiint_{V^{(\gamma)}}\nabla^{\alpha}\cdot[\rho(\theta+\varphi)]\,\boldsymbol{v}dV^{(\gamma)} \tag{69}$$

vanishes if the term $\rho\theta\boldsymbol{v}$ is zero at the boundaries and

$$\begin{aligned}\iiint_{V^{(\gamma)}}(p(\nabla^{\alpha}\cdot\boldsymbol{v}))\,dV^{(\gamma)} &= \oiint_{S^{(\beta)}}(p\boldsymbol{v})\,d\mathbf{S}^{(\beta)} \\ &\quad-\iiint_{V^{(\gamma)}}[\boldsymbol{v}\cdot(\nabla^{\alpha}p)]\,dV^{(\gamma)}.\end{aligned} \tag{70}$$

The mechanical energy equation with no viscous dissipation on Cantor sets is written in the local Cantorian form as

$$\rho\frac{D^{\alpha}}{Dt^{\alpha}}(\theta+\varphi)=-\nabla^{\alpha}p\cdot\boldsymbol{v}+\boldsymbol{v}\cdot(\nabla^{\alpha}\cdot\mathbf{J})+\rho\mathbf{b}\cdot\boldsymbol{v}+K^{2\alpha}\nabla^{\alpha}\cdot\mathbf{q}, \tag{71}$$

where φ is the kinetic energy per unit of mass, $\mathbf{b}$ is the external force per unit of mass, $\mathbf{J}$ is the fractal Cauchy stress tensor, and θ is the internal energy per unit of mass.

5. Systems of Navier-Stokes Equation on Cantor Sets

In this section, we investigate the Navier-Stokes equation on Cantor sets and derive systems of Navier-Stokes equation on Cantor sets.

5.1. Navier-Stokes Equation on Cantor Sets. Substituting the constitutive equation into Cauchy's equation (41) yields

$$\begin{aligned}\rho\frac{D^{\alpha}\boldsymbol{v}}{Dt^{\alpha}} &= -\nabla^{\alpha}\cdot(p\mathbf{I}) \\ &\quad+\nabla^{\alpha}\left[2\mu(\nabla^{\alpha}\cdot\boldsymbol{v}+\boldsymbol{v}\cdot\nabla^{\alpha})-\frac{2}{3}\mu(\nabla^{\alpha}\cdot\boldsymbol{v})\,\mathbf{I}\right]+\rho\mathbf{b},\end{aligned} \tag{72}$$

where viscosity μ in this equation can be a function of the thermodynamic state.

This is a general form of the Navier-Stokes equation on Cantor sets, which is the equation of motion for a Newtonian fluid on Cantor sets.

For incompressible fluids $\nabla^{\alpha}\cdot\boldsymbol{v}=0$, we deduce to

$$\rho\frac{D^{\alpha}\boldsymbol{v}}{Dt^{\alpha}}=-\nabla^{\alpha}p+\mu\nabla^{2\alpha}\boldsymbol{v}+\rho\mathbf{b}. \tag{73}$$

Applying fractal material derivative of the fluid velocity,

$$\frac{D^{\alpha}\boldsymbol{v}}{Dt^{\alpha}}=\frac{\partial^{\alpha}\boldsymbol{v}}{\partial t^{\alpha}}+\boldsymbol{v}(\nabla^{\alpha}\cdot\boldsymbol{v}), \tag{74}$$

in compressible fluids, a general form of the Navier-Stokes equation on Cantor sets is stated as

$$\rho\frac{\partial^{\alpha}\boldsymbol{v}}{\partial t^{\alpha}}=-\nabla^{\alpha}p+\mu\nabla^{2\alpha}\boldsymbol{v}+\rho\mathbf{b}-\rho\boldsymbol{v}(\nabla^{\alpha}\cdot\boldsymbol{v}). \tag{75}$$

For compressible fluid, the Navier-Stokes equation on Cantor sets reads

$$\begin{aligned}\rho\frac{D^{\alpha}\boldsymbol{v}}{Dt^{\alpha}} &= -\nabla^{\alpha}p+\nabla^{\alpha}[\mu(\nabla^{\alpha}\cdot\boldsymbol{v}+\boldsymbol{v}\cdot\nabla^{\alpha})] \\ &\quad+\nabla^{\alpha}(\lambda(\nabla^{\alpha}\cdot\boldsymbol{v}))+\rho\mathbf{b}\end{aligned} \tag{76}$$

which becomes for $\lambda=-(2/3)\mu$

$$\rho\frac{D^{\alpha}\boldsymbol{v}}{Dt^{\alpha}}=-\nabla^{\alpha}p+\frac{1}{3}\mu\nabla^{\alpha}((\nabla^{\alpha}\cdot\boldsymbol{v}))+\mu\nabla^{2\alpha}\boldsymbol{v}+\rho\mathbf{b} \tag{77}$$

which yields by using (55)

$$\begin{aligned}\rho\frac{\partial^{\alpha}\boldsymbol{v}}{\partial t^{\alpha}} &= -\nabla^{\alpha}p+\frac{1}{3}\mu\nabla^{\alpha}((\nabla^{\alpha}\cdot\boldsymbol{v})) \\ &\quad+\mu\nabla^{2\alpha}\boldsymbol{v}+\rho\mathbf{b}-\rho\boldsymbol{v}(\nabla^{\alpha}\cdot\boldsymbol{v}).\end{aligned} \tag{78}$$

5.2. Systems of Navier-Stokes Equation on Cantor Sets. In this section, we consider systems of Navier-Stokes fluid on Cantor sets, which states that the systems consists of the continuity equation, the motion equation, and the energy balance equation on Cantor sets.

By using (29), (71), and (77), systems of compressible Navier-Stokes equations on Cantor sets become as follows:

$$\begin{gathered}\frac{\partial^{\alpha}\rho}{\partial t^{\alpha}}+\nabla^{\alpha}\cdot(\rho\boldsymbol{v})=0, \\ \rho\frac{D^{\alpha}\boldsymbol{v}}{Dt^{\alpha}}=-\nabla^{\alpha}p+\frac{1}{3}\mu\nabla^{\alpha}((\nabla^{\alpha}\cdot\boldsymbol{v}))+\mu\nabla^{2\alpha}\boldsymbol{v}+\rho\mathbf{b}, \\ \rho\frac{D^{\alpha}}{Dt^{\alpha}}(\theta+\varphi)=-\nabla^{\alpha}\cdot(p\boldsymbol{v})+\boldsymbol{v}\cdot(\nabla^{\alpha}\cdot\mathbf{J}) \\ +\rho\mathbf{b}\cdot\boldsymbol{v}+K^{2\alpha}\nabla^{\alpha}\cdot\mathbf{q}.\end{gathered} \tag{79}$$

By applying (31), (71), and (73), systems of incompressible Navier-Stokes equations on Cantor sets are stated as

$$\begin{aligned}
&\nabla^{\alpha}\cdot\boldsymbol{v}=0,\\
&\rho\frac{D^{\alpha}\boldsymbol{v}}{Dt^{\alpha}}=-\nabla^{\alpha}p+\mu\nabla^{2\alpha}\boldsymbol{v}+\rho\mathbf{b},\\
&\rho\frac{D^{\alpha}}{Dt^{\alpha}}(\theta+\varphi)=-\nabla^{\alpha}\cdot(p\boldsymbol{v})+\boldsymbol{v}\cdot(\nabla^{\alpha}\cdot\mathbf{J})\\
&\qquad+\rho\mathbf{b}\cdot\boldsymbol{v}+K^{2\alpha}\nabla^{\alpha}\cdot\mathbf{q},
\end{aligned}\tag{80}$$

or in the 3D Cantorian coordinates, systems of Navier-Stokes equations on Cantor sets can be written for each component, x, y, and z as

$$\begin{aligned}
&\frac{\partial^{\alpha}v_x}{\partial x^{\alpha}}+\frac{\partial^{\alpha}v_y}{\partial y^{\alpha}}+\frac{\partial^{\alpha}v_z}{\partial z^{\alpha}}=0,\\
&\rho\frac{D^{\alpha}v_x}{Dt^{\alpha}}=-\frac{\partial^{\alpha}p}{\partial x^{\alpha}}+\mu\left(\frac{\partial^{2\alpha}v_x}{\partial x^{2\alpha}}+\frac{\partial^{2\alpha}v_x}{\partial y^{2\alpha}}+\frac{\partial^{2\alpha}v_x}{\partial z^{2\alpha}}\right)+\rho b_x,\\
&\rho\frac{D^{\alpha}v_y}{Dt^{\alpha}}=-\frac{\partial^{\alpha}p}{\partial y^{\alpha}}+\mu\left(\frac{\partial^{2\alpha}v_y}{\partial x^{2\alpha}}+\frac{\partial^{2\alpha}v_y}{\partial y^{2\alpha}}+\frac{\partial^{2\alpha}v_y}{\partial z^{2\alpha}}\right)+\rho b_y,\\
&\rho\frac{D^{\alpha}v_z}{Dt^{\alpha}}=-\frac{\partial^{\alpha}p}{\partial z^{\alpha}}+\mu\left(\frac{\partial^{2\alpha}v_z}{\partial x^{2\alpha}}+\frac{\partial^{2\alpha}v_z}{\partial y^{2\alpha}}+\frac{\partial^{2\alpha}v_z}{\partial z^{2\alpha}}\right)+\rho b_z,\\
&\rho\frac{D^{\alpha}}{Dt^{\alpha}}(\theta+\varphi)=-\left(\frac{\partial^{\alpha}(pv_x)}{\partial x^{\alpha}}+\frac{\partial^{\alpha}(pv_y)}{\partial y^{\alpha}}+\frac{\partial^{\alpha}(pv_z)}{\partial z^{\alpha}}\right)\\
&\qquad+v_x\left(\frac{\partial^{\alpha}J_{xx}}{\partial x^{\alpha}}+\frac{\partial^{\alpha}J_{xy}}{\partial y^{\alpha}}+\frac{\partial^{\alpha}J_{xz}}{\partial z^{\alpha}}\right)\\
&\qquad+v_y\left(\frac{\partial^{\alpha}J_{yx}}{\partial x^{\alpha}}+\frac{\partial^{\alpha}J_{yy}}{\partial y^{\alpha}}+\frac{\partial^{\alpha}J_{yz}}{\partial z^{\alpha}}\right)\\
&\qquad+v_z\left(\frac{\partial^{\alpha}J_{zx}}{\partial x^{\alpha}}+\frac{\partial^{\alpha}J_{zy}}{\partial y^{\alpha}}+\frac{\partial^{\alpha}J_{zz}}{\partial z^{\alpha}}\right)\\
&\qquad+\rho b_xv_x+\rho b_yv_y+\rho b_zv_z\\
&\qquad+K^{2\alpha}\left(\frac{\partial^{\alpha}q_x}{\partial x^{\alpha}}+\frac{\partial^{\alpha}q_y}{\partial y^{\alpha}}+\frac{\partial^{\alpha}q_z}{\partial z^{\alpha}}\right).
\end{aligned}\tag{81}$$

In the 2D Cantorian coordinates, systems of incompressible Navier-Stokes equations on Cantor sets are rewritten in the form

$$\begin{aligned}
&\frac{\partial^{\alpha}v_x}{\partial x^{\alpha}}+\frac{\partial^{\alpha}v_y}{\partial y^{\alpha}}=0,\\
&\rho\frac{D^{\alpha}v_x}{Dt^{\alpha}}=-\frac{\partial^{\alpha}p}{\partial x^{\alpha}}+\mu\left(\frac{\partial^{2\alpha}v_x}{\partial x^{2\alpha}}+\frac{\partial^{2\alpha}v_x}{\partial y^{2\alpha}}\right)+\rho b_x,\\
&\rho\frac{D^{\alpha}v_y}{Dt^{\alpha}}=-\frac{\partial^{\alpha}p}{\partial y^{\alpha}}+\mu\left(\frac{\partial^{2\alpha}v_y}{\partial x^{2\alpha}}+\frac{\partial^{2\alpha}v_y}{\partial y^{2\alpha}}\right)+\rho b_y,\\
&\rho\frac{D^{\alpha}}{Dt^{\alpha}}(\theta+\varphi)=-\left(\frac{\partial^{\alpha}pv_x}{\partial x^{\alpha}}+\frac{\partial^{\alpha}pv_y}{\partial y^{\alpha}}\right)\\
&\qquad+v_x\left(\frac{\partial^{\alpha}J_{xx}}{\partial x^{\alpha}}+\frac{\partial^{\alpha}J_{xy}}{\partial y^{\alpha}}\right)\\
&\qquad+v_y\left(\frac{\partial^{\alpha}J_{yx}}{\partial x^{\alpha}}+\frac{\partial^{\alpha}J_{yy}}{\partial y^{\alpha}}\right)+\rho b_xv_x+\rho b_yv_y\\
&\qquad+K^{2\alpha}\left(\frac{\partial^{\alpha}q_x}{\partial x^{\alpha}}+\frac{\partial^{\alpha}q_y}{\partial y^{\alpha}}\right).
\end{aligned}\tag{82}$$

In the 1D Cantorian coordinates, systems of incompressible Navier-Stokes equations on Cantor sets are repeated by the expression

$$\begin{aligned}
&\frac{\partial^{\alpha}v}{\partial x^{\alpha}}=0,\\
&\rho\frac{D^{\alpha}v}{Dt^{\alpha}}=-\frac{\partial^{\alpha}p}{\partial x^{\alpha}}+\mu\frac{\partial^{2\alpha}v}{\partial x^{2\alpha}}+\rho b,\\
&\rho\frac{D^{\alpha}}{Dt^{\alpha}}(\theta+\varphi)=-\frac{\partial^{\alpha}(pv)}{\partial x^{\alpha}}+v\frac{\partial^{\alpha}J}{\partial x^{\alpha}}+\rho bv+K^{2\alpha}\frac{\partial^{\alpha}q}{\partial x^{\alpha}}.
\end{aligned}\tag{83}$$

6. Conclusions

In the present work, we propose the systems of Navier-Stokes equations derived from local fractional vector calculus. These obtained Navier-Stokes equations in one-, two-, and three-dimension Cantorian coordinates are shown to describe the materials as being local fractional continuous and nondifferential functions, which are applied to describe fluid flow in fractal media. Comparing between the fractional result in Navier-Stokes equation in fractal media [39] and local fractional one, the former via fractional calculus is continuous and differential quantities as classical result, however, the latter is local fractional continuous and nondifferential quantities. The classical result is obtained in case of fractal space-time dimension, which is equal to 1.

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