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Dynamics of the Fractional-Order Van der Pol Oscillator

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Abstract – **In this paper we propose a modified version of the classical unforced Van der Pol oscillator that occurs when introducing a fractional-order time derivative in the state space equations that describes its dynamics. The resulting fractional-order Van der Pol oscillator is analyzed in the time and frequency domains, for several values of order's fractional derivative and, consequently, of the total system order. It is shown that the system can exhibit different output behavior depending on the total system order. Several numerical simulations and performance indices illustrate the fractional dynamics.**

I. INTRODUCTION

The study of nonlinear oscillators has been important in the development of the theory of dynamical systems. The Van der Pol oscillator (VPO) represents a nonlinear system with an interesting behavior that arises naturally in several applications. It has been used for study and design of many models/systems including biological phenomena, such as the heartbeat or the generation of action potentials by neurons, acoustic models, radiation of mobile phones, and as model of electrical oscillators.

The VPO was used by Van der Pol in the 1920's to study oscillations in vacuum tube circuits (part of early radios). In the standard form, it is given by a second-order nonlinear differential equation of type:

$$\ddot{y} + \varepsilon\left(y^2 - 1\right)\dot{y} + y = 0 \quad (1)$$

where ε is the control parameter and $\dot{y}$ and $\ddot{y}$ are correspondingly the first and second derivatives of y with respect to time t. It can be regarded as describing an RLC electrical circuit with a nonlinear resistor. The equivalent state space formulation has the form ($y = y_1$, $\dot{y} = y_2$):

$$\begin{aligned} \frac{dy_1}{dt} &= y_2 \\ \frac{dy_2}{dt} &= -y_1 - \varepsilon\left(y_1^{\,2} - 1\right)y_2 \end{aligned} \quad (2)$$

This is an example of a system, with nonlinear damping, that typically possesses limit cycles (a periodic attractor). Limit cycles represent an important phenomenon in nonlinear systems. They can be found in many areas of engineering and nature. The parameter ε reflects the nonlinear behaviour of the system. At $\varepsilon = 0$ the system reduces to a simple harmonic oscillator. For positive ε the VPO exhibits a stable limit cycle and, as ε grows up the system becomes more nonlinear.

A modified version of the classical VPO is now proposed by introducing a fractional time derivative of order α in state space equations (2), yielding:

$$\begin{aligned} \frac{dy_1^{\alpha}}{dt^{\alpha}} &= y_2 \\ \frac{dy_2}{dt} &= -y_1 - \varepsilon\left(y_1^{\,2} - 1\right)y_2 \end{aligned} \quad (3)$$

In this work we take $0 < \alpha < 1$ and $\varepsilon > 0$. A similar approach was performed in [11−14]. Note that the resulting fractional-order Van der Pol oscillator (FrVPO) reduces to the classical VPO (2) when $\alpha = 1$. Also, the total system order is changed from the integer value 2 (for the VPO) to the fractional value $\alpha+1 < 2$ (for the FrVPO).

In this paper we analyse and present simulation results of the dynamics produced from the FrVPO system as the α-order derivative of state equations (3) is varied in the range $0 < \alpha < 1$. We show that α has a large influence upon the overall system dynamics.

The article is organized as follows. Section II gives a brief introduction to the fundamental aspects of fractional-order derivatives. In Section III we propose a simulation scheme for the unforced FrVPO system. Also, we describe a method for obtaining approximated integer-order rational functions to fractional-order systems. Section IV presents the simulation results. Finally, Section V addresses the main conclusions and perspectives of future work.

II. BASICS OF FRACTIONAL-ORDER DERIVATIVES

There are various definitions of fractional-order derivatives that can be adopted for the operator $D^{\alpha}y(t) \equiv d^{\alpha}y(t)/dt^{\alpha}$ [1–4]. The two most commonly used are the Riemann-Liouville and the Grünwald-Letnikov definitions [1, 3]. In all that follows, we consider $y(t)$ being a causal function of t, that is, $y(t) = 0$ for $t < 0$.

The Riemann-Liouville definition of the fractional-order derivative is:

$$\begin{aligned} D^{\alpha}y(t) &= \frac{d^n}{dt^n}D^{\alpha-n}y(t) \\ &= \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_0^t (t-\tau)^{n-\alpha-1}\, y(\tau)\,d\tau \end{aligned} \quad (4)$$

where n is an integer such that $n-1 < \alpha < n$ and Γ is the well known Gamma function.

The Grünwald-Letnikov definition is:

$$D^{\alpha}y(t) = \lim_{h\to 0}\left\{\frac{1}{\Gamma(-\alpha)h^{\alpha}}\sum_{j=0}^{[t/h]}\frac{\Gamma(j-\alpha)}{\Gamma(j+1)}\,y(t-jh)\right\} \quad (5)$$

where h is the time step and $[x]$ means the integer part of x.

For a wide class of functions, important for applications, both definitions are equivalent [1–3]. This allows one to use the Riemann-Liouville definition during problem formulation, and then turn to the Grünwald-Letnikov definition for obtaining the numerical solution.

An important fact revealed by both definitions is that the evaluation of fractional-order derivatives in any instant t requires the whole history of $y(t)$ from $t \in [0, t]$. This means that fractional-order derivatives are "global" operators having a memory of the entire past in opposition with the integer-order derivatives that are "local" operators. While this brings a fresh view in many areas of science and engineering, it poses, however, evaluation problems due to the unlimited memory imposed for their computation (*e.g.*, for large values of t). To overcome this, Podlubny [3] suggested the use of the "short memory" principle, which it takes into account the behaviour of $y(t)$ only in the "recent past", *i.e.* in the interval $[t–L, t]$, where L is the "memory length" and, consequently, maximizing the amount of computation to L seconds. This method was applied successfully for the numerical solution of linear ordinary fractional-order differential equations with constant and non-constant coefficients and non-linear ordinary fractional-order differential equations [3].

Another usual definition of fractional-order derivatives is given through the Laplace transform method. This useful tool is widely used by the control engineering community in the analysis and control of dynamic systems. Considering vanishing initial conditions, it is given by the simple form:

$$D^{\alpha} y(t) = L^{-1}\left\{s^{\alpha} Y(s)\right\}, \quad \alpha \in \mathbb{R} \tag{6}$$

where L denotes the Laplace operator and $Y(s) = L\{y(t)\}$. This expression reveals a straightforward adaptation of the classical frequency-based methods to fractional-order systems. The known interpretation of the integer-order derivatives (and integrals) in the frequency s–domain is easily translated to the fractional-order case. In fact, the Bode diagrams of amplitude and phase of (6) are given by straight lines of 20αdB/dec and $\alpha\pi/2$ rad ($\alpha \in \mathbb{R}$) in all frequency domain, respectively.

III. SIMULATION SCHEME

In section II we show several ways of leading with fractional-order derivatives. Among them, the most intuitive is undoubtedly the one that adopts the frequency domain given by expression (6). The application of the fractional calculus concepts to control theory, in the frequency domain, are well established and studied on a more systematic approach only in the last three decades (for example, see published books in the matter by Oustaloup [5–6] and Podlubny [3]). Besides the effort that has been dedicated, in the recent years, to the study of fractional-order systems in the time domain, much more investigation is needed in this area to produce a more consistent theory and reliable algorithms [3, 10]. In this paper we adopt the frequency domain approach to the study of fractional-order systems.

In this line of thought, Fig. 1 illustrates the block diagram of the unforced FrVPO system. Note that $\alpha = 1$ is the classical VPO system, both in state space formulation (3) and block diagram representation (Fig. 1).

As can be seen from Fig. 1, the unforced FrVPO system is implemented by using an integer integrator $1/s$, an fractional integrator $1/s^{\alpha}$ of order $0 < \alpha < 1$, and the block $\varepsilon f(y_1, y_2)$ that models the nonlinearity. Here, the unusual element is given by the fractional-order integrator that is an irrational transfer function in the Laplace s-variable. This type of systems has an unlimited memory which precludes its direct utilization in time-domain simulations. Therefore, the usual approach is the development of integer-order approximations that approximate (up to a given degree of accuracy) the fractional-order operators. So, in order to effectively analyze the fractional-order system of Fig. 1, we develop rational approximations for the fractional-order integrator $1/s^{\alpha}$. In this perspective, we adopt the approximation frequency method described by Charef *et al.* [15], known as "Singularity Function Method". A survey of both continuous and discrete approximations to fractional-order operators can be found in [7–9].

The singularity function method is based on the transfer function of a single-fractional power pole $1/(1+s/p_T)^{\alpha}$ that models a single-fractal system [15], where $1/p_T$ is the relaxation time constant and $0 < \alpha < 1$. The Bode diagram of magnitude of this system has a constant slope of -20α dB/dec (for $\omega >> p_T$) and, therefore, the basic idea is to approximate this slope with an alternative succession of zeros and poles with slopes of 0 dB/dec and −20 dB/dec, respectively, over the required range of frequency, yielding:

$$\frac{1}{s^{\alpha}} \approx \frac{1}{\left(1+\frac{s}{p_T}\right)^{\alpha}} \approx \frac{\prod_{i=0}^{N-1}\left(1+\frac{s}{z_i}\right)}{\prod_{i=0}^{N}\left(1+\frac{s}{p_i}\right)} \tag{7}$$

where the parameters p_i, z_i and N are chosen according to the desired order α and frequency bandwidth.

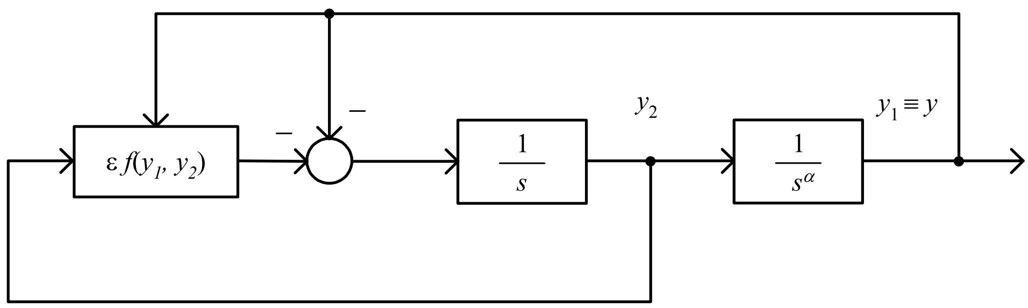

Fig. 1. Block diagram of the unforced FrVPO system; $\alpha = 1$ is the classical VPO system.

Hence, given the order α, the fractional-order pole p_T and the desired maximum discrepancy Δ (in dB) between the actual and the approximate asymptotes of the magnitude Bode plots, the parameters (p_i, z_i, N) of the approximation are obtained as follows: *i*) the first pole is $p_0 = p_T 10^{(\Delta/20\alpha)}$, *ii*) defining $a = 10^{[\Delta/10(1-\alpha)]}$, $b = 10^{(\Delta/10\alpha)}$, $ab = 10^{[\Delta/10\alpha(1-\alpha)]}$, the remaining poles p_i and zeros z_i are calculated from the first pole p_0 using the following algorithm as:

$$p_i = (ab)^i p_0 \tag{8}$$

$$z_i = (ab)^i a p_0 \tag{9}$$

and *iii*) the number of poles N is determined through the frequency bandwidth ω_{max} and of relations (8) and (9), assuming that $p_{N-1} < \omega_{max} < p_N$, yielding:

$$N = Integer\left(\frac{\log\left(\frac{\omega_{\max}}{p_0}\right)}{\log(ab)}\right) + 1 \tag{10}$$

In the study that follows we use approximations of type (7) for the fractional-order integrator $1/s^{\alpha}$ with $\alpha \in\ ]0, 1[$. These were obtained for $p_T = 0.01$, $\omega_{max} = 100$ rad s^{-1} and a maximum discrepancy of $\Delta = 2$ dB. Such kind of approximations can also be found in [13, 14] following the same method described here.

IV. SIMULATION RESULTS

In this section we study the effects of fractional dynamics in the proposed FrVPO model both in the time and frequency domains. We show that the two perspectives are essential for a complete understanding of system dynamics.

The FrVPO system of Fig. 1 is simulated over a time-period of $t_s = 1000$ s with a time step of $h = 0.01$ s while adopting a Runge-Kutta integration scheme using the MATLAB/SIMULINK software package and initial conditions $y_1(0) = 0$ and $y_2(0) = 1.0$.

Fig. 2 illustrates the phase plane (y_1, y_2) plot for the FrVPO when varying the fractional-order α = {0.4, 0.6, 0.7, 0.8, 0.9, 1.0} for a fixed value of the control parameter ε = 1. Alternatively, Fig. 3 shows the phase plane plot for ε = {0.5, 1, 2, 4, 8, 16} and α = 0.8. In both cases, we verify large variations on the limit cycle generation. It reveals that the α-order derivative has a large impact upon the system dynamics, namely in the amplitude and the period of the output oscillation. On the other hand, and as expected, the higher the value of ε the more nonlinear the system becomes.

Fig. 4 shows the period T and the amplitude A of the output oscillation for $0.3 \le \alpha \le 1$ and $\varepsilon_{min}(\alpha) \le \varepsilon \le 10$, where $\varepsilon_{min}(\alpha)$ is the minimum value of the control parameter for which the system oscillates. This limit depends on fractional-order α and its evolution is illustrated in Fig. 5 for $0.3 \le \alpha \le 1$.

Once again, we observe large variations in the limit cycle, particularly in the period of the output oscillation. In fact, the period T increases significantly for small values of α and for α near 1. The amplitude A varies also with α being more sensitive for small values of ε.

Fig. 6 illustrates the steady-state time responses of the output signal $y(t)$ (for $500 \le t \le 530$ s) and the corresponding Fourier spectra for the FrVPO system with fractional-order α = {0.4, 0.6, 0.8, 1.0} and control parameter ε = 1.

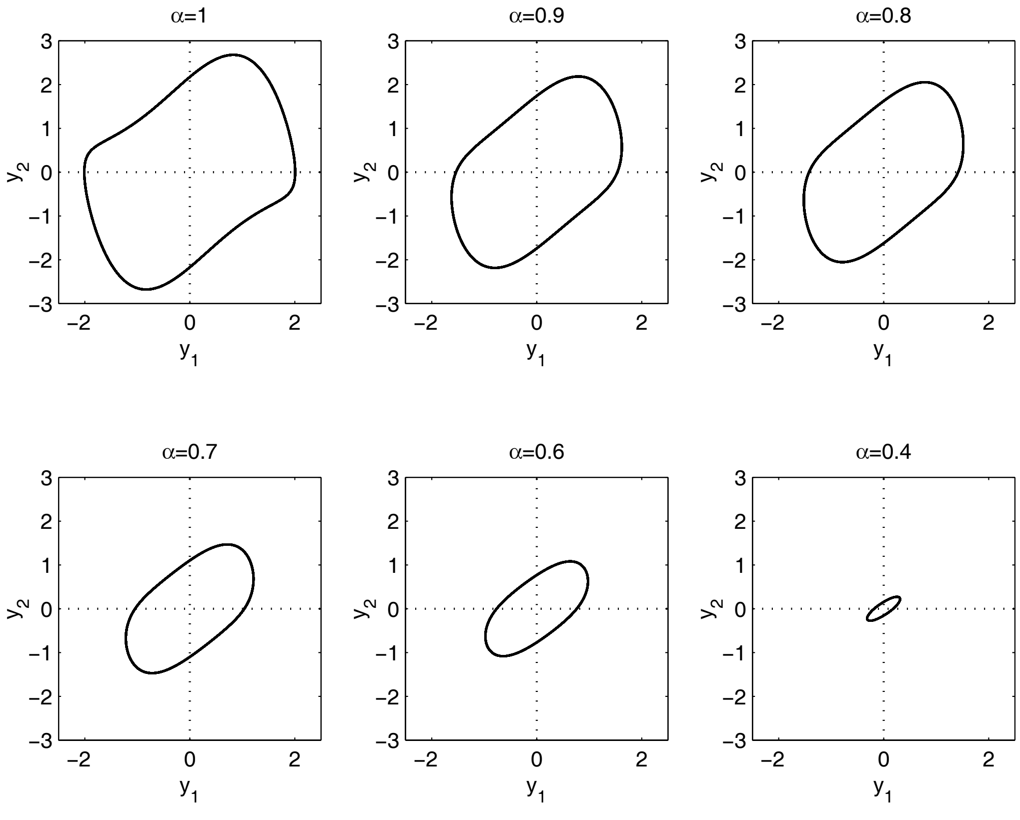

Fig. 2. Phase plane (y_1, y_2) plot for the FrVPO system with fractional-order α = {0.4, 0.6, 0.7, 0.8, 0.9, 1.0} and control parameter ε = 1.

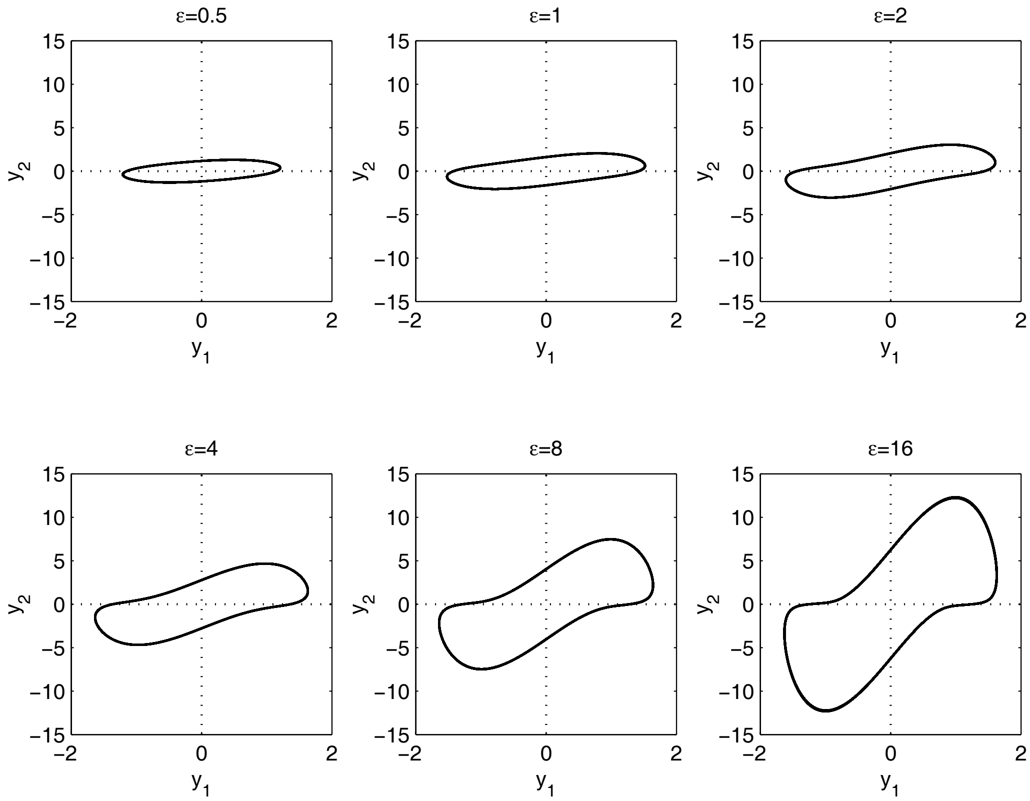

Fig. 3. Phase plane (y_1, y_2) plot for the FrVPO system with fractional-order $\alpha = 0.8$ and control parameter $\varepsilon = \{0.5, 1, 2, 4, 8, 16\}$.

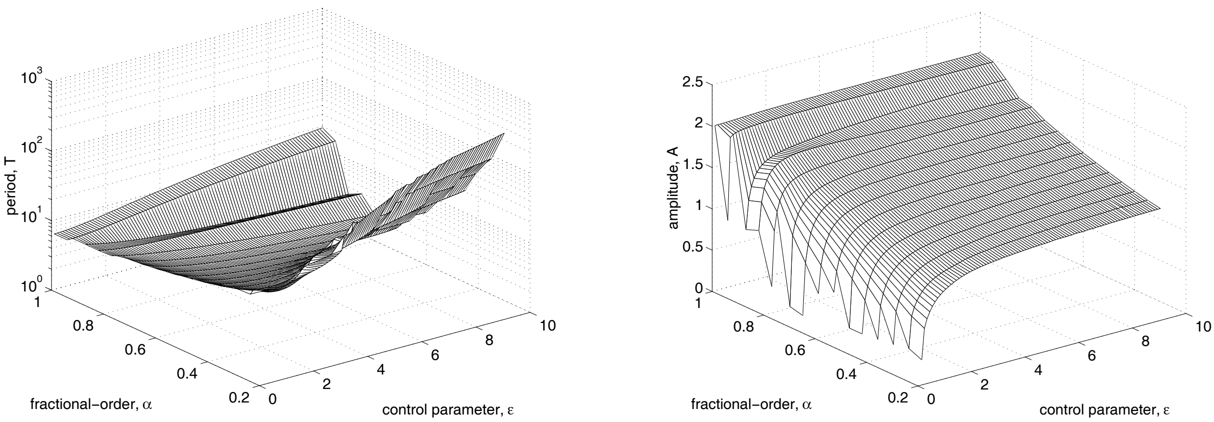

Fig. 4. Period T (left) and amplitude A (right) of limit cycle for $\varepsilon_{min}(\alpha) \le \varepsilon \le 10$ and $0.3 \le \alpha \le 1$.

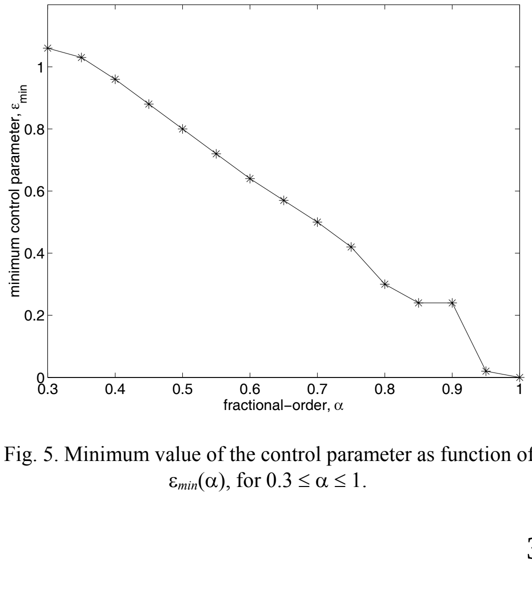

Fig. 5. Minimum value of the control parameter as function of α, $\varepsilon_{min}(\alpha)$, for $0.3 \le \alpha \le 1$.

For the evaluation of the Fourier spectrum we use the *Discrete Fourier Transform* (DFT) implemented through the *Fast Fourier Transform* (FFT) algorithm. Hence, assuming that we take a set of N points from the signal output $y(t)$, $y(nh)$ for $n = 0, \ldots, N\!-\!1$, the DFT yields the frequency spectrum at N points by the formula:

$$c_k = c\left(\frac{k}{Nh}\right) = \frac{1}{N}\sum_{n=0}^{N-1} y(nh)\, e^{\frac{-i2\pi nk}{N}}, \quad k = 0, \ldots, N-1 \qquad (11)$$

Thus, it is obtained a frequency range from $f = 0, \ldots, (N\!-\!1)\,\Delta f$, with the resolution $\Delta f = 1/(Nh)$ Hz. In the experiments, the FFT is evaluated for $N = 2^{15}$ points after elapsing the initial transient up to $T_0 = 100$ s of the output signal $y(t)$. The graphs show the amplitude Fourier spectra (properly scaled) for $0 \le \omega \le 20$ rad s^{-1} ($\omega = 2\pi f$).

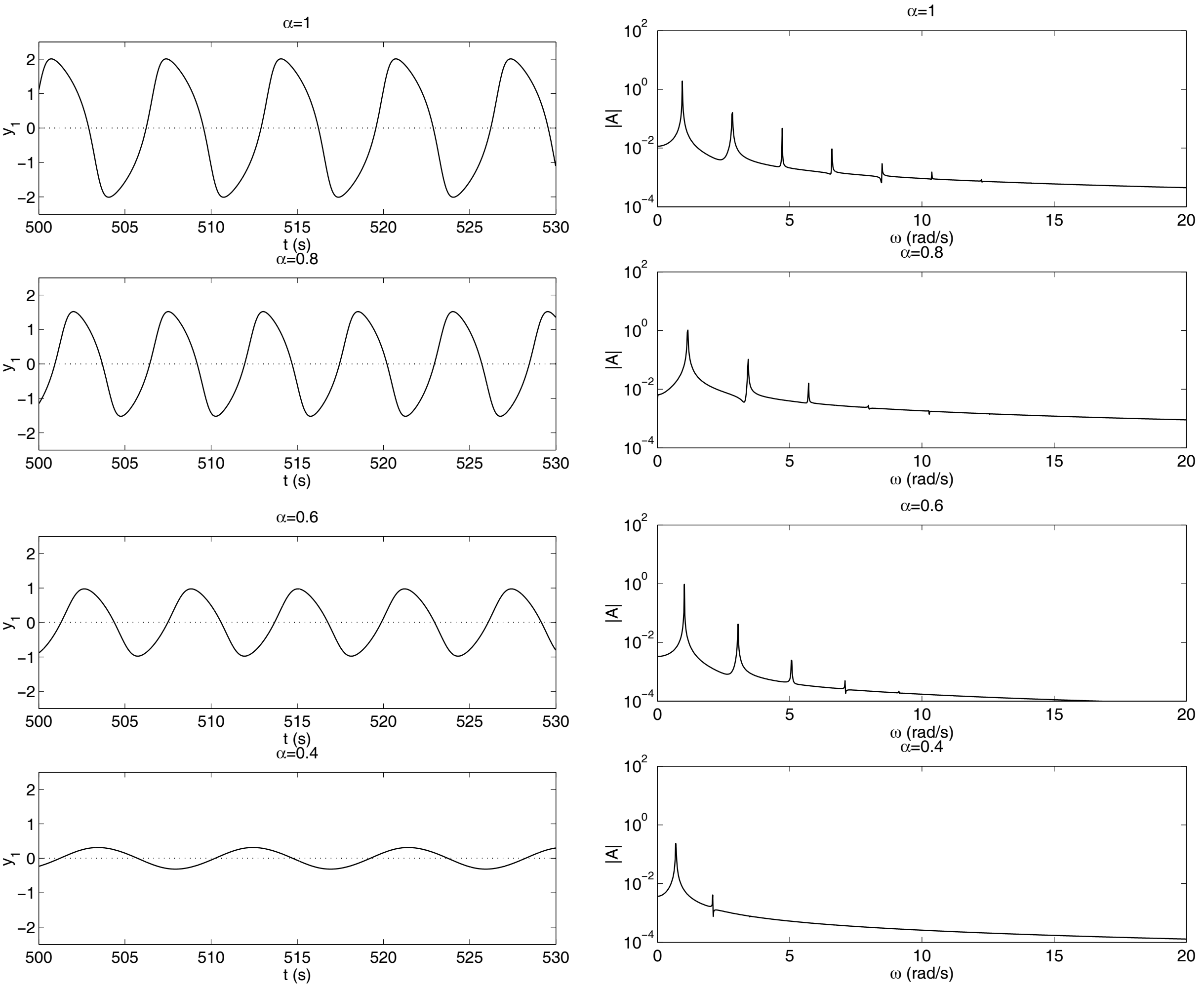

Fig. 6. Time responses of output $y(t)$ (left) and corresponding Fourier spectra (right) for the FrVPO system with fractional-order $\alpha = \{0.4, 0.6, 0.8, 1.0\}$ and control parameter $\varepsilon = 1$.

From the time responses of Fig. 6, it is clear that the period T and amplitude A of the limit cycle vary significantly with fractional-order α. In fact, the amplitude gets smaller as α is decreased until, eventually, the system stops oscillating. The oscillation limit occurs when $\alpha = 0.37$ (with $\varepsilon = 1$). We also note variations in the period of the output oscillation.

On the other hand, from the Fourier spectra, we verify that the amplitude have several peaks. The multiple peaks are typical of nonlinear systems and are due to the presence of the nonlinearity. We also notice that the higher-order harmonics are integer-odd multiples of the fundamental component (*i.e.*, on the first peak seen in Fourier spectrum). By other words, if Y_1 denotes the amplitude of the fundamental harmonic, then the higher-order odd harmonics are Y_k for $k = 3, 5\ldots$. Moreover, the multiplicity and amplitude of these peaks varies with the order α, which is in accordance with the time responses.

At first glance, by varying the order α we can shape the output signal $y(t)$ to resembles an almost sinusoidal ("pure") signal but, in fact, this is not necessarily true. The energy of the signal is, not only concentrated in the peaks, but distributed along all frequency domain. This fact is characteristic of chaotic systems showing that the FrVPO presents chaotic limit cycles (like in the classical VPO [16]). Furthermore, the amplitude spectrum shows a long-term behaviour of type $C(\alpha)\,\omega^{-1}$ indicating different amplitude decays depending on α.

To further illustrate this fact, we develop a percentage ratio criterion η that relates the power concentrated in the harmonics P_h (*i.e.*, in the peaks of the amplitude spectrum) over the total power P_t of the output signal $y(t)$, that is:

$$\eta = \frac{\text{Power in Harmonics}}{\text{Total Power}} \times 100\% = \frac{P_h}{P_t} \times 100\% \qquad (12)$$

A relationship between the signal power in the time and frequency domains is established by *Parseval's theorem*, which states that they must be the same in the two cases. Thus, for the (periodic) output signal $y(t)$ with period T, the total power P_t is given by:

$$P_t = \frac{1}{T}\int_{t_0}^{t_0+T} \left[y(t)\right]^2 dt = \sum_{k=0}^{N-1} \left|c_k\right|^2 \qquad (13)$$

where c_k are the coefficients of the DFT evaluated through expression (11). If we select a finite number of amplitude peaks p (*i.e.*, $p \ll N$), the power of harmonics P_h is then given by:

$$P_h = \sum_{k=1}^{p} \left|c_k\right|^2 \qquad (14)$$

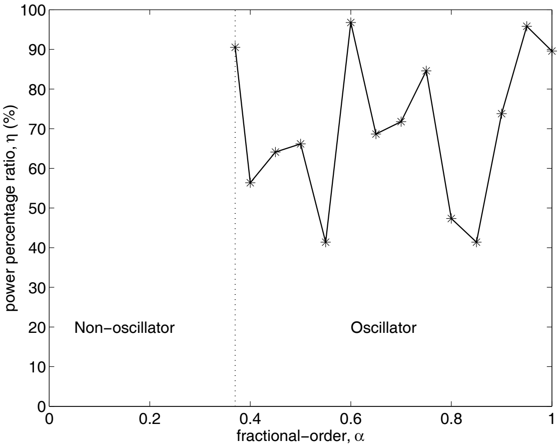

Fig. 7. Power percentage ratio η (%) for $0 < \alpha \leq 1$ and $\varepsilon = 1$.

where the coefficients c_k now correspond to the frequencies when the amplitude peaks occur. The relationship $P_h < P_t$ is verified and, consequently, the index η yields $0 < \eta < 100\%$.

Fig. 7 depicts the plot of the index η (%) for the FrVPO system with $0 < \alpha \leq 1$ and $\varepsilon = 1$. It shows two different regions: *i*) for $0 < \alpha < 0.37$, where no oscillation occurs, and *ii*) for $0.37 \leq \alpha \leq 1$, where the system oscillates. Once more, we verify that the signal energy is distributed along all frequency domain, depending on fractional-order α. However, for some values of α (*e.g.*, $\alpha \approx \{0.37, 0.6, 0.95, 1.0\}$) we have $\eta \approx 100\%$ indicating that all the energy of the system is in the harmonics. In these cases, the major part of the signal energy is concentrated in the first (fundamental) harmonic.

In conclusion, the introduction of a α-order derivative in the classical VPO reveals some interesting characteristics that can be more clearly distinguished in the frequency domain. One gets different regime outputs from those obtained with the classical VPO that may be useful for a better understanding and control of such systems.

V. CONCLUSIONS

In this paper we have introduced a modified version of the classical VPO by inserting a fractional derivative of order α on the state equations that describes its dynamics. The resulting FrVPO system presents characteristics that can be very different from the classical one depending on the α-order derivative. In fact, with the FrVPO system we can distinguish different behaviours of the output going from the oscillator to the non-oscillator regimes.

The additional degree of freedom introduced by the order α as great influence on the system behaviour and deserves a deeper investigation to clarify its implications.

Further developments of this work can be outlined, namely to study the influence of α for a broader range of the control parameter ε (both in the time and frequency domains) and to analyse the system for values of fractional-order $\alpha > 1$. Moreover, the study presented here can be extended to the forced Van der Pol oscillator which may reveal more interesting results.

VI. REFERENCES

[1] K. B. Oldham and J. Spanier, *The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order*, Academic Press, New York: 1974.

[2] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley & Sons, New York: 1993.

[3] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego: 1999.

[4] R. Hilfer, *Application of Fractional Calculus in Physic*s, World Scientific, Singapore: 2000.

[5] A. Oustaloup, *La Commande CRONE: Commande Robuste d'Ordre Non Entier*, Editions Hermès, Paris: 1991.

[6] A. Oustaloup, *La Dérivation Non Entière: Théorie, Synthèse et Applications*, Editions Hermès, Paris: 1995.

[7] J. A. Tenreiro Machado, "Analysis and Design of Fractional-Order Digital Control Systems," *SAMS Journal Systems Analysis, Modelling, Simulation*, vol. 27, 1997, pp. 107−122.

[8] J. A. Tenreiro Machado, "Discrete-Time Fractional-Order Controllers," *FCAA Fractional Calculus & Applied Analysis*, vol. 4, no. 1, 2001, pp. 47−66.

[9] B. M. Vinagre, I. Podlubny, A. Hernández, and V. Feliu, "Some Approximations of Fractional Order Operators Used in Control Theory and Applications," *FCAA Fractional Calculus & Applied Analysis*, vol. 3, no. 3, 2000, pp. 231−248.

[10] Chyi Hwang, Jeng-Fan Leu, and Sun-Yuan Tsay, "A Note on Time-Domain Simulation of Feedback Fractional-Order Systems," *IEEE Transactions on Automatic Control*, vol. 47, no. 4, April 2002, pp. 625−631.

[11] P. Arena, R. Caponetto, L. Fortuna and D. Porto, *Nonlinear Noninteger Order Circuits and Systems-An Introduction*, World Scientific, Singapore: 2000.

[12] P. Arena, R. Caponetto, L. Fortuna and D. Porto, "Chaos in a Fractional Order Duffing System," in *Proceedings of the 1997 European Conference on Circuit Theory and Design (ECCTD'97)*, Budapest, Hungary, September 1997, pp. 1259−1262.

[13] Tom T. Hartley, Carl F. Lorenzo, and Helen Killory Qammer, "Chaos in a Fractional Order Chua's System," *IEEE Transactions on Circuits and Systems-I: Fundamental Theory and Applications*, vol. 42, no. 8, August 1995, pp. 485−490.

[14] Wajdi M. Ahmad and J. C. Sprott, "Chaos in a Fractional-Order Autonomous Nonlinear Systems," *Chaos, Solitons & Fractals*, vol. 16, no. 2, 2003, pp. 339−351.

[15] A. Charef, H. H. Sun, Y. Y. Tsao, and B. Onaral, "Fractal System Represented by Singularity Function," *IEEE Transactions on Automatic Control*, vol. 37, no. 9, September 1992, pp. 1465−1470.

[16] Gamal M. Mahmoud, Ahmed A. M. Farghaly, "Chaos Control of Chaotic Limit Cycles of Real and Complex Van der Pol Oscillators," *Chaos, Solitons & Fractals*, vol. 21, 2004, pp. 915−924.