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IMPLEMENTATION OF FRACTIONAL ELECTRIC POTENTIAL USING GENETIC ALGORITHMS

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Abstract. *A fresh look of several phenomena present in electrical systems induced an approach based in the fractional calculus (FC) viewpoint. The problem of point multipoles with electrical charges is an important subject in the field of electromagnetism. In this work, it is applied the concept of FC to implement the electrical potential of fractional order through a genetic algorithm (GA) optimization scheme. The work is complemented by analyzing the locus of charge $versus$ position distribution of the required electrical charges.*

1 INTRODUCTION

Several phenomena present in electrical systems motivated the development of comprehensive models based on the theory of fractional calculus (FC). Several authors [1, 2, 3] verified that well-known expressions for the electrical potential are related through integer-order integral and derivatives, and proposed its generalization based on the concept of fractional-order poles. Nevertheless, the mathematical generalization towards FC lacks a comprehensive method for its practical implementation.

Bearing these ideas in mind, in this article we address the synthesis of fractional-order multipoles. In section 2 we recall the classical expressions for the static electric potential and we analyze them in the perspective of its generalization through the FC. In section 3 we develop a genetic algorithm (GA) scheme for implementing fractional-order approximations for the electrical potential. Finally, in section 4 we outline the main conclusions.

2 CLASSICAL EXPRESSIONS FOR THE STATIC ELECTRICAL POTENTIAL

For a homogeneous, linear and isotropic media, the electric potential φ at a point P produced by a single charge (1a), a dipole (1b), a quadrupole (1c), an infinite straight filament carrying a charge λ per unit length (2a), two opposite charged filaments (2b), and a planar surface with charge density σ (3), are given by [4, 5]:

$$\varphi = \frac{q}{4\pi\varepsilon_0}\frac{1}{r} + C \tag{1a}$$

$$\varphi = \frac{ql\cos\theta}{4\pi\varepsilon_0}\frac{1}{r^2} + C,\ r >> l \tag{1b}$$

$$\varphi = \frac{ql^2\left(3\cos^2\theta - 1\right)}{4\pi\varepsilon_0}\frac{1}{r^3} + C,\ r >> l \tag{1c}$$

$$\varphi = -\frac{\lambda}{2\pi\varepsilon_0}\ln r + C, \tag{2a}$$

$$\varphi = \frac{\lambda l\cos\theta}{2\pi\varepsilon_0}\frac{1}{r} + C,\ r >> l \tag{2b}$$

$$\varphi = -\frac{\sigma}{2\varepsilon_0}r + C, \tag{3}$$

where ε_0 represents the permittivity, q the electric charge, r the radial distance, θ the corresponding angle with the multipole axis and $C \in \Re$.

Analyzing expressions (1)-(3) we verify the relationship $\varphi \sim r^{-3}, r^{-2}, r^{-1}, \ln r, r$, involving the application of integer-order derivatives and integrals.

3 IMPLEMENTATION OF THE FRACTIONAL POTENTIAL

The integer-order differential nature of the potential expressions (1-3) motivated several authors [3] to propose its generalization in a FC perspective. Therefore, a fractional multipole produces at point P a potential $\varphi \sim r^{\alpha},\ \alpha \in \Re$. Nevertheless, besides the abstract manipulation of mathematical expressions, the truth is that there is no practical method, and physical interpretation, for establishing the fractional potential [2, 3, 6, 7, 8].

Inspired by the integer-order recursive approximation of fractional-order transfer functions [9, 10], we adopt a genetic algorithm (GA) [11, 12, 13] for implementing a fractional order

potential. Similarly to what occurs with integer/fractional transfer functions, the electrical integer-order potential has a *global* nature and fractional-order potentials can have only a *local* nature. By other words, it is only possible to capture a fractional potential in a restricted region of space. This observation leads to an implementation approach conceptually similar to the one described in [6, 9, 10, 14, 15] that is, to an approximation scheme based on a recursive placement of integer-order functions.

In this line of thought, we develop a one-dimensional GA that places n charges at the positions x_i and determines the corresponding values q_i $(i = 1, 2, ..., n)$. Our goal is to compare the approximate potential φ_{app} given by:

$$\varphi_{app} = \sum_{i=1}^{n} \frac{q_i}{4\pi\varepsilon_0 \left| x - x_i \right|} \tag{5}$$

that mimics the desired reference potential $\varphi_{ref} = kx^{\alpha}$ in a given interval $\triangle_x$. The position $x = 0$ corresponds to the center of the multipole.

It is important to refer that a reliable execution and analysis of a GA usually requires a large number of simulations to guarantee that stochastic effects are properly considered [16, 17, 18, 19]. Therefore, in this study the experiments consist on executing the GA several times, for generating a combination of positions and charges that leads to an electrical potential with a fractional slope similar to the desired reference potential. The values of the GA parameters are: population number P = 40, crossover C(%) = 85.0%, mutation M(%) = 1.0% and an elitist strategy ES(%) = 10.0%. The chromosome has $2n$ genes: the first n genes correspond to the charges and the last n genes indicate their positions. The gene codifications adopts a Gray code with a string length of $l = 16$ bits. The optimization fitness function corresponds to the minimization of the index:

$$J = \sum_{k=1}^{m} \left(\varphi_{app} - \varphi_{ref} \right)^2 \tag{6}$$

where m is the number of sampling points along the interval $\triangle_x = x_{mim} < x < x_{max}$. We establish a maximum number of iterations I_{max} = 1000 and a stoping scheme when $J < 10^{-1}$ for the best individual (*i.e.*, solution) of the GA population.

For an approximation with $n = 6$ charges, figure 1 shows φ_{app} and $\varphi_{ref} = 1.0x^{-1.5}$ (with a scale factor of $\times(4\pi\varepsilon_0)^{-1}$), $\triangle_x = 0.2 < x < 0.8$, obtained for two different GA solution:

- case A: $\{q_1, q_2, q_3, q_4, q_5, q_6\}$=$\{0.8081, 0.9969, -0.6544, -0.7510, -0.8635, -0.2998\}$ [C] (with scale factor $\times(4\pi\varepsilon_0)^{-1}$), located at $\{x_1, x_2, x_3, x_4, x_5, x_6\} = \{-0.0928, -0.0066, 1.5163, 1.7269, 1.9466, 1.98366\}$ [m], respectively;

- case B: $\{q_1, q_2, q_3, q_4, q_5, q_6\}$=$\{0.2002, 0.9943, 0.9515, -0.9495, -0.8678, -0.9465\}$ [C] (with scale factor $\times(4\pi\varepsilon_0)^{-1}$), located at $\{x_1, x_2, x_3, x_4, x_5, x_6\} = \{-0.0947, -0.07784, 0.1114, 1.0652, 1.2715, 1.6323\}$ [m], respectively.

In the case A the GA needs $I_A = 398$ iterations to satisfy the adopted fitness function stoping threshold, and in the case B the GA needs $I_B = 348$ iterations. The results show a good fit between φ_{ref} and φ_{app} and we verify that it is possible to find more than one 'good solution'.

With the purpose of evaluating the influence on the charges and positions when we increase the length of the range of the approximations, we execute the GA for $x_{min} = -0.2$ and $x_{max} = \{0.2, 0.6, 1.0, 1.4, 1.8, 2.0\}$. For reducing the stochastic effects the GA is repeated 10 times and

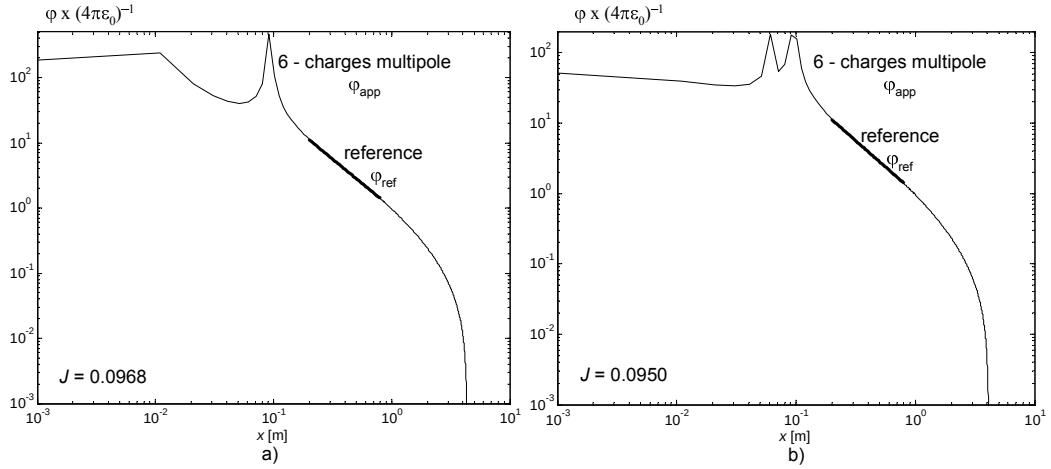

Figure 1: Comparison of the electrical potential φ_{app} and φ_{ref} *versus* the position x for $\varphi_{ref} = 1.0\ x^{-1.5}$ [volt], $\triangle_x = 0.2 < x < 0.8$ [m], and a $n = 6$ charges approximation, in both cases.

we analyze the average of the results. Figure 2 depicts the average of the charges $versus$ the average of locations for $n = \{1, ..., 6\}$.

We conclude that the AG places the first charge approximately in the same value and with the same location for all the cases. The pattern for the rest of the cases is to decrease the value of the charges and to extend their location when the total number of charges increases.

We study also the statistics of the charge values and positions for $n = \{1, ..., 6\}$. Tables 1 and 2 show the averages μ_q and μ_x of the charges and their positions. Tables 3 and 4 depict the standard deviation σ_q and σ_x of the charge values and positions, respectively. Finally, figure 3 shows the minimum and maximum errors, $Min(J)$ and $Max(J)$, between φ_{ref} and φ_{app}, for the different number of charges.

n	μ_{q1}	μ_{q2}	μ_{q3}	μ_{q4}	μ_{q5}	μ_{q6}
1	0.9999					
2	0.8027	0.3518				
3	0.8566	0.7568	−0.9999			
4	0.8427	0.7864	−0.5467	−0.8761		
5	0.4865	0.7759	0.2741	−0.4917	−0.7807	
6	0.4680	0.6554	0.5087	0.1037	−0.7047	−0.7824

Table 1: Average of the charges μ_{qi} for $\varphi_{ref} = 1.0x^{-1.5}$, $\triangle_x = -0.2 < x < 1.0$ and $n = \{1, ..., 6\}$.

Figure 3 reveals clearly that, as expected, the errors, both the minimum and the maximum, decrease with n. Therefore, for a given application, the larger the precision, the higher the required number of charges. We verify also that it is difficult to reveal a clear pattern in the solution space.

This lack of 'order' is due to the large number of possible solutions. Therefore, the GA has a high freedom, choosing solutions that are almost not correlated. However, we believe that further study imposing more strict restrictions may lead to the emergence of a comprehensive pattern.

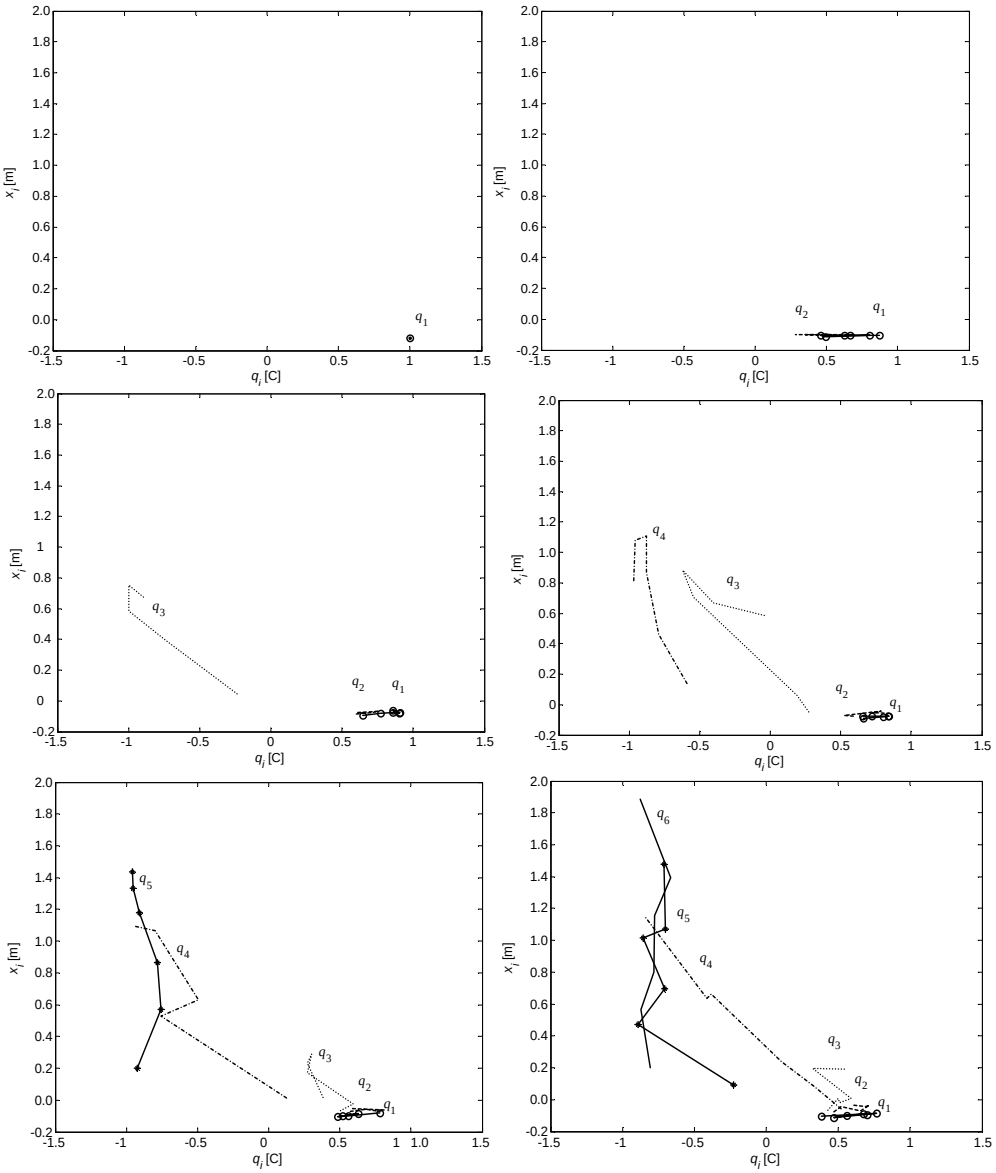

Figure 2: Locus of (μ_{qi}, μ_{xi}): average of the charges μ_{qi} $versus$ average of the position μ_{xi}, $n = \{1, .., 6\}$.

4 CONCLUSIONS

This paper addressed the implementation of a fractional-order electrical potential through a GA. The results reveal the necessity of a large number of charges for decreasing the approximation error. On the other hand, it was verified that increasing the number of charges leads to an higher dispersion of their location.

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n	μ_{x1}	μ_{x2}	μ_{x3}	μ_{x4}	μ_{x5}	μ_{x6}
1	-0.1194					
2	-0.1018	-0.1014				
3	-0.0772	-0.0683	0.5826			
4	-0.0764	-0.0427	0.7046	0.8718		
5	-0.1050	-0.0727	0.1713	0.6308	0.8629	
6	-0.1129	-0.0707	-0.0206	0.2384	0.6941	0.7997

Table 2: Average position of the μ_{xi} for $\varphi_{ref} = 1.0x^{-1.5}$ and $n = \{1, ..., 6\}$.

n	σ_{q1}	σ_{q2}	σ_{q3}	σ_{q4}	σ_{q5}	σ_{q6}
1	0.0000					
2	2.1441×10^{-1}	2.1443×10^{-1}				
3	9.0187×10^{-2}	1.0524×10^{-1}	0.0000			
4	1.5941×10^{-1}	2.6547×10^{-1}	6.2958×10^{-1}	1.8293×10^{-1}		
5	4.6258×10^{-1}	2.2596×10^{-1}	5.3356×10^{-1}	5.7923×10^{-1}	2.8197×10^{-1}	
6	4.0120×10^{-1}	2.5499×10^{-1}	2.2519×10^{-1}	5.6018×10^{-1}	3.0550×10^{-1}	2.4746×10^{-1}

Table 3: Standard deviation of the charges σ_{qi} for $\varphi_{ref} = 1.0x^{-1.5}$ and $n = \{1, ..., 6\}$.

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n	σ_{x1}	σ_{x2}	σ_{x3}	σ_{x4}	σ_{x5}	σ_{x6}
1	$2.9257\text{x}10^{-17}$					
2	$1.2181\text{x}10^{-4}$	$1.2482\text{x}10^{-4}$				
3	$2.8798\text{x}10^{-3}$	$4.5942\text{x}10^{-3}$	$9.2774\text{x}10^{-2}$			
4	$6.4455\text{x}10^{-3}$	$3.4725\text{x}10^{-2}$	$4.1038\text{x}10^{-1}$	$1.7505\text{x}10^{-1}$		
5	$4.1370\text{x}10^{-2}$	$3.5468\text{x}10^{-2}$	$4.1704\text{x}10^{-1}$	$3.9215\text{x}10^{-1}$	$1.4630\text{x}10^{-1}$	
6	$4.5569\text{x}10^{-2}$	$2.7564\text{x}10^{-2}$	$3.9774\text{x}10^{-2}$	$2.7234\text{x}10^{-1}$	$1.6503\text{x}10^{-1}$	$1.5832\text{x}10^{-1}$

Table 4: Standard deviation of the positions σ_{xi} for $\varphi_{ref} = 1.0x^{-1.5}$ and $n = \{1, ..., 6\}$.

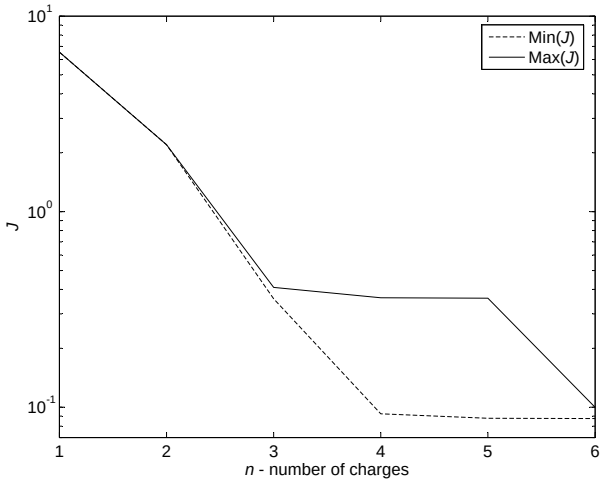

Figure 3: Minimum and maximum errors between φ_{app} and $\varphi_{ref} = 1.0x^{-1.5}$ for a sample of ten GA executions and $n = \{1, ..., 6\}$.

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