Arc exchange systems and renormalization

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We exhibit the construction of stable arc exchange systems from the stable laminations of hyperbolic diffeomorphisms. We prove a one-to-one correspondence between (i) Lipshitz conjugacy classes of C^{1+H} stable arc exchange systems that are C^{1+H} fixed points of renormalization and (ii) Lipshitz conjugacy classes of C^{1+H} diffeomorphisms f with hyperbolic basic sets L that admit an invariant measure absolutely continuous with respect to the Hausdorff measure on L. Let $HD^s(L)$ and $HD^u(L)$ be, respectively, the Hausdorff dimension of the stable and unstable leaves intersected with the hyperbolic basic set L. If $HD^u(L)=1$, then the Lipschitz conjugacy is, in fact, a C^{1+H} conjugacy in (i) and (ii). We prove that if the stable arc exchange system is a $C^{1+HD^s+\alpha}$ fixed point of renormalization with bounded geometry, then the stable arc exchange system is smooth conjugate to an affine stable arc exchange system.

Keywords: hyperbolic dynamics; renormalization; Markov maps; minimal sets

1. Introduction

The works of Masur [17], Penner [22], Thurston [39,40] and Veech [38] show a strong link between affine interval exchange maps and Anosov and pseudo-Anosov maps. Pinto et al. [33] developed a smooth version of the above link proving that every C^{1+H} diffeomorphism f on a surface, with a codimension 1 hyperbolic attractor, induces a C^{1+H} interval exchange system F_f accordingly with Williams' approach [39]. Here, we extend the result in Ref. [33] to diffeomorphisms f on a surface with a hyperbolic basic set that includes codimension 1 hyperbolic attractors, and also Smale horseshoes Anosov diffeomorphisms (see also Pinto et al. [34]). The extension presented in this paper forces us to consider arc exchange systems with a junction sets of maps. E. Ghys and D. Sullivan (see Cawley [4]) observed that Anosov diffeomorphisms on the torus determine circle diffeomorphisms that have an associated renormalization operator. In the same direction, we prove that every C^{1+H} diffeomorphism f on a surface, with a hyperbolic basic set, determines a renormalization operator acting on the topological conjugacy class $[F_f]_{C^0}$ of F_f. Then, we go one step further proving that every Lipschitz conjugacy class of C^{1+H} interval exchange systems $F \in [F_f]_{C^0}$ that are C^{1+H} fixed points of renormalization $[R_f F]_{C^{1+H}} = [F]_{C^{1+H}}$ determines a unique Lipschitz conjugacy class of C^{1+H} diffeomorphisms g, topologically conjugate to f, with an invariant measure absolutely continuous with respect to the Hausdorff measure on their invariant sets. Furthermore, there is a Teichmüller space of measure solenoid functions (as introduced in Ref. [28]) which characterizes the set of all Lipschitz conjugacy classes of C^{1+H} arc

exchange systems $F \in [F_f]_{C^0}$ that are C^{1+H} fixed points of renormalization $[R_f F]_{C^{1+H}} = [F]_{C^{1+H}}$. Denjoy [5] has shown the existence of upper bounds for the smoothness of Denjoy maps (see related results of Harrison [12] and Norton [21]). If L is a codimension 1 hyperbolic attractor, then, as proved in Ref. [33], there is no $C^{1+HD(L)+\alpha}$ s-arc exchange system, with bounded geometry, that is a $C^{1+HD(L)+\alpha}$ fixed point of renormalization with regularity α greater than zero. Poincare proved that every $C^{1+\alpha}$ circle diffeomorphism is C^0 conjugate to a rigid rotation.

Arnol'd [1], Herman [13] and Yoccoz [40] proved that every smooth enough circle diffeomorphism with Diophantine rotation number is smooth conjugate to a rigid rotation. In the same spirit, we prove that if the i-arc exchange system is a $C^{1+HD^i+\alpha}$ fixed point of renormalization with bounded geometry, then the i-arc exchange system is smooth conjugate to an affine i-arc exchange system, where HD^i is the Hausdorff dimension of its minimal set.

1.1 *Hyperbolic diffeomorphisms*

Throughout this paper, $(f, L, \mathcal{M})$ is a C^{1+H} diffeomorphism f with a hyperbolic basic set L and with a Markov partition $\mathcal{M}$ on L satisfying the disjointness property as we pass to describe. We say that (f, L) is a C^{1+} *hyperbolic diffeomorphism* if it has the following properties:

(i) $f : M \to M$ is a $C^{1+\alpha}$ diffeomorphism of a compact surface M with respect to a $C^{1+\alpha}$ structure C_f on M, for some $\alpha > 0$.

(ii) L is a hyperbolic invariant subset of M such that $f|L$ is topologically transitive and L has a local product structure.

We allow both the case where $L = M$ and the case where L is a proper subset of M. If $L = M$ then f is Anosov and M is a torus (see Franks [10,11], Manning [16] and Newhouse [19]). Examples where L is a proper subset of M include the Smale horseshoes and the codimension one attractors such as the Plykin and derived-Anosov attractors.

Let $i \in \{s, u\}$ and let i' denote the element of $\{s, u\}$ that is not i. Let $HD^i(L)$ be the Hausdorff dimension of the i-leaves intersected with the basic set (see Appendix A.1). Furthermore, (f, L) has a Markov partition $\mathcal{M}$ on L with the following *disjointness property*: If $HD^i(L) < 1$, then the i'-leaf boundaries of any two Markov rectangles do not intersect except, possibly, at their endpoints (see also Appendix A.3 and Bonatti and Langevin [2]).

2. Arc exchange systems

A *train track* $T = \sqcup_{j=1}^{n} I_j$ is the disjoint union of non-trivial sets I_j, topologically non-trivial closed intervals, with a given endpoints equivalence relation. Let $\sqcup_{j=1}^{n} I_j$ be a finite disjoint union of non-trivial compact intervals. An *endpoints equivalence relation* consists in fixing pairwise disjoint equivalence classes $E_1, \ldots, E_i$ such that $\bigcup_{j=1}^{i} E_j$ is equal to the set of all endpoints of the intervals $I_1, \ldots, I_n$, and any two endpoints x and y are equivalent if, and only if, they belong to a same set E_j. We allow the case where some, or all, equivalence classes are singletons. If all the equivalence classes are singletons, then the endpoints equivalence relation is trivial.

The closed (respectively, open) intervals contained in $\sqcup_{j=1}^{n} I_j$ are called *closed* (respectively, *open*) *arcs* of the train track T. If T has junctions, then one fix a *set of junction arcs* $K_1, \ldots, K_m \subset T$ that are images of intervals $J_1, \ldots, J_m \subset \mathbb{R}$ by homeomorphisms $k_i : J_i \to K_i$ with the property that $k_i(\operatorname{int} J_i)$ intersects only one junction.

From now on, a train track $\mathcal{T}$ has always associated to a fixed set of junction arcs allowed. If I is closed (respectively, open), we say that $k(I)$ is a closed (respectively, open) arc in $\mathcal{T}$. A *chart* in $\mathcal{T}$ is the inverse of a parametrization. A *topological atlas* $\mathcal{B}$ on the train track $\mathcal{T}$ is a given set of charts $\{(j, J)\}$ on the train track covering locally every arc. A $C^{1+\alpha}$, with $\alpha > 0$, atlas $\mathcal{B}$ on the train track $\mathcal{T}$ is a topological atlas such that the overlap maps are $C^{1+\alpha}$ and have uniformly $C^{1+\alpha}$ bounded norm. A C^{1+H} atlas $\mathcal{B}$ is a $C^{1+\alpha}$ atlas, for some $\alpha > 0$.

Definition 2.1. *The quadruple $(F, \mathcal{J}_F, \mathcal{T}_F, \mathcal{B}_F)$ is a C^{1+H} arc exchange system if the following properties are satisfied:*

(i) $\mathcal{T}_F$ *is a train track with a set* $\{L_{F,1}, \ldots, L_{F,m}\}$ *of junction arcs, and* $\mathcal{B}_F$ *is a* $C^{1+\alpha}$ *train track atlas, for some* $\alpha > 0$.

(ii) F *is a set of homeomorphisms* $f_i : I_{F,i} \to J_{F,i}$ *such that* $f_i|\mathrm{int}(I_{F,i})$ *is a* $C^{1+\alpha}$ *diffeomorphism, and* $I_{F,i}$ *and* $J_{F,i}$ *are non-trivial closed arcs.*

(iii) $\mathcal{J}_F$ *is a set of* $C^{1+\alpha}$ *diffeomorphisms* $e_j = e_{F,j} : L_{F,j} \to K_{F,j}$, *for* $j = 1, \ldots, m$, *with the following properties:* (*a*) $L_{F,j}$ *is a junction arc,* (*b*) *there are closed arcs* $I^L_{F,j}$ *and* $I^R_{F,j}$ *such that* $I^L_{F,j} \cup I^R_{F,j} = L_{F,j}$ *and* $I^L_{F,j} \cap I^R_{F,j}$ *is a junction and* (*c*) *there are maps* $f^L_{j,i_1}, \ldots, f^L_{j,i_{n(j,L)}}$ *and* $f^R_{j,i_1}, \ldots, f^R_{j,i_{n(j,R)}}$ *in* F *such that* $e_j|I^L_{F,j} = f^L_{j,i_{n(j,L)}} \circ \cdots \circ f^L_{j,i_1}$ *and* $e_j|I^R_{F,j} = f^R_{j,i_{n(j,R)}} \circ \cdots \circ f^R_{j,i_1}$.

For simplicity, (a) we assume that if $f_i : I_{F,i} \to J_{F,i}$ is in F, then there is $f_j : I_{F,j} \to J_{F,j}$ in F such that $I_{F,j} = J_{F,i}$, $J_{F,j} = I_{F,i}$ and $f_j = f_i^{-1}$, and (b) for every $x \in \mathcal{T}_F$, there exist at most two distinct intervals $I_{F,i}$ and $I_{F,j}$ containing x. For simplicity of notation, we will denote by F the C^{1+H} exchange system $(F, \mathcal{J}_F, \mathcal{T}_F, \mathcal{B}_F)$. We will call $\mathcal{J}_F$ the *junction exchange set* of the C^{1+H} arc exchange system F.

We say that a finite sequence $\{f_{i_n} \in F\}_{n=1}^{m}$ or an infinite sequence $\{f_{i_n} \in F\}_{n \geq 1}$ is *admissible* with respect to x, if $f_{i_n} \circ \cdots \circ f_{i_1}(x) \in I_{F,i_{n+1}}$ and $f_{i_n} \neq f_{i_{n-1}}^{-1}$ for all $n \geq 1$. We define the invariant set V_F of F as being the set of all points $x \in \mathcal{T}_F$ for which there are two distinct infinite admissible sequences $\{f^F_{i_n} \in F\}_{n \geq 1}$ and $\{f^B_{i_n} \in F\}_{n \geq 1}$ with respect to x. The forward orbit $O^F(x)$ of a point $x \in V_F$ is the set $\{f^F_{i_n}(x) : n \geq 1\}$, and the backward orbit $O^B(x)$ of x is the set $\{f^B_{i_n}(x) : n \geq 1\}$. We will assume that the invariant set V_F is *minimal*, i.e. for every $x \in V_F$, the closure $O^F(x)$ is equal to the invariant set V_F and that the closure $O^B(x)$ is also equal to the invariant set V_F. Furthermore, we will assume that the endpoints of the intervals $I_{F,i}$ belong to the invariant set V_F and $V_F \subset \cup_{i=1}^{n} I_{F,i}$. We denote the Hausdorff dimension of V_F by $HD(V_F)$. If $0 < HD(V_F) < 1$, we call F a C^{1+H} *arc exchange system*. If $HD(V_F) = 1$, we call F a C^{1+H} *interval exchange system*.

We say that an arc exchange system F is determined by a map $f : I_f \to J_f$ if all the maps $f_i : I_{F,i} \to J_{F,i}$ contained in F are the restriction of the map f or its inverse f^{-1} to $I_{F,i}$. In this case, we call f an *arc exchange map*. We note that not all arc exchange systems are determined by arc exchange maps.

Let $F = \{f_i : I_{F,i} \to J_{F,i};\ i = 1, \ldots, n\}$ and $C = \{c_i : I_{C,i} \to J_{C,i};\ i = 1, \ldots, n\}$ be $C^{1+\alpha}$ arc exchange systems with junction sets $\mathcal{J}_F = \{e_{F,j} : L_{F,j} \to K_{F,j};\ j = 1, \ldots, m\}$ and $\mathcal{J}_C = \{e_{C,j} : L_{C,j} \to K_{C,j};\ j = 1, \ldots, m\}$, respectively. We say that F and C are C^0 *conjugate*, if there is a homeomorphism $h : V_F \to V_C$ with the following properties:

(i) h has a homeomorphic extension $j : \mathcal{T}_F \to \mathcal{T}_C$ such that $I_{C,i} = j(I_{F,i})$, $J_{C,i} = j(J_{F,i})$, $L_{C,i} = j(L_{F,i})$ and $K_{C,i} = j(K_{F,i})$.

(ii) For every $1 \leq i \leq n$, $h \circ f_i(x) = c_i \circ h(x)$, where $x \in V_F \cap I_{F,i}$.

(iii) For every $1 \le j \le m$, $h \circ e_{F;j}(x) = e_{C;j} \circ h(x)$, where $x \in V_F \cap L_{F;j}$. By minimality of V_F, h is uniquely determined and the arcs $\mathbf{j}(I_{F;j})$, $\mathbf{j}(J_{F;j})$, $\mathbf{j}(L_{F;j})$ and $\mathbf{j}(K_{F;j})$ do not depend upon the extension $\mathbf{j}$ of h. We say that F and C are *Lipschitz conjugate*, if there is a Lipschitz homeomorphic extension $\mathbf{j} : \mathcal{T}_F \to \mathcal{T}_C$ of h satisfying property (i) above. We say that F and C are $C^{1+\alpha}$ *conjugate*, for some $\alpha > 0$, if there is a $C^{1+\alpha}$ homeomorphic extension $\mathbf{j} : \mathcal{T}_F \to \mathcal{T}_C$ of h satisfying property (i) above. We say that F and C are C^{1+H} *conjugate*, if F and C are $C^{1+\alpha}$ conjugate, for some $\alpha > 0$. We denote by $[F]_{C^{1+\alpha}}$ the set of all $C^{1+\alpha}$ arc exchange systems that are $C^{1+\alpha}$ conjugate to F, and we denote by $[F]_{C^{1+H}}$ the set $\cup_{\alpha > 0}[F]_{C^{1+\alpha}}$.

2.1 Induced arc exchange systems

Let $g \in \mathcal{F}$. Suppose that M and N are Markov rectangles of g, and $x \in M$ and $y \in N$. We say that x and y are *stable holonomically related* if (i) there is an unstable leaf segment $\ell^u(x, y)$ such that $\partial \ell^u(x, y) = \{x, y\}$, and (ii) $\ell^u(x, y) \subset \ell^u(x, M) \cup \ell^u(y, N)$. Let $\mathcal{P} = \mathcal{P}_{\mathcal{M}}$ be the set of all pairs (M, N) such that there are points $x \in M$ and $y \in N$ stable holonomically related.

For every Markov rectangle $M \in \mathcal{M}$, choose a spanning leaf segment ℓ_M in M. Let $\mathcal{I} = \{\ell_M : M \in \mathcal{M}\}$. For every pair $(M, N) \in \mathcal{P}$, there are maximal leaf segments $\ell^D_{(M,N)} \subset \ell_M$, $\ell^C_{(M,N)} \subset \ell_N$ such that the holonomy $h_{(M,N)} : \ell^D_{(M,N)} \to \ell^C_{(M,N)}$ is well-defined (see Appendices A.3 and A.5). We call such holonomies $h_{(M,N)} : \ell^D_{(M,N)} \to \ell^C_{(M,N)}$ the (stable) primitive holonomies associated to the Markov partition $\mathcal{M}$.

Definition 2.2. *The complete set $\mathcal{H}^s$ of stable holonomies consists of all primitive holonomies $h_{(M,N)}$ and their inverses $h^{-1}_{(M,N)}$, for every $(M, N) \in \mathcal{P}^s$. The complete set $\mathcal{H}^u$ is defined similarly to $\mathcal{H}^s$.*

Let $f : T \to T$ be the Anosov automorphism defined by $f(x, y) = (x + y, y)$, where $T = \mathbb{R}^2 \setminus (\mathbb{Z}v \times \mathbb{Z}w)$. We exhibit the complete set of holonomies $\mathcal{H}_{f;\mathcal{M}} = \{h_{(A,A)}, h_{(A,B)}, h_{(B,A)}, h^{-1}_{(A,A)}, h^{-1}_{(A,B)}, h^{-1}_{(B,A)}\}$ associated to the Markov partition $\mathcal{M} = \{A, B\}$ of f. We consider a derived-Anosov diffeomorphism $g : T \to T$ semi-conjugated, by a map $p : T \to T$, to the Anosov automorphism f. The derived-Anosov diffeomorphism g admits a Markov partition $\mathcal{M}_g = \{A_1, A_2, B_1\}$ with the property that $A = p(A_1) \cup p(A_2)$ and $B = p(B_1)$. The complete sets of holonomies $\mathcal{H}_{g;\mathcal{M}_g}$ and $\mathcal{H}_{f;\mathcal{M}}$ are related by the following equalities: $h_{(A,B)} \circ p|p(\ell^D_{(A_1,B_1)}) = p \circ h_{(A_1,B_1)}$, $h_{(A,A)} \circ p|p(\ell^D_{(A_2,A_1)}) = p \circ h_{(A_2,A_1)}$, $h_{(B,A)} \circ p|p(\ell^D_{(B_1,A_1)}) = p \circ h_{(B_1,A_1)}$ and $h_{(B,A)} \circ p|p(\ell^D_{(B_1,A_2)}) = p \circ h_{(B_1,A_2)}$ (see Figure 1).

Lemma 1. *The triple $(f, \mathcal{L}, \mathcal{M})$ induces a train track $\mathcal{T}^\iota_f$ with a set of junction arcs. Furthermore, the atlas $\mathcal{A}^\iota(f, \rho)$ induces a $C^{1+\alpha}$ atlas $\mathcal{B}^\iota(f, \rho)$ on $\mathcal{T}^\iota_f$.*

Proof. For every ι-leaf segment $\ell^\iota_M \in \mathcal{I}^\iota$, let $\hat{\ell}^\iota_M$ be the smallest full ι-leaf segment containing ℓ^ι_M (see definition in Appendix Section). If $HD(\Lambda^\iota) = 1$, then $\ell^\iota_M = \hat{\ell}^\iota_M$. By the Stable Manifold theorem, there are C^{1+H} diffeomorphisms $j_{\iota;M} : \hat{\ell}^\iota_M \to \hat{J}^\iota_M$. We choose the C^{1+H} diffeomorphisms $j_{\iota;M} : \hat{\ell}^\iota_M \to \hat{J}^\iota_M$ with the extra property that their images are pairwise disjoint, i.e. $\hat{J}^\iota_M \cap \hat{J}^\iota_N = \emptyset$ for all $M, N \in \mathcal{M}$ such that $M \neq N$. Let

$$\hat{L}^\iota_{\mathcal{M}} = \bigsqcup_{i=1}^{n} \hat{\ell}^\iota_{M_i}, \quad \text{and} \quad L^\iota_{\mathcal{M}} = \hat{\ell}^\iota_{\mathcal{M}} \cap \Lambda^\iota_f. \tag{1}$$

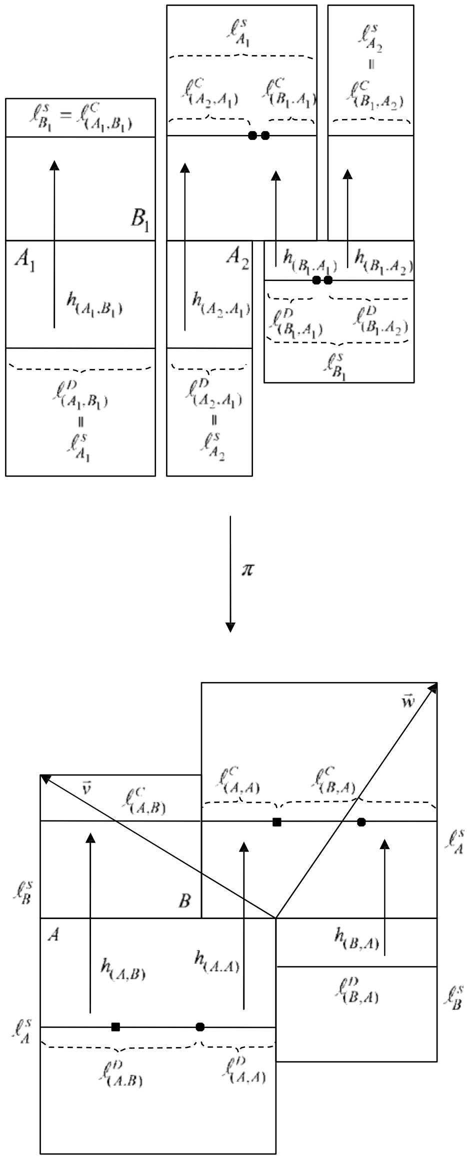

Figure 1. The complete set of holonomies $H_{g;\mathcal{M}_g} = \{h_{(A_1,B_1)}, h_{(A_2,A_1)}, h_{(B_1,A_1)}, h_{(B_1,A_2)}, h^{21}_{(A_1,B_1)}, h^{21}_{(A_2,A_1)}\}$ for the derived-Anosov diffeomorphism $g : T \to T$ semi-conjugated, by a map $p : T \to T$, to the Anosov automorphism $f : T \to T$ defined by $f(x, y) = (x + y, y)$. The complete set of holonomies for the Anosov automorphism $f : T \to T$ associated to the Markov partition $\mathcal{M} = \{A, B\}$ is given by $H_{f;\mathcal{M}} = \{h_{(A,A)}, h_{(A,B)}, h_{(B,A)}, h^{21}_{(A,A)}, h^{21}_{(A,B)}, h^{21}_{(B,A)}\}$. The complete set of holonomies $H_{g;\mathcal{M}}$ is related to $H_{f;\mathcal{M}}$ as follows: $h_{(A,B)} \circ p|p(\ell^D_{(A_1,B_1)}) = p \circ h_{(A_1,B_1)}$, $h_{(A,A)} \circ p|p(\ell^D_{(A_2,A_1)}) = p \circ h_{(A_2,A_1)}$, $h_{(B,A)} \circ p|p(\ell^D_{(B_1,A_1)}) = p \circ h_{(B_1,A_1)}$ and $h_{(B,A)} \circ p|p(\ell^D_{(B_1,A_2)}) = p \circ h_{(B_1,A_2)}$.

Let $j_\iota : L^\iota_{\mathcal{M}} \to \mathcal{J}^\iota_{\mathcal{M}}$ be the map defined by $j_\iota|\ell^\iota_M = j_{\iota;M}$, for every $M \in \mathcal{M}$. Let $\ell^\iota_M(x)$ be the spanning ι'-leaf segment of the Markov rectangle $M \in \mathcal{M}$ passing through x. Let

$$\pi_\iota : \bigcup_{i=1}^{n} M_i \to L^\iota_{\mathcal{M}} \tag{2}$$

be the projection defined by $p_i(x_i)$ ¼ y_i, where y_i [‘$^i_{M_i}$ ðx_iÞ > L^i_M, for every x_i [M_i. If $HD(L^i)$, 1, then the endpoints equivalence relation is trivial. If $HD(L^i)$ ¼ 1, then the endpoints equivalence relation is non-trivial, as we pass to describe. The endpoints

$\hat{x}_i$ [$\hat{L}^i_{M_i}$ and $\hat{x}_j$ [$\hat{L}^i_{M_j}$ are in the same endpoints equivalence class, if ‘$^{i'}_{M_i}$ ðx_iÞ > ‘$^{i'}_{M_j}$ ðx_jÞ is non-empty. The endpoints equivalence class in $\hat{L}^i_M$ is the minimal equivalence class satisfying the above property. Let the i-train track T^i_f ¼ $\hat{L}^i_M$ = , be the set $\hat{L}^i_M$ with the endpoints equivalence class as defined above.

If $HD(L^i)$, 1, the charts $k_{i,M}$, for every M [M, form a $C^{1\text{þ}a}$ atlas B$^i(f,r)$ for the train track T^i_f.

If $HD(L^i)$ ¼ 1, for every pair (M,N) [P$^{i'}$, we define $\hat{L}^i_{ðM;NÞ}$ ¼ $\hat{L}^i_M$ < $\hat{L}^i_N$, T^i_f as a *junction arc*. We fix an i-leaf segment $L^i_{ðM;NÞ}$ that is the union of two spanning i-leaf segments L^i_M and L^i_N. For every i-leaf segment, let $\tilde{L}^i_{ðM;NÞ}$ be the smallest full i-leaf segment containing $L^i_{ðM;NÞ}$, and a chart $\tilde{j}_{ðM;NÞ}$: $\tilde{L}^i_{ðM;NÞ}$! $J^i_{ðM;NÞ}$ in the atlas A^i (f, r). By Ref. [25], the holonomies h_M : ‘^i_M ! ‘$^i_{ðM;NÞ}$ > M and h_N : ‘^i_N ! ‘$^i_{ðM;NÞ}$ > N have $C^{1\text{þ}a}$ extension $\tilde{h}_M$: $\hat{L}^i_M$! $\tilde{L}^i_{ðM;NÞ}$ and $\tilde{h}_N$: $\hat{L}^i_N$! $\tilde{L}^i_{ðM;NÞ}$ onto their images. We define the *junction stable chart* $j_{ðM;NÞ}$: $\hat{L}^i_{ðM;NÞ}$! $J_{ðM;NÞ}$ in B^s(f,r) by $j_{ðM;NÞ}$j$\hat{L}^i_M$ ¼ $\tilde{j}_{ðM;NÞ}$ 8 $\tilde{h}_M$ and $j_{ðM;NÞ}$j$\hat{L}^i_N$ ¼ $\tilde{j}_{ðM;NÞ}$ 8 $\tilde{h}_N$. By construction, the charts k_M, for every M [M, and $k_{(M,N)}$, for every (M,N) [P$^{i'}$, form a $C^{1\text{þ}a}$ atlas B$^i(f,r)$ for T^i_f. A

Let $A_{ðM;NÞ}$; $B_{ðM;NÞ}$ [M be the Markov rectangles such that there is a i'-leaf segment $T^{i'}_{ðM;NÞ}$ that (i) passes through x, (ii) has endpoints a ¼ $a_{ðM;NÞ}$ [int $A_{ðM;NÞ}$ and b ¼ $b_{ðM;NÞ}$ [int $B_{ðM;NÞ}$, and (iii) $L^{i'}_{ðM;NÞ}$nð‘$^{i'}$ða; $A_{ðM;NÞ}$Þ < ‘$^{i'}$ðb; $B_{ðM;NÞ}$ÞÞ is contained in the i'-boundaries of Markov rectangles, where ‘$^{i'}$ða; $A_{ðM;NÞ}$Þ is the spanning leaf of $A_{(M,N)}$ passing through a, and ‘$^{i'}$ðb; $B_{ðM;NÞ}$Þ is the spanning leaf of $B_{(M,N)}$ passing through b. Let ‘$_{(A,M,N)}$ be an i-spanning leaf of $A_{(M,N)}$ passing through a, and let ‘$_{(B,M,N)}$ be an i-spanning leaf of $B_{(M,N)}$ passing through b. For i [{A,B}, fix $K_{ði;M;NÞ}$, ‘$_{ði;M;NÞ}$ and $L_{ði;M;NÞ}$, $L_{ðM;NÞ}$ such that the basic holonomy $h_{ði;M;NÞ}$: $K_{ði;M;NÞ}$ > L ! $L_{ði;M;NÞ}$ is well-defined. Let $\hat{K}_{ði;M;NÞ}$, $\hat{L}_{ði;M;NÞ}$, $\hat{‘}^D_{ðM;NÞ}$ and $\hat{‘}^C_{ðM;NÞ}$ be the smallest full i-leaf segments that contain $K_{ði;M;NÞ}$, $L_{ði;M;NÞ}$, ‘$^D_{ðM;NÞ}$ and ‘$^C_{ðM;NÞ}$, respectively. The set of all basic holonomies $h_{ði;M;NÞ}$: $\hat{K}_{ði;M;NÞ}$! $L_{ði;M;NÞ}$, with i [{A; B} and ðM; NÞ [P$^{i'}$, form the *i-primitive junction set* (see Figure 2).

Lemma 2. *The triple* $(f, L,$ M) *induces a* $C^{1\text{þ}H}$ *i-arc exchange system*

$$(\Phi^\iota_{f,\mathcal{M}}, \mathcal{J}^\iota_\Phi, \mathbb{T}^\iota_\Phi, \mathcal{B}^\iota_\Phi(f,\rho)),$$

with the following properties:

(i) *The set* F^i ¼ $F^i_{f;M}$ *consists of all* $C^{1\text{þ}a}$ *diffeomorphisms* $f^i_{ðM;NÞ}$: $\hat{‘}^D_{ðM;NÞ}$! $\hat{‘}$

$\tilde{C}_{(M,N)}$, with $i \in \{A,B\}$ and $(M,N) \in P^i$ such that $f^i_{(M,N)}|_{\tilde{D}_{(M,N)}} = h_{(M,N)}$.

(ii) *The junction set* J_F *consists of all* $C^{1+\alpha}$ *diffeomorphisms* $e_{(i;M,N)} : \hat{K}_{(i;M,N)} \to \hat{L}_{(i;M,N)}$, *with* $i \in \{A,B\}$ *and* $(M,N) \in P^{i'}$, *such that* $e_{(i;M,N)}|K_{(i;M,N)} = h_{(i;M,N)}$, *for every* $i \in \{A,B\}$ *and* $(M,N) \in P^{i'}$.

Proof. Since the holonomies are $C^{1+\alpha}$ diffeomorphisms with respect to $A^i(f,r)$, (a) there are $C^{1+\alpha}$ diffeomorphic extensions $f^i_{(M,N)} : \hat{D}_{(M,N)} \to \hat{C}_{(M,N)}$ of the holonomies $h^i_{(M,N)} : \tilde{D}_{(M,N)} \to \tilde{C}_{(M,N)}$ with respect to the atlas $B^i(f;r)$, for $(M,N) \in P^i$, and (b) there are $C^{1+\alpha}$ diffeomorphic extensions $e_{(i;M,N)} : \hat{K}_{(i;M,N)} \to \hat{L}_{(i;M,N)}$ of the holonomies $h^i_{(i;M,N)} : K_{(i;M,N)} \to L_{(i;M,N)}$ with respect to the atlas $B^i(f;r)$, for $(M,N) \in P^{i'}$ and $i \in \{A,B\}$. ∎

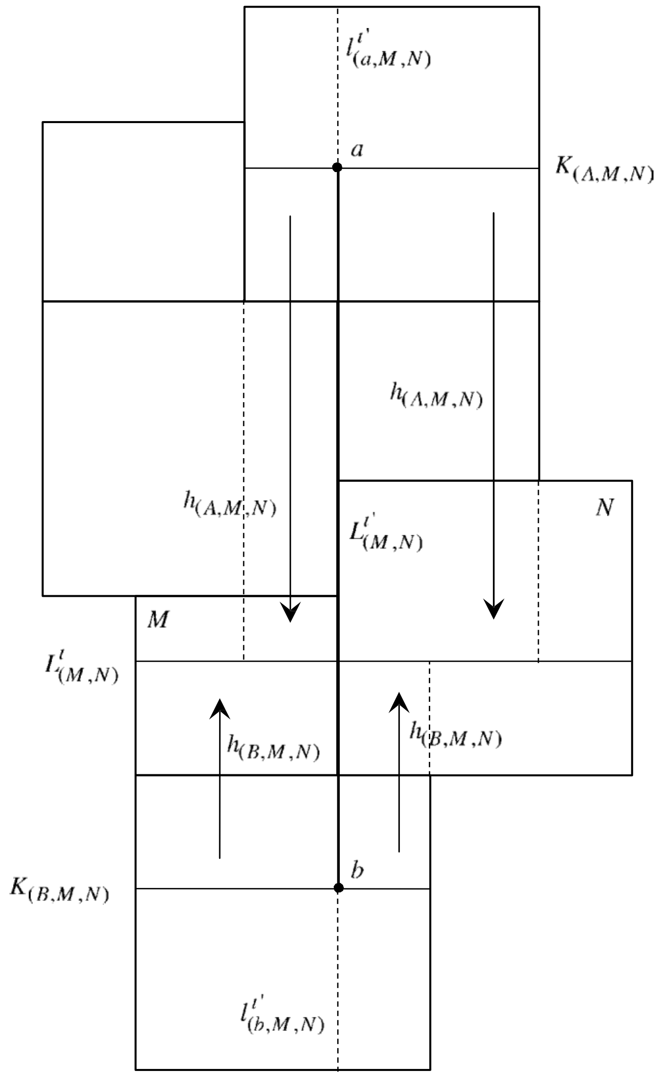

Figure 2. The construction of the elements of the junction set.

3. Renormalization of arc exchange systems

Let $F = \{f_i : \Gamma_{F,i} \to \Gamma_{F,j} : i = 1, \ldots, n\}$ and $C = \{c_i : \Gamma_{C,i} \to \Gamma_{C,j} : i = 1, \ldots, m\}$ be C^{1+H} arc exchange systems. We say that C is a *renormalization of* F if there is a *renormalization sequence set* $S = S(F, C) = \{\underline{s}^1, \ldots, \underline{s}^m\}$ with the following properties:

(i) For every $i \in \{1, \ldots, n\}$, we have that

$$\psi_i = \phi_{s^i_{k(\underline{s}^i)}} \circ \cdots \circ \phi_{s^i_1} | I_{\Psi,i},$$

where $\underline{s}^i = s^i_{k(\underline{s}^i)} \cdots s^i_1 \in S$. In particular, $V_C \subset V_F$ and $I_{C_i} \subset I_{F_{s^i_1}}$.

(ii) For every $x \in V_F \setminus V_C$, there are exactly two distinct sequences $\underline{s}^i, \underline{s}^j \in \mathcal{S}$ with the property that there are points $y_i \in I_{C;i}$, $y_j \in I_{C;j}$ such that

$$x = f_{s^i_{k(x;i)}} \circ \cdots \circ f_{s^i_1}(y_i) \quad \text{and } x = f_{s^j_{k(x;j)}} \circ \cdots \circ f_{s^j_1}(y_j);$$

for some $0 \leq k(x;i) \leq k(\underline{s}^i)$ and $0 \leq k(x;j) \leq k(\underline{s}^j)$.

For every $\underline{F} \in [F]_{C^0}$, let $j_{\underline{F}} : T_F \to T_{\underline{F}}$ be an extension of the topological conjugacy h between the C^{1+H} arc exchange systems F and $\underline{F}$. Since h is unique, by minimality of V_F, for every $\underline{s}^i \in \mathcal{S}$, $j(I_{C_i})$ and $j(J_{C_i})$ are the smallest closed arcs containing $h(I_{C_i})$ and $h(J_{C_i})$, respectively, and, so, are uniquely determined. Define the C^{1+H} arc exchange system $\underline{C}$ by

$$\underline{\Psi} = \left\{ \underline{\psi}_i = \underline{\phi}_{s^i_{k(\underline{s}^i)}} \circ \cdots \circ \underline{\phi}_{s^i_1} : \xi(\tilde{I}_{\psi_i})\, \xi(\tilde{J}_{\psi_i}), \text{ for every } \underline{s}^i \in \mathcal{S}(\Phi, \Psi) \right\}.$$

For every $e_{F;j} : L_{f_j} \to K_{f_j}$, let l_C, l_C, $c_{j;1}; \ldots; c_{j;n(j;L)}$ and $c_{j;1}; \ldots; c_{j;n(j;L)}$ be as in property (ii) of definition of $C^{1+\alpha}$ arc exchange system, in Section 2. We define the junction set $J_{\underline{C}} = \{e_{\underline{C};1}; \ldots; e_{\underline{C};m}\}$ of $\underline{C}$ as follows: $e_{\underline{C};j} : L_{\underline{C};j} \to K_{\underline{C};j}$ is given by $e_{\underline{C};j}|j(\underline{f}(l^L_{C;j})) = \underline{c}^L_{j;i_{n(j;L)}} \circ \cdots \circ \underline{c}^L_{j;i_1}$ and $e_{\underline{C};j}|j(\underline{f}(l^R_{C;j})) = \underline{c}^R_{j;i_{n(j;R)}} \circ \cdots \circ \underline{c}^R_{j;i_1}$. By construction, $\underline{C}$ is topologically conjugate to C and does not depend on the extension j of h considered in the sets $j(I_{c_1}); \ldots; j(I_{c_n})$. Furthermore, $\underline{C}$ is a C^{1+H} arc exchange system that is a renormalization of $\underline{F}$ with respect to the renormalization sequence set $\mathcal{S}(\underline{F}; \underline{C}) = \mathcal{S}(F; C)$. Hence, the *renormalization operator* R is well-defined by $R\underline{F} = \underline{C}$.

Definition 3.1. *Let $R : [F]_{C^0} \to [C]_{C^0}$ be a renormalization operator. We say that a $C^{1+\alpha}$ arc exchange system $G \in [F]_{C^0}$ is a $C^{1+\alpha}$ fixed point of the renormalization operator R, if RG is $C^{1+\alpha}$ conjugated to G, i.e. $[RG]_{C^{1+\alpha}} = [G]_{C^{1+\alpha}}$. We say that a C^{1+H} arc exchange system $G \in [F]_{C^0}$ is a C^{1+H} fixed point of the renormalization operator R, if G is $C^{1+\alpha}$ fixed point of the renormalization operator R, for some $\alpha > 0$.*

3.1 Renormalization of induced arc exchange systems

We present an explicit construction of a renormalization operator $R = R_{f,\mathcal{M}}$ acting on the topological conjugacy class of the C^{1+H} arc exchange system $F_{\mathcal{M}}$ induced by $(f, L, \mathcal{M})$. Let the Markov partition $\mathcal{N} = f_*\mathcal{M}$ be the pushforword of the Markov partition $\mathcal{M}$, i.e. for every $M \in \mathcal{M}$, $N = f(M) \in \mathcal{N}$.

Lemma 3. *Let $F_{f;\mathcal{M}}$ and $F_{f;\mathcal{N}}$ be the C^{1+H} arc exchange systems induced (as in Lemma 2), respectively, by $(f, L, \mathcal{M})$ and $(f, L, \mathcal{N})$.*

(a) *There is a well-defined renormalization operator*

$$R = R_{f,\mathcal{M}} : [\Phi_{f,\mathcal{M}}]_{C^0} \to [\Phi_{f,\mathcal{N}}]_{C^0}.$$

(b) *Let $C = RF$. For every $e_{F;j} : L_{f_j} \to K_{f_j}$ and $e_{C;j} : L_{c_j} \to K_{c_j}$, let $l^L_{f_j}$, $l^R_{f_j}$, $l^L_{c_j}$ and $l^R_{c_j}$ be as in property (iii) of the Definition 2.1. If $e_{F;j}|l^L = f^L_{j;i_n} \circ \cdots \circ f^L_{j;i_1}$ and $e_{F;j}|l^R = f^R \circ \cdots \circ f^R$, then $e_{C;j}|l^L = c^L \circ \cdots \circ c^L$ and $e_{C;j}|l^R = c^R \circ \cdots \circ c^R$.*

f_j $j;i_n$ $j;i_1$ c_j $j;i_n$ $j;i_1$ c_j $j;i_n$ $j;i_1$

Proof. For simplicity of notation, let us denote $k_{\mathcal{M}}$ by k (see (1)). We choose a map

$$\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\} \qquad (3)$$

with the property that $N_i > M_{\sigma(i)} - \emptyset$, where $N_i \in \mathcal{N}$ and $M_{\sigma(i)} \in \mathcal{M}$. For each $N_i \in \mathcal{N}$, let ℓ_{N_i} be the stable spanning leaf segment $\ell_{M_{\sigma(i)}} > p(N_i)$, and let $\hat{\ell}_{N_i}$ be the corresponding full stable spanning leaf (i.e. $\hat{\ell}_{N_i} > L = \ell_{N_i}$), where $p : \cup_{i=1}^{n} M_i \to L_{\mathcal{M}}$ is the natural projection as defined in (1). Set

$$\mathcal{L}_{\mathcal{N}} = \bigcup_{i=1}^{n} \ell_{N_i} \quad \text{and} \quad \hat{\mathcal{L}}_{\mathcal{N}} = \bigcup_{i=1}^{n} \hat{\ell}_{N_i}.$$

The set $\hat{\mathcal{L}}_{\mathcal{N}}$ determines the train track $\mathcal{T}_{\mathcal{N}}$ with atlas $\mathcal{B}(f, r)$ as constructed in Lemma 1. Let $\mathcal{H}_{\mathcal{N}} = \{h_{(N_i;N_j)} : \ell^D_{(N_i;N_j)} \to \ell^C_{(N_i;N_j)} | (N_i; N_j) \in \mathcal{P}_{\mathcal{N}}\}$ be the (stable) primitive holonomic system associated to the Markov partition $\mathcal{N}$. By construction, for every $(N_i; N_j) \in \mathcal{P}_{\mathcal{N}}$ there is a sequence $h_{\alpha_1}; \ldots; h_{\alpha_n}$ of holonomies in $\mathcal{H}_{\mathcal{M}}$ such that

$$h_{(N_i,N_j)} = h_{\alpha_n} \circ \cdots \circ h_{\alpha_1} | \hat{\ell}^D_{N_i}.$$

Let

$$\psi_{(N_i,N_j)} : \hat{\ell}^D_{(N_i,N_j)} \to \hat{\ell}^D_{(N_i,N_j)}$$

be given by $c_{(N_i;N_j)} = f_{\alpha_n} + \cdots + f_{\alpha_1}$, where $f_{\alpha_i} \in F_{f;\mathcal{M}}$ and $f_{\alpha_i} | \ell^D_{(N_i;N_j)} = h_{\alpha_i} | \ell^D_{(N_i;N_j)}$. Set

$$\Psi = \left\{ \psi_{(N_i,N_j)} : \hat{\ell}^D_{(N_i,N_j)} \to \hat{\ell}^C_{(N_i,N_j)} | (N_i, N_j) \in \mathcal{P}_{\mathcal{N}} \right\}.$$

Let $F_{f;\mathcal{N}}$ be as constructed in Lemma 2. Hence, $C = F_{f;\mathcal{N}}$, and, so, C is a C^{1+H} arc exchange system. Since the set $S(F_{f;\mathcal{M}}; F_{f;\mathcal{N}})$ of all sequences $\alpha_1 \cdots \alpha_n$ such that $c_{(N_i;N_j)} = f_{\alpha_n} + \cdots + f_{\alpha_1}$, for some $(N_i; N_j) \in \mathcal{P}_{\mathcal{N}}$, form a renormalizable sequence set, the C^{1+H} arc exchange system $F_{f,\mathcal{N}}$ is a renormalization of $F_{f,\mathcal{M}}$. Therefore, by Section 3, there is a well-defined renormalization operator $R = R_{f;\mathcal{M}} : [F_{f;\mathcal{M}}]_{C^0} \to [F_{f;\mathcal{N}}]_{C^0}$. Since $\mathcal{N} = f_* \mathcal{M}$ and $RF_{f;\mathcal{M}} = F_{f;\mathcal{N}}$, property (b) holds. $\square$

Lemma 4. *The C^{1+H} arc exchange system $F_{f,\mathcal{M}}$ is a C^{1+H} fixed point of renormalization, i.e. $[RF_{f;\mathcal{M}}]_{C^0} = [F_{f;\mathcal{M}}]_{C^0}$, where $R = R_{f;\mathcal{M}} : [F_{f;\mathcal{M}}]_{C^0} \to [F_{f;\mathcal{N}}]_{C^0}$ is the renormalization operator.*

Proof. We construct a $C^{1+\alpha}$ conjugacy $Q : \mathcal{T}_{\mathcal{N}} \to \mathcal{T}_{\mathcal{M}}$ between $F_{f,\mathcal{M}}$ and $F_{f,\mathcal{N}}$. For every $N \in \mathcal{N}$ and $M = f^{-1}(N)$, there is a holonomy u_N between the spanning leaf segments $f^{-1}(\ell_N)$ and ℓ_M. By Theorem 2.1 in Ref. [25], the holonomy u_N has a $C^{1+\alpha}$ diffeomorphic extension $\hat{u}_N : f^{-1}(\hat{\ell}_N) \to \hat{\ell}_M$. Let $Q : \mathcal{T}_{\mathcal{N}} \to \mathcal{T}_{\mathcal{M}}$ be the $C^{1+\alpha}$ diffeomorphism given by

$$\Theta | \hat{\ell}_N = \hat{\theta}_N \circ f^{-1}, \qquad (4)$$

for every N [N. We observe that each pair

ðN_i;N_jÞ [P_{N}

determines a unique pair $(M_i, M_j) = (f^{-1}(N_i), f^{-1}(N_j)) \in P^D_{\mathcal{M}}$, and vice-versa. By Lemma 3(b), it is enough to prove that Q conjugates $\psi_{(N_i,N_j)}|\ell^D_{(N_i,N_j)}$ with $\psi_{(M_i,M_j)}|\ell^D_{(M_i,M_j)}$, for every $(N_i,N_j) \in P_{\mathcal{N}}$, to show that $F_{f,\mathcal{M}}$ is a C^{1+H} fixed point of renormalization.

By construction of the maps u_{N_i} and u_{N_j}, we have that

$$h_{(M_i,M_j)}|\ell^D_{(M_i,M_j)} = \theta_{N_i} \circ f^{-1} \circ h_{(N_i,N_j)} \circ f \circ \theta^{-1}_{N_j},$$

and so

$$\Theta \circ \psi_{(N_i,N_j)} \circ \Theta^{-1}|\ell^D_{(M_i,M_j)} = \theta_{N_j} \circ f^{-1} \circ h_{(N_i,N_j)} \circ f \circ \theta^{-1}_{N_j} = h_{(M_i,M_j)} = \psi_{(M_i,M_j)},$$

which ends the proof.

4. Markov maps versus renormalization

The map $F : \hat{\mathcal{T}} \subset \mathcal{T} \to \mathcal{T}$ determines a $C^{1+\alpha}$ *Markov map*, with respect to the atlas $\mathcal{B}$ and with invariant set $V \subset \hat{\mathcal{T}}$, if the following properties are satisfied:

(i) $\hat{\mathcal{T}} = \mathcal{T}$ or $\hat{\mathcal{T}}$ is a union of closed intervals.

(ii) $F : \mathcal{T} \to \mathcal{T}$ is a $C^{1+\alpha}$ diffeomorphism, for every (small) arc, with respect to the $C^{1+\alpha}$ atlas $\mathcal{B}$ on the train track $\mathcal{T}$.

(iii) There exist $c > 0$ and $\lambda > 1$ such that, for every $x \in V$,

$$|d(j_n \circ F^n \circ i^{-1})(x)| > c\lambda^n, \tag{5}$$

with respect to charts $i, j_n \in \mathcal{B}$.

(iv) The map F admits a Markov partition $\{K_1, \ldots, K_m\}$, i.e. there exists a finite set of arcs $\{\hat{K}_1, \ldots, \hat{K}_m\}$ such that (*a*) $K_i = \hat{K}_i \cap V$, (*b*) $\cup_{i=1}^m \hat{K}_i \supset V$ and (*c*) $F(\partial \hat{K}_j) \subset \cup_{i=1}^m \partial \hat{K}_i$, for every $j = 1, \ldots, m$.

Let $\underline{F} : L_{\mathcal{M}} \to L_{\mathcal{M}}$ be the map induced by the action of f^{-1} on stable leaf segments, i.e. $\underline{F}(x) = p_\dagger f^{-1}(x)$ for every $x \in L$ (see (2)). Since f is a local diffeomorphism, the map $\underline{F}$ is a local homeomorphism. Let $\tilde{F} : k_{\mathcal{M}}(L_{\mathcal{M}}) \to k_{\mathcal{M}}(L_{\mathcal{M}})$ be the map defined by $\tilde{F} = k_{\mathcal{M}} \circ \underline{F} \circ k_{\mathcal{M}}^{-1}$. Since the holonomies have C^{1+} extensions (see Theorem 2.1 in Ref. [25]), and the map f is $C^{1+\alpha}$, for some $\alpha > 0$, the map $\tilde{F}$ has a $C^{1+\alpha}$ extension $F_{f,\mathcal{M}} : \mathcal{T}_f \to \mathcal{T}_f$, with respect to the atlas $\mathcal{B}(f,r)$, (not uniquely determined) that is a $C^{1+\alpha}$ Markov map with Markov partition $\{k_{\mathcal{M}} + p(M_1), \ldots, k_{\mathcal{M}} + p(M_l)\}$, where $\mathcal{M} = \{M_1, \ldots, M_l\}$ is the Markov partition of f (see also Pinto and Rand [26]). Hence, the map $F_{f,\mathcal{M}} : \mathcal{T}_f \to \mathcal{T}_f$ constructed above is a $C^{1+\alpha}$ Markov map.

Definition 4.1. *Let $h : V_F \to V_C$ be the topological conjugacy between a C^{1+H} arc exchange system $C = \{c_i : I_{c_i} \to J_{c_i}; i = 1, \ldots, m\}$ and $F_{f,\mathcal{M}} = \{f_i : I_{f_i} \to J_{f_i}; i = 1, \ldots, n\}$. We say that C induces a C^{1+H} Markov map*

$$F_C : \mathcal{T}_C \to \mathcal{T}_C,$$

if F_C is a $C^{1+\alpha}$ Markov map, for some $\alpha > 0$, and $F_C \circ h(x) = h \circ F_{f;\,M}(x)$, for every $x \in V_C$.

Let us suppose that the C^{1+H} arc exchange system C is a C^0 fixed point of renormalization $[RC]_{C^0} = [C]_{C^0}$. In this case, C is an infinitely renormalizable C^{1+H} arc exchange system, i.e. there is an infinite sequence

$$\left(R^m\Psi = \left\{\psi_i^{(m)} : \bar{I}_{\psi_i}^{(m)} \to \bar{J}_{\psi_i}^{(m)};\ i=1,\dots,n(m)\right\}\right)_{m\ge 1}$$

of arc exchange systems inductively determined, for every $m \geq 1$, by $R^m\mathcal{C} = R(R^{m-1}\mathcal{C})$.
Set

$$L_m^{(1)} = \left\{\psi_{s_k^i}^{(m)}\circ\dots\circ\psi_{s_1^i}^{(m)}\left(I_{\psi_i}^{(m+1)}\right) : I_{\psi_i}^{(m+1)} \subset I_{\psi_{s_1^i}}^{(m)}, 0\le k\le k(\underline{s}_i), \underline{s}_i\in\mathcal{S}\right\}.$$

Set, inductively on $j \geq 1$, the sets

$$L_m^{(j)} = \left\{\psi_{s_k^i}^{(m)}\circ\dots\circ\psi_{s_1^i}^{(m)}(I) : I \in L_{m+1}^{(j-1)}, I\subset I^{(m)}_{\psi_{s_1^i}}, 0\le k\le k(\underline{s}_i), \underline{s}_i\in\mathcal{S}\right\}.$$

By construction, $L_m^{(j+1)} \subset L_m^{(j)}$ and $V_{R^m\mathcal{C}} = \cap_{j\ge 1} L_m^{(j)}$. We call $L_m^{(j)}$ the *j-th level of the partition of* $R^m\mathcal{C}$. Let the *j-gap set* $G_m^{(j)}$ of $R^m\mathcal{C}$ be the set of all maximal closed intervals I such that $I\subset J$ for some $J\in L_m^{(j-1)}$ and int $I \cap K = \emptyset$, for every $K\in L_m^{(j)}$. We say that the C^{1+H} arc exchange system $\mathcal{C}$ *has bounded geometry*, if there are constants $0 < c_1; c_2 < 1$ such that, for all $j\ge 1$ and all intervals $I \in L_0^{(j)} \cup G_0^{(j)}$ contained in a same interval $K\in L_0^{(j-1)}$, we have $c_1 < |z(I)|/|z(K)| < c_2$, where the length is measured with respect to any chart z in the $C^{1+\alpha}$ atlas $\mathcal{B}_\mathcal{C}$.

Lemma 5. *Let $F_{f,\mathcal{M}}$ be a C^{1+H} arc exchange system induced by (f, L, M). A C^{1+H} arc exchange system $\mathcal{C}\in[F_{f,\mathcal{M}}]_{C^0}$, with bounded geometry, determines a C^{1+H} Markov map $F_\mathcal{C}$ topologically conjugate to $F_{f,\mathcal{M}}$ if, and only if, $\mathcal{C}$ is a C^{1+H} fixed point of the renormalization operator $R_{f,\mathcal{M}}$.*

Remark 1. Lemma 5 also holds for $C^{1,\alpha}$ regularities.

Proof of Lemma 5. For simplicity of notation, let us denote $k_\mathcal{M}$ by k (see (1)). Let $\Theta: K_\mathcal{N} \to K_\mathcal{M}$ be the $C^{1+\alpha}$ diffeomorphism as constructed in (4). For every $\mathcal{N}\in\mathbf{N}$, let $\mathcal{M} = f^{-1}(\mathcal{N})\in\mathbf{M}$. Recall that $\ell_\mathcal{N} \subset \ell_\mathcal{M}^{s(\mathcal{N})}\subset L_\mathcal{M}$ (see (3)). By construction of $F=F_{f,\mathcal{M}}$ and Θ, the spanning leaf segment $\ell_\mathcal{N}\subset L_\mathcal{N}$ has the property that $F\circ k(\ell_\mathcal{N}) = k(\ell_\mathcal{M})$ and $F|k(\ell_\mathcal{N}) = \Theta$. Therefore,

$$F|K_\mathcal{N} = \Theta. \tag{6}$$

Every leaf segment $\ell\subset L_\mathcal{M}$ with the property that $F\circ k(\ell) = k(\ell_\mathcal{M})$ is a spanning leaf segment of $\mathcal{N}$. Therefore, there is a sequence $e_{\alpha_1};\dots;e_{\alpha_p}$ of arc exchange maps in $F = F_{f,\mathcal{M}}$ such that

$$e_{\alpha_p}\circ\dots\circ e_{\alpha_1}(k(\ell)) = k(\ell_\mathcal{N}).$$

Furthermore,

$$F|k(\ell) = \Theta\circ e_{\alpha_p}\circ\dots\circ e_{\alpha_1}. \tag{7}$$

Let $j: \cup_{i=1}^n I_{f_i} \to \cup_{i=1}^n I_{c_i}$ be a homeomorphic extension of the conjugacy between F and

C. For every e **[** F, there is a unique $\underline{e}$ **[** C such that $\underline{e}$ ¼ $\boldsymbol{j}$+e+$\boldsymbol{j}^{21}$. Since F_C is topologically conjugate to F, by (6), we have that

$$F_\Psi|\xi(K_{\mathcal{N}}) = \Theta_\Psi, \tag{8}$$

where $Q_C : \xi(K_{\mathcal{N}}) \to \xi(K_{\mathcal{M}})$ is a homeomorphic extension of the conjugacy between C and its renormalization RC. Letting $\tilde{\ell}_{\mathcal{N}}$, $\tilde{\ell}$ and $e_{a_1}; \ldots; e_{a_p}$ be as above, by (7), we obtain that

$$F_\Psi|\xi \circ k(\tilde{\ell}) = \Theta_\Psi \circ \underline{e}_{\alpha_p} \circ \cdots \circ \underline{e}_{\alpha_1} \tag{9}$$

By (8), if F_C is $C^{1+\alpha}$ then Q_C is $C^{1+\alpha}$ (also along arcs containing junctions). By (9), if Q_C is $C^{1+\alpha}$ then F_C is locally a $C^{1+\alpha}$ diffeomorphism.

Let $L_0^{(j)}$ be the jth level of the partition of C. By construction, every interval $I \in L_0^{(j)}$ has the property that $F_C^{j-1}(I)$ is an element of the Markov partition of F_C (this property characterizes $L_0^{(j)}$). In particular, the map F_C sends each interval $I \in L_0^{(j)}$ onto an interval $F_C(I) \in L_0^{(j-1)}$ for every $j \geq 0$. Hence, if C has bounded geometry we obtain that the length of the sets in $L_0^{(j)}$ converge exponentially fast to 0 when j tends to infinity. Therefore, using the Mean Value theorem, we obtain that if C has bounded geometry then F_C satisfies property (ii) and, conversely, if F_C satisfies property (ii) we obtain that C has bounded geometry. So, we conclude that if C is a $C^{1+\alpha}$ arc exchange system, with bounded geometry, then F_C is a $C^{1+\alpha}$ Markov map, and vice-versa. □

5. C^{1+H} flexibility

Let $(f, \Lambda, \mathcal{M})$ be a C^{1+H} hyperbolic diffeomorphism. Let $\mathcal{C}^\iota_{f,\mathcal{M}}$ be the topological conjugacy class of $\Phi^\iota_{f,\mathcal{M}}$. Let $\mathcal{F}$ be the set of all C^{1+H} hyperbolic diffeomorphisms topologically conjugate to f (see Appendix A.6).

Theorem 1. *There is a unique map*

$$\mathcal{T}^\iota_{f,\mathcal{M}} : \mathcal{F} = \{[g]_{C^{1+H}} : g \in \mathcal{F}\} \to \mathcal{C}^\iota = \left\{[\Phi^\iota]_{C^{1+H}} : \Phi^\iota \in \mathcal{C}^\iota_{f,\mathcal{M}}\right\}$$

defined by $\mathcal{T}^\iota_{f,\mathcal{M}}([g]_{C^{1+H}}) = [\Phi^\iota_{g,\mathcal{M}_g}]_{C^{1+H}}$, *where* $\mathcal{M}_g$ *is the pushforword of the Markov partition* $\mathcal{M}$ *of* f *by the topological conjugacy between* f *and* g. *The map* $\mathcal{T}_\iota = \mathcal{T}^\iota_{f,\mathcal{M}} : \mathcal{F} \to \mathcal{C}^\iota$ *has the following properties:*

(a) *If* $[\Phi^\iota]_{C^{1+H}} = \mathcal{T}_\iota([g]_{C^{1+H}})$, *then* $HD(V^\iota) = HD(\Lambda^\iota)$;

(b) $\mathcal{T}_\iota(\mathcal{F}) = \mathcal{C}^\iota_R$, *where* $\mathcal{C}^\iota_R \subset \mathcal{C}^\iota$ *is the set of all* C^{1+H} *conjugacy classes* $[\Phi^\iota]_{C^{1+H}} \in \mathcal{C}^\iota$ *that are* C^{1+H} *fixed points of renormalization,* $[R^\iota\Phi^\iota]_{C^{1+H}} = [\Phi^\iota]_{C^{1+H}}$;

(c) *For every pair* $([\Phi^s]_{C^{1+H}}, [\Phi^u]_{C^{1+H}}) \in \mathcal{C}^s_R \times \mathcal{C}^u_R$, *there is a unique* C^{1+H} *conjugacy class of* C^{1+H} *hyperbolic diffeomorphisms*

$$g \in \mathcal{T}_s^{-1}([\Phi^s]_{C^{1+H}}) \cap \mathcal{T}_u^{-1}([\Phi^u]_{C^{1+H}});$$

(d) *For every* $[F^i]_{C^{1+H}} \in \mathcal{C}^i_R$*, there is a unique* C^{1+H} *conjugacy class of Lipschitz hyperbolic diffeomorphisms* $g \in \mathcal{T}_i^{-1}([F^i]_{C^{1+H}})$ *that admits an invariant measure absolutely continuous with respect to the Hausdorff measure on* Λ_g*;*

(e) *The set* $\mathcal{C}^i_R$ *is characterized by a moduli space consisting of solenoid functions;*

(f) *The set* $\mathcal{C}^i_L$ *consisting of all Lipschitz conjugacy classes in* $\mathcal{C}^i_R$ *is also characterized by a moduli space consisting of measure solenoid functions.*

The above solenoid functions and measure solenoid functions are introduced in Refs. [26,29] where they are used to construct moduli spaces for the set of all C^{1+H} and Lipschitz conjugacy classes of C^{1+H} hyperbolic diffeomorphisms (see Appendix A.10 - A.16). If $HD(L^i) = 1$, then, in Theorem 1, the Lipschitz conjugacy classes coincide with the C^{1+H} conjugacy classes, and, so, $\mathcal{C}^i_L = \mathcal{C}^i_R$.

Remark 2. We note that in Theorem 1, if the *i*-lamination of the hyperbolic basic set L is orientable, then the *i*-arc exchange systems in $\mathcal{C}^i_{f,M}$ are determined by *i*-arc exchange maps.

Proof of Theorem 1. By Ref. [25], the basic holonomies are $C^{1+\alpha}$ diffeomorphisms with respect to the $C^{1+\alpha}$ atlases $\mathcal{A}^i(g_1; r_1)$ and $\mathcal{A}^i(g_2; r_2)$, for some $\alpha > 0$. Hence, there is a $C^{1+\alpha}_i$ diffeomorphism $\underline{u}: \mathbb{T}_{g_1} \to \mathbb{T}_{g_2}$ with respect to the atlases $\mathcal{B}^i(g_1; r_1)$ on $\mathbb{T}_{g_1}$ and $\mathcal{A}(g_2; r_2)$ on $\mathbb{T}_{g_2}$, such that $\underline{u} \circ p_{g_1} = p_{g_2} \circ \underline{u}$, where $p_{g_1}: L_{g_1} \to \mathbb{T}_{g_1}$ and $p_{g_2}: L_{g_2} \to \mathbb{T}_{g_2}$ are the natural projections. Hence, the $C^{1+\alpha}$ induced arc exchange system F_{g_1} is $C^{1+\alpha}$ conjugate to the $C^{1+\alpha}$ induced arc exchange system F_g.

Proof of statement (*a*): Since the holonomies are $C^{1+\alpha}$ (see Pinto and Rand [25]), the Hausdorff dimension of the stable leaf segments ℓ is the same independently of the stable leaf segment considered, and so equal to $HD(L^s_g)$. In particular, all leaf segments $\ell_{g,M} \in \mathbf{I}_g$ have the same Hausdorff dimension which is equal to the Hausdorff dimension of L_g. Since the arc invariant set $\mathbb{T}_{F_{g,M_g}}$ is equal to $k(L_g)$, the Hausdorff dimension $HD(\mathbb{T}_{F_{g,M_g}})$ is equal to $HD(L^s_g)$.

Proof of statement (*b*): By Lemma 4, if $g \in \mathcal{F}$, then the C^{1+H} arc exchange system F_{g,M_g} is a fixed point of the renormalization operator R_{g,M_g} that, by construction, is the same as $R_{f,M}$. Hence, $\mathcal{T}(\mathcal{F}) \subset \mathcal{C}_R$.

The proof that $\mathcal{T}(\mathcal{F}) \supset \mathcal{C}_R$ follows from the proof of the statement (c) below.

Proof of statement (*c*): Let F be a C^{1+H} arc exchange system such that $[RF]_{C^{1+H}} = [F]_{C^{1+H}}$. Since $[RF]_{C^{1+H}} = [F]_{C^{1+H}}$, by Lemma 5, the C^{1+H} arc exchange system F induces a Markov map F_F. Therefore, (F, F_F) is equivalent to a $C^{1+\alpha}$ self-renormalizable structure as defined in Ref. [30].

By Theorem 1.14 in Ref. [28] (see also Pinto and Rand [29]), there is a one-to-one correspondence between C^{1+H} conjugacy classes of (F, F_F) and C^{1+H} conjugacy classes of C^{1+H} diffeomorphisms $g(F, F_F)$ with hyperbolic invariant set L_g, and with an invariant measure absolutely continuous with respect to the Hausdorff measure.

Proof of statement (*d*): Let F be a C^{1+H} arc exchange system such that $[RF]_{C^{1+H}} = [F]_{C^{1+H}}$. Since $[RF]_{C^{1+H}} = [F]_{C^{1+H}}$, by Lemma 5, the C^{1+H} arc exchange system F induces a Markov map F_F. Let $\mathcal{C}_F$ be the set of all C^{1+H} conjugacy classes of pairs (F, F_F). Hence, there is a one-to-one map $m_1: \mathcal{C}_R \to \mathcal{C}_F$ given by $m_1(F) = (F; F_F)$. By Lemma 4.2 in Ref. [28] (see also Pinto and Rand [29]), there is a well-defined Teichmüller space TS consisting of solenoid functions, and a one-to-one map $m_2: TS \to \mathcal{C}_F$ given by $m_2(s) = (F; F_F)$. Therefore, $m_1^{-1} \circ m_2: TS \to \mathcal{C}_R$ is a one-to-one map. ∎

6. $C^{1,HD}$ rigidity

Let us present the following notion of $C^{1,HD}$ regularity of a function.

Definition 6.1. *Let $f: I \to J$ be a homeomorphism between open sets $I \subset \mathbb{R}$ and $J \subset \mathbb{R}$. If $0 < \alpha < 1$, then f is said to be $C^{1,\alpha}$ if f is differentiable and for all points $x, y \in I$*

$$|\phi'(y) - \phi'(x)| \leq \chi_\phi(|y - x|), \tag{10}$$

where the positive function $\chi_f(t)$ satisfies $\lim_{t\to 0}\chi_f(t)/t^{\alpha}=0$. f is said to be $C^{1,\alpha}$, if, for all points $x, y\in I$,

$$\left|\log\phi'(x)+\log\phi'(y)-2\log\phi'\left(\frac{x+y}{2}\right)\right|\le\chi(|y-x|),$$

where the positive function $\chi(t)$ satisfies $\lim_{t\to 0}\chi(t)/t=0$.

In particular, for every $\beta>\alpha>0$, a $C^{1+\beta}$ diffeomorphism is $C^{1,\alpha}$, and, for every $\gamma>0$, a $C^{2+\gamma}$ diffeomorphism is $C^{1,1}$. We note that the regularity $C^{1,1}$ (also denoted by $C^{1+\text{zigmund}}$) of a diffeomorphism u used in this paper is stronger than the regularity $C^{1+\text{Zigmund}}$ (see Ref. [18]). The importance of these $C^{1,\alpha}$ smoothness classes for a diffeomorphism $u: I\to J$ follows from the fact that if $0<\alpha<1$ the map u will distort ratios of lengths of short intervals in an interval $K\subset I$ by an amount that is $o(|I|^{\alpha})$, and if $\alpha=1$ the map u will distort the cross-ratios of quadruples of points in an interval $K\subset I$ by an amount that is $o(|I|)$ (see Ref. [27]).

An *arc exchange system* $(F; \mathcal{J}_F; \mathcal{T}_F; \mathcal{B}_F)$ is *affine*, if $\mathcal{B}_F$ is an affine atlas and the maps in F and in $\mathcal{J}_F$ are affine with respect to the charts in $\mathcal{B}_F$.

Theorem 2. *Let $\mathcal{C}^i_{f;\Lambda}$ be the topological conjugacy class of C^{1+H} i-arc exchange systems determined by a C^{1+H} hyperbolic diffeomorphism (f, L, Λ) (as in Theorem 1). Every $C^{1,HD(L^i)}$ arc exchange system $F\in\mathcal{C}^i_{f,\Lambda}$ with bounded geometry, that is a $C^{1,HD(V_F)}$ fixed point of renormalization operator, i.e. $[R_{f;\Lambda}F]_{C^{1,HD(V_F)}}=[F]_{C^{1,HD(V_F)}}$, is $C^{1,HD(L^i)}$ conjugate to an affine i-arc exchange system.*

By Ref. [33], if $HD(L^i)<1$ and $HD(L^{i'})=1$, then there are no $C^{1,HD(V_F)}$ arc exchange systems $F\in\mathcal{C}_{f,\Lambda}$, with bounded geometry, that are $C^{1,HD(V_F)}$ fixed points of renormalization operator.

Proof of Theorem 2. Let us suppose that the arc exchange system C is a $C^{1,\alpha}$ fixed point of the renormalization operator $R_{f,\Lambda}$ with $\alpha=HD(T_C)$ and with bounded geometry. Hence, by Lemma 5, C induces a $C^{1,\alpha}$ Markov map F_C. Let j be the homeomorphic extension of the conjugacy between F and C, and set $h=j+k+p$. We will consider the following two distinct cases: (a) $HD(L^i)<1$ and (b) $HD(L^i)=1$.

Case $HD(L^i)<1$. Let T_n be the set of all pairs (I,J) such that (i) I is a stable leaf n-cylinder, (ii) J is a stable leaf n-cylinder or a stable n-gap cylinder and (iii) I and J have a unique common endpoint (see Appendix A.4). Using the Mean Value theorem and that F_C is a $C^{1,\alpha}$ Markov map, the function $r:\cup_{n\ge 1}T_n\to\mathbb{R}^+$ given by

$$r(I,J)=\lim_{m\to+\infty}\frac{|\eta\circ f^m(J)|}{|\eta\circ f^m(I)|}$$

is well-defined, where $|L|$ means the length of the smallest interval containing $L\subset\mathbb{R}$. By bounded geomatry of C, we obtain that r is bounded away from zero. Furthermore, using that F_C is a $C^{1,\alpha}$ Markov map, for every pair $(I,J)\in T_n$, we get

$$\frac{|\eta(J)|}{|\eta(I)|}(1-C_n(|\eta(I\cup J)|^{\alpha}))\le r(I,J)\le\frac{|\eta(J)|}{|\eta(I)|}(1+C_n(|\eta(I\cup J)|^{\alpha})),\qquad(11)$$

where $C_n\in\mathbb{R}^+_0$ converges to zero when n tends to infinity.

Let $h = h_{(M,N)} : \tilde{I}^D_{(M,N)} \to \tilde{I}^C_{(M,N)}$ be a i-primitive holonomy. Since the arc exchange system is $C^{1,\alpha}$, for every $(I,J) \in T_n$ such that $I < J \subset \tilde{I}^D_{(M,N)}$, we get

$$1 - C_n|\eta(I \cup J)|^\alpha \le \frac{|\eta(I)|}{|\eta(J)|}\frac{|\eta \circ h(J)|}{|\eta \circ h(I)|} \le 1 + C_n|\eta(I \cup J)|^\alpha, \qquad (12)$$

where $C_n \in \mathbb{R}^+_0$ converges to zero when n tends to infinity.

From (11), we obtain that

$$\frac{|\eta(I)|}{|\eta(J)|}\frac{|\eta \circ h(J)|}{|\eta \circ h(I)|}(1 - C_n|\eta(I \cup J)|^\alpha) \le \frac{r(h(I),h(J))}{r(I,J)}$$
$$\le \frac{|\eta(I)|}{|\eta(J)|}\frac{|\eta \circ h_\alpha(J)|}{|\eta \circ h_\alpha(I)|}(1 + C_n|\eta(I \cup J)|^\alpha).$$

Thus, using (12) we get

$$1 - C'_n|\eta(I \cup J)|^\alpha \le \frac{r(h(I),h(J))}{r(I,J)} \le 1 + C'_n|\eta(I \cup J)|^\alpha,$$

where $C'_n \in \mathbb{R}^+_0$ converges to zero when n tends to infinity.

Since $\alpha = HD(T_C)$, by the Rigidity Lemma 4.1 in Ref. [27], we obtain that r is a stable transversely affine ratio function (see definition in Appendix A.7).

Case $HD(L^i) = 1$. Let J_0, J_1 and J_2 be distinct leaf segments such that J_0 and J_1 have a common endpoint, and J_1 and J_2 have also a common endpoint. Let the *cross-ratio* $cr(J_0, J_1, J_2)$ be given by

$$cr(J_0, J_1, J_2) = \frac{1 + r(J_1, J_0)}{r(J_2, J_0 \cup J_1 \cup J_2)}.$$

A similar argument to the one above gives that

$$1 - C_n|\eta(J_0 \cup J_1 \cup J_2)| \le \frac{cr(h(J_0), h(J_1), h(J_2))}{cr(J_0, J_1, J_2)} \le 1 + C_n|\eta(J_0 \cup J_1 \cup J_2)|,$$

where $C_n \in \mathbb{R}^+_0$ converges to zero when n tends to infinity. Hence, by the Rigidity Lemma 4.1 in Ref. [27], we obtain that r is a stable transversely affine ratio function. Therefore, the ratio function r determines an affine atlas $A(r)$ on the i-leaf segments such that the holnomies and f are affine. Thus, the atlas $B(r)$, on the train track T_f, induced by $A(r)$ is an affine atlas such that the arc exchange system is affine and the Markov map is also affine. Therefore, the arc exchange system is an affine fixed point of renormalization. ∎

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Appendix

In this appendix, we present some basic facts for C^{1+H} hyperbolic diffeomorphisms (f, Λ), that we include for clarity of the exposition. We say that (f, Λ) is a C^{1+H} *hyperbolic diffeomorphism*, if (f, Λ) has the following properties:

(a) $f: S \to S$ is a $C^{1+\alpha}$ diffeomorphism of a compact surface S with respect to a $C^{1+\alpha}$ structure on S, for some $\alpha > 0$.
(b) Λ is a hyperbolic invariant subset of S such that $f|\Lambda$ is topologically transitive and Λ has a local product structure.

In particular, a C^{1+H} diffeomorphism f with a codimension 1 attractor Λ is a C^{1+H} hyperbolic diffeomorphism.

A.1 Leaf segments

Let d be a metric on M, and define the map $f_i = f$ if $i = u$, or $f_i = f^{-1}$ if $i = s$. For $i \in \{s,u\}$, if $x \in \Lambda$ we denote the local i-manifolds through x by

$$W^{\iota}(x,\varepsilon) = \left\{ y \in M : d(f_{\iota}^{-n}(x), f_{\iota}^{-n}(y)) \le \varepsilon, \ \text{for all } n \ge 0 \right\}.$$

By the Stable Manifold theorem [36] (see Refs. [14,36]), these sets are respectively contained in the stable and unstable immersed manifolds

$$W^{\iota}(x) = \bigcup_{n \ge 0} f_{\iota}^{n}\left(W^{\iota}\left(f_{\iota}^{-n}(x), \varepsilon_0 \right) \right),$$

which are the image of a $C^{1+\gamma}$ immersion $\kappa_{i,x} : \mathbb{R} \to M$. An *open* (respectively, *closed*) *full i-leaf segment* I is defined as a subset of $W^{i}(x)$ of the form $\kappa_{i,x}(I_1)$ where I_1 is an open (respectively, closed) subinterval (non-empty) in R. An *i-open* (respectively, *closed*) *leaf segment* is the intersection with L of a full open (respectively, closed) i-leaf segment such that the intersection contains at least two distinct points. If the intersection is exactly two points we call this i-closed leaf segment an *i-leaf gap*. An *i-full leaf segment* is either an open or closed i-full leaf segment. An *i-leaf segment* is either an open or closed i-leaf segment. The *endpoints* of a full i-leaf segment are the points $k_{i,x}(u)$ and $k_{i,x}(v)$ where u and v are the endpoints of I_1. The *endpoints* of an i-leaf segment I are the points of the minimal closed full i-leaf segment containing I. The *interior* of a i-leaf segment I is the complement of its boundary. In particular, a i-leaf segment I has empty interior if, and only if, it is an i-leaf gap. A map $c : I \to \mathbb{R}$ is an *i-leaf chart* of an i-leaf segment I if has an extension $c_E : I_E \to \mathbb{R}$ to a full i-leaf segment I_E with the following properties: $I \subset I_E$ and c_E is a homeomorphism onto its image. An *i-full leaf segment* is either an open or close full leaf segment.

A.2 *Rectangles*

Since L is a hyperbolic invariant set of a diffeomorphism $f : M \to M$, for $0 < \epsilon < \epsilon_0$ there is $d = d(\epsilon) > 0$, such that for all points $w, z \in L$ with $d(w, z) < d$, $W^u(w, \epsilon)$ and $W^s(z, \epsilon)$ intersect in an unique point that we denote by $[w, z]$. Since we assume that the hyperbolic set has a *local product structure*, we have that $[w, z] \in L$. Furthermore, the following properties are satisfied: (i) $[w, z]$ varies continuously with $w, z \in L$; (ii) the bracket map is continuous on a d-uniform neighbourhood of the diagonal in $L \times L$; and (iii) whenever both sides are defined $f([z, w]) = [f(z), f(w)]$. Note that the bracket map does not really depend on d provided it is sufficiently small.

Let us underline that it is a standing hypothesis that all the hyperbolic sets considered here have such a local product structure.

A *rectangle* R is a subset of L which is (i) closed under the bracket i.e. $x, y \in R \Rightarrow [x, y] \in R$, and (ii) proper i.e. is the closure of its interior in L. This definition imposes that a rectangle has always to be proper which is more restrictive than the usual one which only insists on the closure condition.

If γ^s and γ^u are respectively stable and unstable leaf segments intersecting in a single point then we denote by $[\gamma^s, \gamma^u]$ the set consisting of all points of the form $[w, z]$ with $w \in \gamma^s$ and $z \in \gamma^u$. We note that if the stable and unstable leaf segments γ and γ' are closed then the set $[\gamma, \gamma']$ is a rectangle. Conversely in this two-dimensional situations, any rectangle R has a product structure in the following sense: for each $x \in R$ there are closed stable and unstable leaf segments of L, $\gamma^s(x, R) \subset W^s(x)$ and $\gamma^u(x, R) \subset W^u(x)$ such that $R = [\gamma^s(x, R), \gamma^u(x, R)]$. The leaf segments $\gamma^s(x, R)$ and $\gamma^u(x, R)$ are called *stable and unstable spanning leaf segments* for R. For $i \in \{s, u\}$, we denote by $\partial\gamma^i(x, R)$ the set consisting of the endpoints $\gamma^i(x, R)$, and we denote by $\mathrm{int}\,\gamma^i(x, R)$ the set of $\gamma^i(x, R) \setminus \partial\gamma^i(x, R)$. The *interior of R* is given by $\mathrm{int}\,R = [\mathrm{int}\,\gamma^s(x, R), \mathrm{int}\,\gamma^u(x, R)]$, and the *boundary of R* is given by $\partial R = [\partial\gamma^s(x, R), \gamma^u(x, R)] \cup [\gamma^s(x, R), \partial\gamma^u(x, R)]$.

A.3 *Markov partitions*

By Theorem 3.12 in page 79 of Ref. [3] (see also Sinai [37]), a *Markov partition of f* is a collection $\mathcal{R} = \{R_1, \ldots, R_k\}$ of such rectangles such that (i) $L \subset \cup_{i=1}^{k} R_i$; (ii) $R_i \cap R_j = \partial R_i \cap \partial R_j$ for all i and j; (iii) if $x \in \mathrm{int}\,R_i$ and $fx \in \mathrm{int}\,R_j$ then

(a) $f(\gamma^s(x, R_i)) \subset \gamma^s(fx, R_j)$ and $f^{-1}(\gamma^u(fx, R_j)) \subset \gamma^u(x, R_i)$

(b) $f(\gamma^u(x, R_i)) \cap R_j = \gamma^u(fx, R_j)$ and $f^{-1}(\gamma^s(fx, R_j)) \cap R_i = \gamma^s(x, R_i)$.

The last condition means that $f(R_i)$ goes across R_j just once. In fact, it follows from condition (a) providing the rectangles R_j are chosen sufficiently small (see Mañé [15]). The rectangles which make up the Markov partition are called *Markov rectangles*.

We note that there is a Markov partition R of f with the following *disjointness property* (see Bowen [3], Newhouse and Palis [20], Sinai [37]):

(i) if $0 < d_{f,s} < 1$ and $0 < d_{f,u} < 1$ then the stable and unstable leaf boundaries of any two Markov rectangles do not intesect.
(ii) if $0 < d_{f,i} < 1$ and $0 < d_{f,i'} = 1$ then the i'-leaf boundaries of any two Markov rectangles do not intersect except, possibly, at their endpoints.

If $d_{f,s} = d_{f,u} = 1$, the disjointness property does not apply and so we consider that it is trivially satisfied for every Markov partition. For simplicity of our exposition, we considerMarkov partitions that satisfy the disjointness property. This result is also used in [7–9,23,24,31,32].

A.4 Cylinders and gaps

For $i = s$ or u, an *i-leaf primary cylinder of a Markov rectangle R* is a spanning i-leaf segment of R. For $n \geq 1$, an *i-leaf n-cylinder of R* is an i-leaf segment I such that

(i) $f_i^n I$ is an i-leaf primary cylinder of a Markov rectangle M;
(ii) $f_i^n(i'(x, R)) \subset M$ for every $x \in I$.

For $n \geq 2$, an *i-leaf n-gap G of R* is an i-leaf gap $\{x, y\}$ in a Markov rectangle R such that n is the smallest integer such that both leaves $f^{n-1} i'(x, R)$ and $f^{n-1} i'(y, R)$ are contained in i'-boundaries of Markov rectangles; An *i-leaf primary gap G* is the image $f_i G'$ by f_i of an i-leaf two-gap G'.

Let f be a diffeomorphisms with a codimension 1 hyperbolic attractor and p be the projection as constructed in (2). The projection $p(I)$ of a stable leaf n-cylinder I is in the n-level $L_0^{(n)}$ of the partition of $F_{|M|}$ (see definition of $L_0^{(n)}$ in Section 4).

A.5 Basic holonomies

Suppose that x and z are two points inside any rectangle R of L. Let I and J be two stable leaf segments respectively containing x and z and inside R. Then we define $h: I \to J$ by $h(w) = [w, z]$. Such maps are called the *basic stable holonomies*. They generate the pseudo-group of all stable holonomies. Similarly we define the basic unstable holonomies.

A.6 Conjugacies

Let (f, L) be a C^{1+H} hyperbolic diffeomorphism. Somewhat unusually we also desire to highlight the C^{1+H} structure on M in which f is a diffeomorphism. By a *C^{1+H} structure on M* we mean a maximal set of charts with open domains in M such that the union of their domains cover M and whenever U is an open subset contained in the domains of any two of these charts i and j then the overlap map $j \circ i^{-1} : i(U) \to j(U)$ is $C^{1+\alpha}$, where $\alpha > 0$ depends on i, j and U. We note that by compactness of M, given such a C^{1+H} structure on M, there is an atlas consisting of a finite set of these charts which cover M and for which the overlap maps are $C^{1+\alpha}$ compatible and uniformly bounded in the $C^{1+\alpha}$ norm, where $\alpha > 0$ just depends upon the atlas. We denote by $\mathcal{C}_f$ the C^{1+H} structure on M in which f is a diffeomorphism. Usually one is not concerned with this as, given two such structures, there is a homeomorphism of M sending one onto the other and thus, from this point of view, all such structures can be identified. For our discussion, it will be important to maintain the identity of the different smooth structures on M.

We say that a map $h : L_f \to L_g$ is a *topological conjugacy* between two C^{1+H} hyperbolic diffeomorphisms (f, L_f) and (g, L_g) if there is a homeomorphism $h : L_f \to L_g$ with the following properties:

(i) $g \circ h(x) = h \circ f(x)$ for every $x \in L_f$.
(ii) The pull-back of the i-leaf segments of g by h are i-leaf segments of f.

Definition 7.1. *Let* F *be the set of all* C^{1+H} *hyperbolic diffeomorphisms* (g, L_g) *such that* (g, L_g) *and* (f, L) *are topologically conjugate by* h.

A.7 HR-Hölder ratios

A *HR-structure* associates an affine structure to each stable and unstable leaf segment in such a way that these vary Hölder continuously with the leaf and are invariant under f.

An affine structure on a stable or unstable leaf is equivalent to a *ratio function* $r(I : J)$ which can be thought of as prescribing the ratio of the size of two leaf segments I and J in the same stable or unstable leaf. A *ratio function* $r(I : J)$ is positive (we recall that each leaf segment has at least two distinct points) and continuous in the endpoints of I and J. Moreover,

$$r(I : J) = r(J : I)^{-1} \quad \text{and} \quad r(I_1 \cup I_2 : K) = r(I_1 : K) + r(I_2 : K) \tag{13}$$

provided I_1 and I_2 intersect at most in one of their endpoints.

We say that r is a *i-ratio function* if (i) for all i-leaf segments K, $r(I : J)$ $(I; J \subset K)$ defines a ratio function on K; (ii) r is invariant under f, i.e. $r(I : J) = r(fI : fJ)$ for all i-leaf segments; and (iii) for every basic i-holonomy map $u : I \to J$ between the leaf segment I and the leaf segment J defined with respect to a rectangle R and for every i-leaf segment $I_0 \subset I$ and every i-leaf segment or gap $I_1 \subset I$,

$$\left|\log \frac{r(\theta I_0 : \theta I_1)}{r(I_0 : I_1)}\right| \le \mathcal{O}((d_\Lambda(I, J))^{\varepsilon}), \tag{14}$$

where $e \in (0,1)$ depends upon r and the constant of proportionality also depends upon R, but not on the segments considered. Since r satisfies the condition (14) and defines an affine structure on each leaf that is f-invariant we say that r is a tranversely Hölder i-ratio function. A *HR-structure* is a pair (r_s, r_u) consisting of a stable and an unstable ratio function.

Definition 7.2. *If an i-ratio function r is invariant under holonomies h* (*i.e.* $r(I : J) = r(h(I) : h(J))$), *then we say that r is a tranversely affine i-ratio function.*

A.8 Realised ratio functions

Let $g \in$ F and $r = r_g$ be a C^{1+} Riemannian metric on the manifold containing L. The *i- lamination atlas* $A^i(g, r)$ *determined by* r is the set of all maps $e : I \to \mathbb{R}$ where $I = L \cap \hat{I}$ with $\hat{I}$ a full i-leaf segment, such that e extends to an isometry between the induced

Riemannian metric on $\hat{I}$ and the Euclidean metric on the reals. We call the maps $e \in A(g; r)$ the *i-lamination charts*. If I is an i-leaf segment (or a full i-leaf segment) then by $|I| = |I|_r$ we mean the length in the Riemannian metric r of the minimal full i-leaf containing I. By hyperbolicity of g in L, there are $0 < \nu < 1$ and $C > 0$ such that for all i- leaf segments I and all $m \geq 0$ we get $|g^m I| \leq C\nu^m |I|$. Thus, using the mean value theorem and the fact that g_i is C^r, for all short leaf segments K and all leaf segments I and J contained in it, the i-realised ratio function $r_{g,i}$ given by

$$
\begin{aligned}
r_{g,\iota}(I:J) &= \lim_{n\to\infty} \frac{|g_\iota^n I|}{|g_\iota^n J|} \\
&= \frac{|g_\iota^m I|}{|g_\iota^m J|} \prod_{n=m}^{\infty} \left(\frac{|g_\iota^{n+1} I|}{|g_\iota^{n+1} J|} \frac{|g_\iota^n J|}{|g_\iota^n I|} \right) \\
&\in \frac{|g_\iota^m I|}{|g_\iota^m J|} \prod_{n=m}^{\infty} (1 \pm \mathcal{O}(\nu^n |K|^\alpha)) \\
&\subset \frac{|g_\iota^m I|}{|g_\iota^m J|} (1 \pm \mathcal{O}(\nu^n |K|^\alpha))
\end{aligned}
$$

is well-defined, where $\alpha = \min\{1, r - 1\}$. This construction gives the HR-structure on L determined by g. By Ref. [26], we get the following equivalence:

Theorem 3. *The map* $g \to (r_{g;s}; r_{g;u})$ *determines a one-to-one correspondence between* C^{1+H} *conjugacy classes in* F *and HR-structures.*

A.9 Lamination atlas

Given an i-ratio function r, we define the embeddings $e: I \to \mathbb{R}$ by

$$
e(x) = r(\ell(\xi, x), \ell(\xi, R)), \tag{15}
$$

where ξ is an endpoint of the i-leaf segment I and R is a Markov rectangle containing ξ (see Figure 3). For this definition it is not necessary that R contains I. We denote the set of all these embeddings e by $A(r)$.

Let $g \in F$ and $A(g, r)$ the i-lamination atlas determined by a Riemmanian metric r. Putting together Propositions 2.5 and 3.5 of Ref. [26], we get that the overlap map $e_1 \circ e_2^{-1}$ between a chart $e_1 \in A(g, r)$ and a chart $e_2 \in A(r_{g;i})$ has a C^{1+H} diffeomorphic extension to the reals. Therefore, for all short leaf segments K and all leaf segments I and J contained

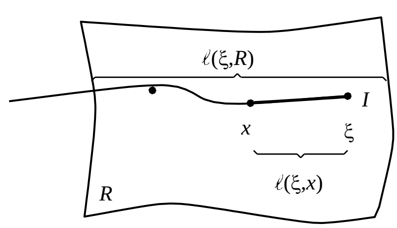


Figure 3. The embedding $e : I \to R$.

in it, we obtain that

$$r_{g,\iota}(I:J) = \lim_{n\to\infty} \frac{|g_\iota^n I|_\rho}{|g_\iota^n J|_\rho} = \lim_{n\to\infty} \frac{|g_\iota^n I|_{i_n}}{|g_\iota^n J|_{i_n}},$$

where i_n is any chart in $\mathcal{A}(r_{g,\iota})$ containing the segment $g_\iota^n K$ in its domain.

A.10 Realised solenoid functions

For $\iota = s$ and u, let $\mathbf{S}^\iota$ denote the set of all ordered pairs (I, J) of ι-leaf segments with the following properties:

(i) The intersection of I and J consists of a single endpoint.
(ii) if $d_{f,\iota} = 1$ then I and J are primary ι-leaf cylinders.
(iii) if $0 < d_{f,\iota} < 1$ then $f_\iota I$ is an ι-leaf two-cylinder of a Markov rectangle R and $f_\iota J$ is an ι-leaf two-gap also of the same Markov rectangle R.

See Section A.4 for the definitions of leaf cylinders and gaps. Pairs (I, J) where both are primary cylinders are called *leaf-leaf pairs*. Pairs (I, J) where J is a gap are called *leaf-gap pairs* and in this case we refer to J as a *primary gap*. The set $\mathbf{S}^\iota$ has a very nice topological structure. If $d_{f;\iota} = 1$ then the set $\mathbf{S}^\iota$ is isomorphic to a finite union of intervals, and if $d_{f;\iota} < 1$ then the set $\mathbf{S}^\iota$ is isomorphic to an embedded arc set.

We define a pseudo-metric $d_{\mathbf{S}^\iota} : \mathbf{S}^\iota \times \mathbf{S}^\iota \to \mathbb{R}^+$ on the set $\mathbf{S}^\iota$ by

$$d_{\mathbf{S}^\iota}((I,J),(I',J')) = \max\{d_\Lambda(I,I'), d_\Lambda(J,J')\}.$$

Let $g \in \mathcal{T}(f, L)$. For $\iota = s$ and u, we call the restriction of an ι-ratio function $r_{g;\iota}$ to $\mathbf{S}^\iota$ a *realised solenoid function* $s_{g,\iota}$. By construction, for $\iota = s$ and u, the restriction of an ι- ratio function to $\mathbf{S}^\iota$ gives an Hölder continuous function satisfying the matching condition, the boundary condition and the cylinder-gap condition as we now proceeed to describe.

A.11 Hölder continuity of solenoid functions

This means that for $t = (I, J)$ and $t' = (I', J')$ in $\mathbf{S}^\iota$, $|s_\iota(t) - s_\iota(t')| \le O(d_{\mathbf{S}^\iota}(t, t')^\alpha)$. The Hölder continuity of $s_{g,\iota}$ and the compactness of its domain imply that $s_{g,\iota}$ is bounded away from zero and infinity.

A.12 Matching condition

Let $(I; J) \in \mathbf{S}^\iota$ be a pair of primary cylinders and suppose that we have pairs

$$(I_0, I_1), (I_1, I_2), \ldots, (I_{n-2}, I_{n-1}) \in \mathbf{S}^\iota$$

of primary cylinders such that $f_\iota I = \cup_{j=0}^{k-1} I_j$ and $f_\iota J = \cup_{j=k}^{n-1} I_j$. Then

$$\frac{|f_\iota I|}{|f_\iota J|} = \frac{\sum_{j=0}^{k-1} |I_j|}{\sum_{j=k}^{n-1} |I_j|} = \frac{1 + \sum_{j=1}^{k-1} \prod_{i=1}^{j} |I_i|/|I_{i-1}|}{\sum_{j=k}^{n-1} \prod_{i=1}^{j} |I_i|/|I_{i-1}|}.$$

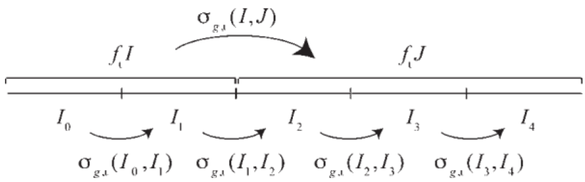


Figure 4. The *f*-matching condition for *i*-leaf segments.

Hence, noting that *g*j*L* ¼ *f* j*L*, the realised solenoid function $s_{g,i}$ must satisfy the *matching condition* (see Figure 4) for all such leaf segments:

$$\sigma_{g,\iota}(I:J) = \frac{1+\sum_{j=1}^{k-1}\prod_{i=1}^{j}\sigma_{g,\iota}(I_i:I_{i-1})}{\sum_{j=k}^{n-1}\prod_{i=1}^{j}\sigma_{g,\iota}(I_i:I_{i-1})}. \tag{16}$$

A.13 Boundary condition

If the stable and unstable leaf segments have Hausdorff dimension equal to 1, then leaf segments *I* in the boundaries of Markov rectangles can sometimes be written as the union of primary cylinders in more than one way. This gives rise to the existence of a boundary condition that the realised solenoid functions have to satisfy as we pass to explain.

If *J* is another leaf segment adjacent to the leaf segment *I* then the value of j*I*j/j*J*j must

be the same whichever decomposition we use. If we write *J* ¼ I_0 ¼ K_0 and *I* as$_{i¼1}$ I_i and ⋞^m_n

$_{j¼1}K_j$ where the I_i and K_j are primary cylinders with I_i – K_j for all *i* and *j*, then the above two ratios are

$$\sum_{i=1}^{m}\prod_{j=1}^{i}\frac{|I_j|}{|I_{j-1}|} = \frac{|I|}{|J|} = \sum_{i=1}^{n}\prod_{j=1}^{i}\frac{|K_j|}{|K_{j-1}|}.$$

Thus, noting that *g*j*L* ¼ *f* j*L*, a realised solenoid function $s_{g,i}$ must satisfy the following *boundary condition* (see Figure 5) for all such leaf segments:

X Y

$$\sum_{i=1}^{m}\prod_{j=1}^{i}\sigma_{g,\iota}(I_j:I_{j-1}) = \sum_{i=1}^{n}\prod_{j=1}^{i}\sigma_{g,\iota}(K_j:K_{j-1}). \tag{17}$$

A.14 Scaling function

If the *i*-leaf segments have Hausdorff dimension less than one and the *i*^{0}-leaf segments have Hausdorff dimension equal to 1, then a primary cylinder *I* in the *i*-boundary of a

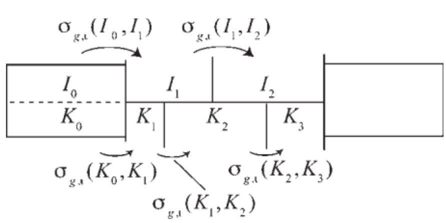


Figure 5. The boundary condition for *i*-leaf segments.

Markov rectangle can also be written as the union of gaps and cylinders of other Markov rectangles. This gives rise to the existence of a cylinder-gap condition that the i-realised solenoid functions have to satisfy.

Before defining the cylinder-gap condition, we will introduce the scaling function that
will be useful to express the cylinder-gap condition.

Let scl^i be the set of all pairs (K, J) of i-leaf segments with the following properties:

(i) K is a leaf n_1-cylinder or an n_1-gap segment for some n_1 - 1;
(ii) J is a leaf n_2-cylinder or an n_2-gap segment for some n_2 - 1;
(iii) $m^{n_1 21}K$ and $m^{n_2 21}J$ are the same primary cylinder.

Lemma 6. *Every fuction $s_i : S^i$! R^b has a canonical extension s_i to scl^i. Furthermore, if s_i is the restriction of a ratio function r_ijS^i to S^i then s_i ¼ r_ijscl^i.*

See proof of Lemma 6 in Ref. [28].

Remark 3. The above map $s_i : scl^i$! R^b is the *scaling function* determined by the solenoid function $s_i : S^i$! R^b.

A.15 Cylinder-gap condition

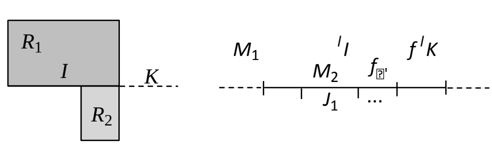
Let (I,K) be a leaf-gap pair such that the primary cylinder I is the i-boundary of a Markov rectangle R_1. Then the primary cylinder I intersects another Markov rectangle R_2 giving rise to the existence of a cylinder-gap condition that the realised solenoid functions have to satisfy as we proceed to explain. Take the smallest l $ 0 such that $f^l\, I < f^l\, K$ is contained
in the intersection of the boundaries of two Markov rectangles M_1 and M_2. Let M_1 be the Markov rectangle with the property that $M_1 > f^l\, R_1$ is a rectangle with non-empty interior (and so $M_2 > f^l\, R_2$ also has non-empty interior). Then, for some positive n, there are distinct n-cylinder and gap leaf segments $J_1; \ldots ; J_m$ contained in a primary cylinder of M_2
such that $f^l\, K$ ¼ J_m and the smallest full i-leaf segment containing $f^l\, I$ is equal to the union $<_{i¼1}^{m21} \mathcal{J}_i$, where $\mathcal{J}_i$ is the smallest full i-leaf segment containing J_i. Hence,

$$\frac{|f_\iota^l I|}{|f_\iota^l K|} = \sum_{i=1}^{m-1} \frac{|J_i|}{|J_m|}.$$

Hence, noting that gjL ¼ fjL, a realised solenoid function $s_{g;i}$ must satisfy the *cylinder- gap condition* (see Figure 6) for all such leaf segments:

$$\sigma_{g,\iota}(I, K) = \sum_{i=1}^{m-1} s_{g,\iota}(J_i, J_m),$$

where $s_{g;i}$ is the scaling function determined by the solenoid function $s_{g;i}$.



?' J_m ?1 J_m

Figure 6. The cylinder-gap condition for *i*-leaf segments.

A.16 Solenoid functions

Now, we are ready to present the definition of an i-solenoid function.

Definition 7.3. *An Hölder continuous function $s_i : \mathrm{S}^i \to \mathbb{R}^+$ is an i-solenoid function if s_i satisfies the matching condition, the boundary condition and the cylinder-gap condition.*

We denote by $\mathrm{PS}(f)$ the set of pairs (s_s, s_u) of stable and unstable solenoid functions.

Remark 4. Let $s_i : \mathrm{S}^i \to \mathbb{R}^+$ be an i-solenoid function. The matching, the boundary and the cylinder-gap conditions are trivially satisfied except in the following cases:

(i) The matching condition if $d_{f,i} = 1$.
(ii) The boundary condition if $d_{f;s} = d_{f;u} = 1$.
(iii) The cylinder-gap condition if $d_{f;i} \geq 1$ and $d_{f;i'} = 1$.

Theorem 4. *The map $r_i \to r_i|\mathrm{S}^i$ gives a one-to-one correspondence between i-ratio functions and i-solenoid functions.*

See proof of Theorem 4 in Ref. [28].

The set $\mathrm{PS}(f)$ of all pairs (s_s, s_u) has a natural metric. Combining Theorem 3 with Theorem 4, we obtain that the set $\mathrm{PS}(f)$ forms a moduli space for the C^{1+H} conjugacy classes of C^{1+H} hyperbolic diffeomorphisms $g \in \mathrm{T}(f, L)$.

Corollary 1. *The map $g \to (r_{g;s}|\mathrm{S}^s; r_{g;u}|\mathrm{S}^u)$ determines a one-to-one correspondence between C^{1+H} conjugacy classes of $g \in \mathrm{T}(f, L)$ and pairs of solenoid functions in $\mathrm{PS}(f)$.*