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IMPLEMENTING AN ELECTRICAL FRACTIONAL POTENTIAL THROUGH A GENETIC ALGORITHM

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Several phenomena present in electrical systems, motivated the development of comprehensive models based on the theory of fractional calculus (FC). Bearing these ideas in mind, in this work, are applied the FC concepts to define, and to evaluate, the electrical potential of fractional order based in a genetic algorithm optimization scheme. Research on the approximation feasibility and convergence is also developed.

Keywords: electrical potential, fractional order systems, electromagnetism, multipoles, genetic algorithms, optimization.

1. Introduction

A new look of several phenomena present in electrical systems,[1] induced an approach based in the fractional calculus (FC) viewpoint. Some authors[2,3] verified that well-known expressions for the electrical potential are related through integer-order integral and derivatives and have proposed its generalization based on the concept of fractional-order poles. Nevertheless, the mathematical generalization towards FC lacks a comprehensive method for its practical implementation.

This article addresses the synthesis of fractional-order multipoles. In section 2 we recall the classical expressions for the static electric potential and we analyze them in the perspective of FC. Based on this re-evaluation we develop a GA scheme for implementing fractional-order electrical potential approximations. Finally, in section 3 we outline the main conclusions.

2. Integer and Fractional Electrical Potential

For a homogeneous, linear and isotropic media, the electric potential φ at a point P produced by a single charge (1a), a dipole (1b), a quadrupole (1c),

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an infinite straight filament carrying a charge λ per unit length (2a), two opposite charged filaments (2b), and a planar surface with charge density σ (3), are given by:[4]

$$\varphi = \frac{q}{4\pi\varepsilon_0}\frac{1}{r} + C \tag{1a}$$

$$\varphi = \frac{ql\cos\theta}{4\pi\varepsilon_0}\frac{1}{r^2} + C,\ r >> l \tag{1b}$$

$$\varphi = \frac{ql^2\left(3\cos^2\theta - 1\right)}{4\pi\varepsilon_0}\frac{1}{r^3} + C,\ r >> l \tag{1c}$$

$$\varphi = -\frac{\lambda}{2\pi\varepsilon_0}\ln r + C, \tag{2a}$$

$$\varphi = \frac{\lambda l\cos\theta}{2\pi\varepsilon_0}\frac{1}{r} + C,\ r >> l \tag{2b}$$

$$\varphi = -\frac{\sigma}{2\varepsilon_0}r + C, \tag{3}$$

where $C \in \Re$, ε_0 represents the permittivity, q the electric charge, r the radial distance and θ the corresponding angle with the axis.

Analyzing expressions (1)−(3) we verify the relationship $\varphi \sim r^{-3}, r^{-2}$, $r^{-1}, \ln r, r$, corresponds to the application of integer-order derivatives and integrals.

The integer-order differential nature of the potential expressions (1-3) motivated several authors[3] to propose its generalization in a FC perspective. Therefore, a fractional multipole produces at point P a potential $\varphi \sim r^{\alpha},\ \alpha \in \Re$. Nevertheless, besides the abstract manipulation of mathematical expressions, the truth is that there is no practical method, and physical interpretation, for establishing the fractional potential.[2,3,5–7]

Inspired by the integer-order recursive approximation of fractional-order transfer functions,[8,9] we adopt a genetic algorithm (GA)[10–12] for implementing a fractional order potential. Similarly to what occur with transfer function, the electrical integer-order potential has a *global* nature and fractional-order potentials can have only a *local* nature, that is, possible to capture only in a restricted region. This observation leads to an implementation approach conceptually similar to the one described in[5,8,9,13] that is,

to an approximation scheme based on a recursive placement of integer-order functions.

In this line of thought, we develop a one-dimensional GA that places n charges at the positions x_i and determines the corresponding values q_i. Our goal is to compare the approximate potential φ_{app} given by:

$$\varphi_{app} = \sum_{i=1}^{n} \frac{q_i}{4\pi\varepsilon_0 \left| x - x_i \right|} \tag{5}$$

that mimics the desired reference potential $\varphi_{ref} = kx^{\alpha}$ in a given interval $x_{mim} < x < x_{max}$.

It is important to refer that a reliable execution and analysis of a GA usually requires a large number of simulations to provide that stochastic effects have been properly considered. Therefore, in this study the experiments consist on executing the GA several times, in order to generate a combination of positions and charges that lead to an electrical potential with fractional slope similar to the desire reference potential. In the first case of study, the values of GA parameters are: population number P = 40, crossover C(%) = 85.0%, mutation M(%) = 1.0% and an elitist strategy ES(%) = 10.0%. The chromosome has $2n$ genes: the first n genes correspond to the charges and the last n genes indicate their positions. The gene codifications adopts a Gray Code with a string length of $l = 16$ bits. The optimization fitness function corresponds to the minimization of the index:

$$J = \sum_{k=1}^{m} \left(\ln \left| \frac{\varphi_{app}}{\varphi_{ref}} \right| \right)^2, \quad \min_{i} (J), \quad i = 0,\ 1, \ldots,\ n-1 \tag{6}$$

where m is the number of sampling points along the interval $x_{mim} < x < x_{max}$. We establish a maximum number of iterations $I_{Max} = 100$ and a stoping scheme when $J < 10^{-10}$ for the best individual (*i.e.*, solution) of the GA population.

Figure 1a) shows a pre-defined number of $n = 5$ charge approximation and $\varphi_{ref} = 1.0\ x^{-1.5}$, $0.2 < x < 0.8$, leading to $\{q_1, q_2, q_3, q_4, q_5\}$=$\{0.737,$ $0.846,\ -0.777,\ 0.382,\ -0.225\}$ [C] (with scale factor $\times(4\pi\varepsilon_0)^{-1}$), located at $\{x_1, x_2, x_3, x_4, x_5\}$ =$\{-0.06, 0.092, 0.147, -0.106, 0.117\}$ [m], respectively. In this case, the GA needs $I = 51$ iterations to satisfy the adopted fitness function stoping threshold.

The results show a good fit between φ_{ref} and φ_{app} and we verify that it is possible to find more than one 'good' solution (Figure 1b). Nevertheless, for a given application, a superior precision may be required and, in that

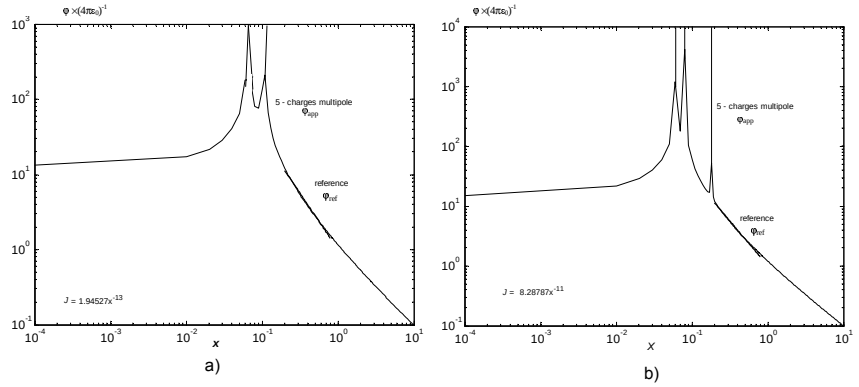

Fig. 1. Comparison of the electric potential φ_{app} and φ_{ref} *versus* the position x for $\varphi_{ref} = 1.0\ x^{-1.5}$ [volt], $0.2 < x < 0.8$ [m], and a $n = 5$ charge approximation, in both cases.

case, a larger number of charges must be used. In this line of thought, we study the performance of this method for different number of charges, namely from $n = 1$ up to $n = 10$ charges, and we compare the necessary number of GA iterations when the number of charges increases. In order to analyze the precision of this distribution of charges, we study the require number of iterations I and the computational time T when the number of charges varies from $n = \{1, \ldots, 10\}$.

Figure 2 shows the values of the charges q_i and the corresponding positions x_i, for $n = \{1, \ldots, 10\}$. We verify that the value of the charge and the location pattern *versus* the number of charges is not clear.

Figure 3*a*) depict the minimum, average and maximum of the number of required GA iterations I *versus* n. This chart reveals clearly that the required number of iterations increases with n.

We can also evaluate the GA computational time T for different number of charges. Therefore, we test the GA scheme for identical parameters and fitness function J (6). Figure 4*b*) illustrates the corresponding minimum, average and maximum of T *versus* n. We verify that we get a smaller approximation error J but a larger computational time T for larger values of n.

With the proposed method it is also possible to have a reference potential with other slope values α. Figure 5 shows a five charge approximation for $0.2 < x < 0.8$ and $\varphi_{ref} = 1.0\ x^{-1.3}$, leading to $\{q_1, q_2, q_3, q_4, q_5\} = \{0.471, 0.464, 0.578, -0.371, -0.173\}$ [C] (with scale factor $\times(4\pi\varepsilon_0)^{-1}$), lo-

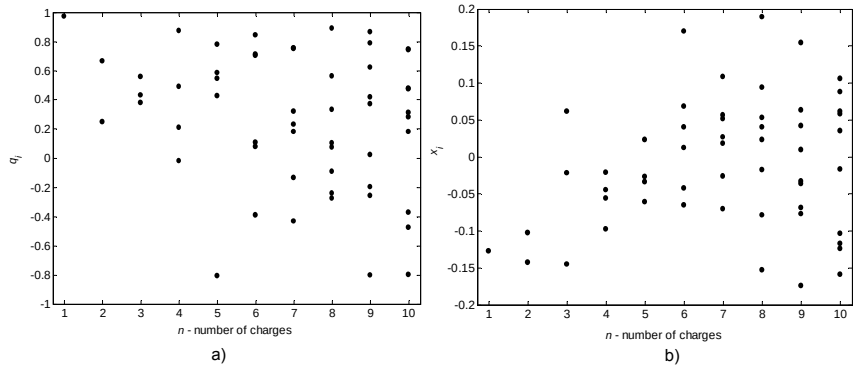

Fig. 2. Values of a) charges q_i and the b) corresponding positions x_i *versus* n, for a distribution of charges with $n = \{1, \ldots, 10\}$, $\varphi_{ref} = 1.0\ x^{-1.5}$ [volt], $0.2 < x < 0.8$ [m].

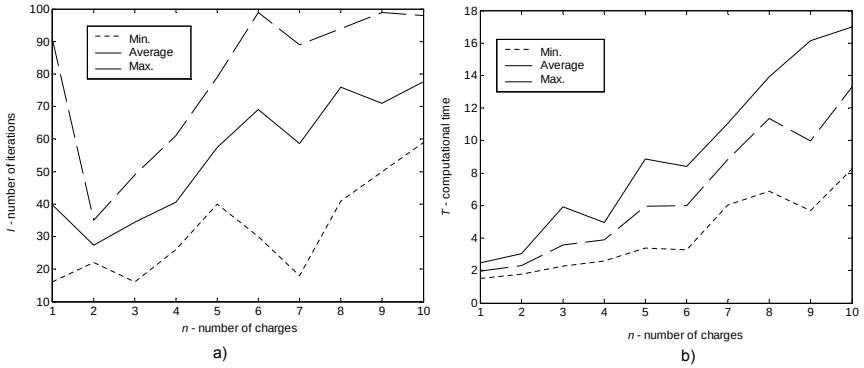

Fig. 3. Performance of the GA scheme *versus* the number charges $n = \{1, \ldots, 10\}$ for $\varphi_{ref} = 1.0\ x^{-1.5}$ [volt], $0.2 < x < 0.8$ [m], *a*) number of required iterations I, *b*) computational time T.

cated at $\{x_1, x_2, x_3, x_4, x_5\} = \{-0.125, 0.029,\ 0.037, 0.132, 0.152\}$ [m] and for $\varphi_{ref} = 1.0\ x^{-1.7}$, $0.2 < x < 0.8$, leading to $\{q_1, q_2, q_3, q_4, q_5\} = \{0.753, 0.535, 0.429, -0.218, -0.681\}$ [C] (with scale factor $\times(4\pi\varepsilon_0)^{-1}$), located at $\{x_1, x_2, x_3, x_4, x_5\} = \{-0.157, -0.070, 0.171, 0.188, 0.200\}$ [m], respectively.

The charges are also function of the slope α and, therefore, we apply the GA with identical parameters, for $0.2 < x < 0.8$ [m] while varying α. Figure 5 depicts q_i and x_i *versus* α, without revealing again any clear pattern. On the other hand, the number of iterations I, the GA computational time T

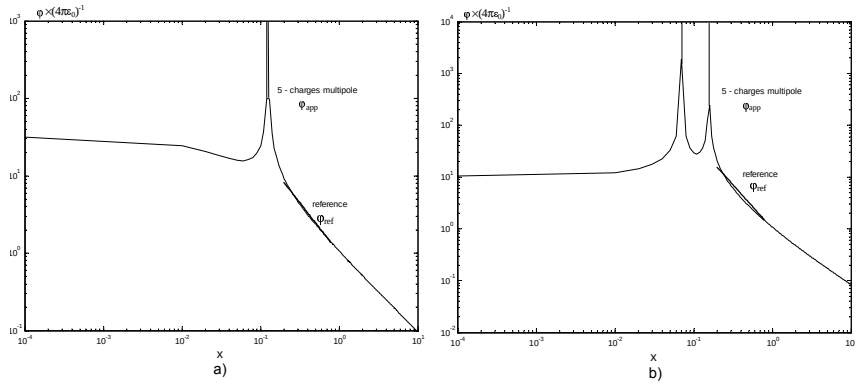

Fig. 4. Comparison of the electrical potential φ_{app} and φ_{ref} *versus* the position x for a) $\varphi_{ref} = 1.0\ x^{-1.3}$ [volt] and b) $\varphi_{ref} = 1.0\ x^{-1.7}$ [volt], $0.2 < x < 0.8$ [m] and a $n = 5$ charge approximation.

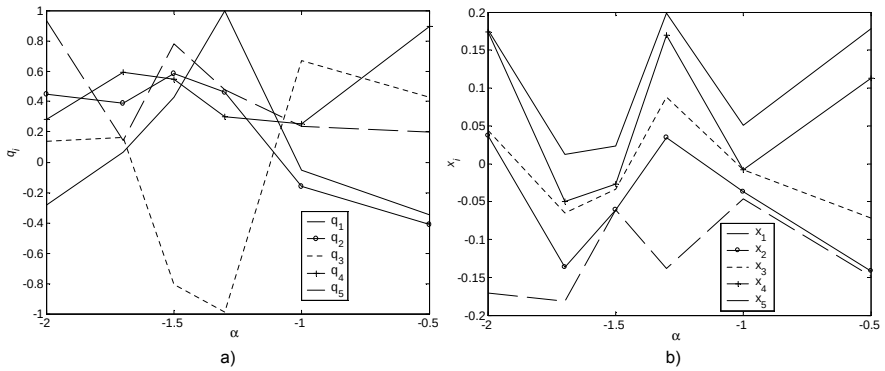

Fig. 5. Values of a) charges q_i and the b) corresponding positions x_i *versus* α, for a $n = 5$ distribution of charges and for $0.2 < x < 0.8$ [m].

and the error J *versus* α, reveal a smooth evolution. Figure 6 illustrates the corresponding minimum, average and maximum of I, T and J as function of α.

In conclusion, the fit between φ_{app} and φ_{ref} is adequate and there is no obvious pattern for the charge distribution as n increases.

This lack of 'order' is due to the large number of possible solutions. Therefore, the GA has a high freedom, choosing solutions that are almost not correlated. However, we believe that further study imposing more strict restrictions may lead to the emergence of a comprehensive scheme.

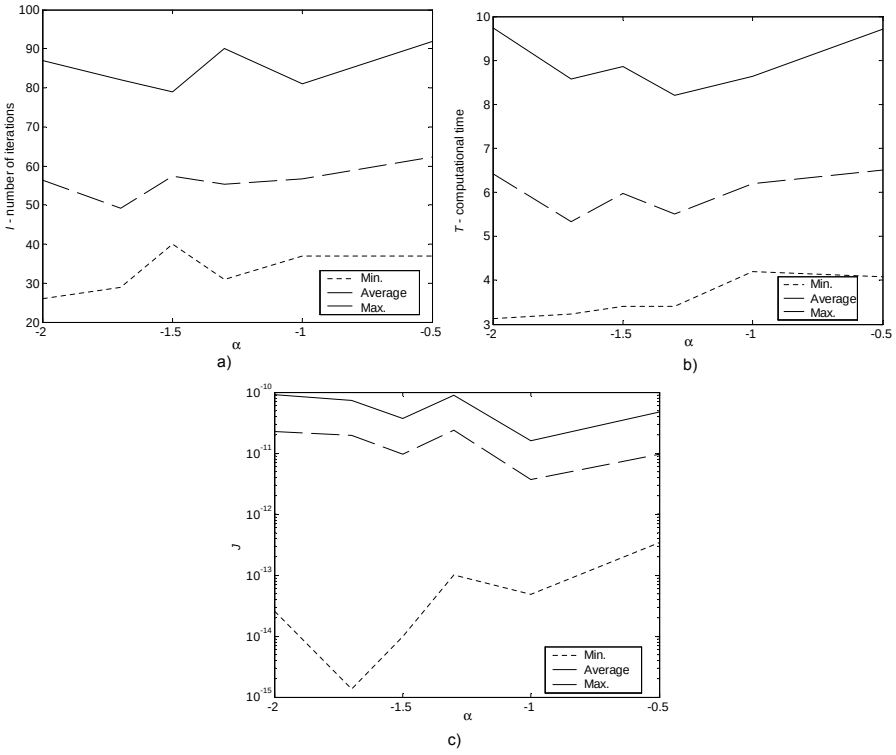

Fig. 6. Performance of the GA scheme *versus* α for $0.2 < x < 0.8$ [m], *a*) number of required iterations I, *b*) computational time T, *c*) error J for $n = 5$.

3. Conclusions

This paper addressed the problem of implementing a fractional-order electric potential through a genetic algorithm. The results reveal the necessity of a larger number of iterations when the number of charges increases. The GA reveals a good compromise between the accuracy and computational time. The GA approach constitutes a step towards the development of a simple design technique and, consequently, several of its aspects must be further evaluated.

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