Analysis of financial data series using fractional Fourier transform and multidimensionalscaling

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Abstract The goal of this study is the analysis of the dynamical properties of financial data series from worldwide stock market indexes during the period 2000–2009. We analyze, under a regional criterium, ten main indexes at a daily time horizon. The meth- ods and algorithms that have been explored for the description of dynamical phenomena become an ef- fective background in the analysis of economical data. We start by applying the classical concepts of signal analysis, fractional Fourier transform, and methods of fractional calculus. In a second phase we adopt the multidimensional scaling approach. Stock market in- dexes are examples of complex interacting systems for which a huge amount of data exists. Therefore, these indexes, viewed from a different perspectives, lead to new classification patterns.

Keywords Financial data series · Fractional Fourier transform · Multidimensional scaling · Fractional calculus

1 Introduction

The study of fractional systems has received consider- able attention due to the fact that many physical sys- tems are well characterized by fractional models. The importance of fractional order modeling is that it can be used to make a more accurate description and it can give a deeper insight into the processes underly- ing long range memory behavior [14, 21, 26]. It seems that there are many distinct analogies between the dy- namics of complex physical and economical or even social systems. The methods and algorithms that have been explored for description of physical phenomena become an effective background and inspiration for very productive methods used in the analysis of eco- nomical data [10, 24, 29, 31].

In this paper we study several national indexes at a daily time horizon at the closing and the continuously compounded return.

Indexes are used to measure the performance of segments of the stock market. They are normally used to benchmark the performance of individual and stock portfolios. There are several possible classifications of stock market.

- A global stock market index includes companies without regard for where they are domiciled or traded (e.g., S&P Global 100).
- A regional index represents the performance of the stock market of defined world regions (e.g., Eu- ronext 100).
- A national index represents the performance of the stock market of a given country (e.g., DAX).

Another is classification by **investment strategy**:

- A value index contains stocks which appear to be underpriced by some form of fundamental analysis (e.g., trade at discount to book value or have low price-to-earnings ratios) (e.g., Russell 1000 Value).
- A growth index contains stocks that exhibit signs of above-average growth. (e.g., S&P 500 Pure Growth).

Some also classify stock market indexes by the **company capitalization**. There is no official definition of either categories or their exact cutoff, but a rule of thumb may look like:

- Large-cap indexes include companies with a market capitalization value of more than $10 billion (e.g., Russell Top 50).
- Mid-cap indexes include companies with a market capitalization between $2 and $10 billion (e.g., S&P 400 MidCaps).
- Small-cap indexes include companies with a market capitalization below $2 billion (e.g., Dow Jones U.S. Small-Cap).

There are also indexes that track the performance of specified sectors or group of sectors of the market. This can be done at varying scale levels, ranging from an **industry level to a subsector level** as for example:

- Utilities (industry level) (e.g., Dow Jones Utilities).
- Chemicals (supersector level) (e.g., PHLX Chemicals Sector).
- Biotechnology (subsector level) (e.g., NASDAQ OMX Global Biotechnology Index).

Indexes can also be classified according to the **method used to determine its price**:

- In a price-weighted index the price of each compo- nent stock is the only consideration when determin- ing the value of the index (e.g., Dow Jones Indus- trial Average). Thus, price movement of even a sin- gle security will heavily influence the value of the index, ignoring the relative size of the company.

- A market-value weighted index factors the size of the company (e.g., Hang Seng). Thus, a relatively small shift in the price of a large company will heav- ily influence the value of the index.

There is a variety of indexes classifications and types. Indexes can track almost anything—for in- stance, there is even an index published by the Linux Weekly News that tracks stocks of companies that sell products based on the Linux operating environment.

This paper uses a regional classification and an- alyzes the next ten national indexes arranged by re- gions:

- **Americas** region:
  - **S&P 500**—includes 500 large-cap common stocks actively traded in the United States.
  - **DJI**—The Dow Jones Industrial Average is an index that shows how 30 large, publicly-owned companies based in the United States have traded during a standard trading session in the stock market.
  - **NYA**—The NYSE Composite is a stock market index covering all common stock listed on the New York Stock Exchange.
- **European** region:
  - **DAX**—measures the performance of the 30 larg- est German companies in terms of order book volume and market capitalization.
  - **CAC 40**—represents a measure of the 40 most significant values among the 100 highest market caps on the Paris Bourse.
- **Asian/Pacific** region:
  - **Nikkei 225**—index that represents the Tokyo Stock Exchange and it is the most widely quoted average of Japanese equities.
  - **SSEC**—is an index of all stocks that are traded at the Shanghai Stock Exchange.
  - **HSI**—is a freefloat-adjusted market capitaliza- tion-weighted stock market index in Hong Kong.
  - **KS11**—is the index of all common stocks traded on the Stock Market index of South Korea.
- **African** region:
  - **CCSI**—Egypt's Stock Exchange, formerly known as Cairo and Alexandria Stock Exchange (CASE), comprises two exchanges, Cairo and Alexandria, both of which are governed by the same board of directors and share the same trad- ing, clearing and settlement systems.

In this study, our particular interest consists in the application of classical tools of signal analysis, frac- tional Fourier transform (FrFT) and Fractional Calcu- lus to reveal the stock indexes proprieties. The remain- der of this paper is as follows. In Sect. 2, we present briefly the fundamental concepts underlying the FrFT and the Multidimensional Scaling (MDS). In Sect. 3 we discuss the dynamical analysis and we present the results of the application of FrFT and MDS to the stock signals. Finally, in Sect. 4, we draw the main conclusions, and we address perspectives toward fu- ture developments.

2 Fundamental concepts

In this section we present a review of fundamental con- cepts involved in the experiments.

2.1 Fractional Fourier transform and power law approximation

The FrFT is a generalization of the ordinary Fourier transform with an order parameter α. Mathematically, the αth order fractional Fourier transform (FT) FrFT^{α} is the αth power of the ordinary FT operation.

With the development of the FrFT and related con- cepts, we see that the ordinary frequency domain is

merely a special case of a continuum of fractional Fourier domains, which are intimately related to time-frequency (or space-frequency) representations. Every property and application of the ordinary FT becomes a special case of the FrFT. In all areas where FTs and frequency-domain concepts are used, there exists the potential for generalization and improvement by using the FrFT.

The FrFT has been found to play an important role in the study of optical systems known as Fourier op- tics, with applications in optical information process- as 1929 [2]. Later on it was used in quantum mechan- ics and signal processing [9, 25], but it was mainly the optical interpretation and the applications in optics that gave a burst of publications since the nineties that culminated in the book of Ozaktas et al. [11].

In [11] are presented several definitions of the FrFT. All of them were suggested to be used in different con- texts like the voice, images or signal processing and work well with the FC models, but there is no known direct connection between these definitions and the FC. The answer to the question what definition to use depends mainly on the problem we are dealing with. There is not one best definition of the FrFT, and one should rather try to take the suitable one while model- ing a process or considering a mathematical problem [13, 15, 30].

The FT of a function can be considered as a linear differential operator acting on that function, while the FrFT generalizes this differential operator by letting it depend on a continuous parameter α. Mathematically, the αth order FrFT is the αth power of FT operator.

Several FrFT definitions are found in the litera- ture, which converge to the original definition. Among them the most commonly used the αth order fractional Fourier transform of a function $s(t)$ is a

$$\mathrm{FrFT}^{\alpha}\left(s(u)\right)=\int_{-\infty}^{\infty}K_{\alpha}(u,t)s(t)\,dt \tag{1}$$

linear opera- tion defined by

where α indicates the rotation angle in the time-frequency plane, u is the frequency domain, $K_\alpha(u, t)$ is the kernel function shown in the equation:

$$K_\alpha(u,t)=\begin{cases}\sqrt{\frac{1-i\cot(\alpha)}{2\pi}}\times e^{i\left(\frac{t^2+u^2}{2}\cot(\alpha)-\csc(\alpha)ut\right)}, & \alpha\neq n\pi\\ \delta(t-u) & \alpha=2n\pi\\ \delta(t+u) & \alpha=2n\pi\pm\pi\end{cases} \tag{2}$$

where $n\in\mathbb{Z}$ and $i=\sqrt{-1}$ [17].

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ing, allowing a reformulation of this area in a much more general way. It has also generalized the notion of the frequency domain and extended our understanding of the time-frequency plane, two central concepts in signal analysis and signal processing. FrFT is expected to have an impact in the form of deeper understanding or new applications in every area in which the FT plays a significant role, and to take its place among the stan- dard mathematical tools of physics and engineering.

FrFT in the form of fractional powers of the Fourier operator appears in the mathematical literature as early

The αth order transform is sometimes referred to as the αth order transform, a practice which will occa- sionally be found convenient when no confusion can arise.

The FrFT has the following special cases:

$$\mathrm{FrFT}^{2n}\ s(u) = s(u) \quad (3)$$

$$\mathrm{FrFT}^{2n\pi+\frac{\pi}{2}}\ s(u) = \mathrm{FT}\ s(u) \tag{4}$$

$$\mathrm{FrFT}^{2n\pi\pm\pi}\ s(u) = s(-u) \tag{5}$$

$$\mathrm{FrFT}^{2n\pi-\frac{\pi}{2}}\ s(u) = \mathrm{FT}\ s(-u) \tag{6}$$

The time domain is the FrFT domain with $\alpha = 2n\pi$, while the frequency domain is the FrFT domain with $\alpha = 2n\pi + \frac{\pi}{2}$. Since the FrFT is periodic, with period 2π, α can be limited in the interval $[-\pi, \pi]$ [25].

Were proposed several algorithms for the discrete implementation of the FrFT [17, 22, 25]. Nevertheless, since calculation speed is not the main issue in the study, the authors decided to program directly the continuous version defined by (1)–(2).

2.2 Multidimensional scaling representation of complex data

MDS is an approach to multivariate analysis aimed at producing a spatial or geometrical representation of complex data. It has its origins in psychometrics, where it was proposed to help understand people's judgments about the similarity between elements of a set of objects [3]. However, it has become a gen- eral data-analysis technique used in a wide variety of fields such as marketing, sociology, physics, political science, biology and biomedical [6, 7, 20, 27] and re- cently in wireless network sensors [16, 19, 28].

MDS is a generic term that includes many different specific types. These types can be classified according to whether the similarities data are qualitative (called nonmetric MDS) or quantitative (metric MDS). The number of similarity matrices and the nature of the MDS model can also classify MDS types. This classification yields classical MDS (one matrix, un- weighted model), replicated MDS (several matrices, unweighted model), and weighted MDS (several ma- trices, weighted model) [5, 12].

MDS represents measurements of similarity among pairs of objects as distances between points of a low- dimensional multidimensional space. Given a matrix of perceived similarities between various items, it plots the items on a map such that those which are perceived to be similar are placed near one another; contrarily, those perceived as very different are placed far away from each other. Therefore, MDS provides a simple visual representation of a complex set of re- lationships which can be analyzed at a glance—the graphical display is much easier to understand than an array of numbers and it shows the essential information in the data, smoothing out noise. However, it has its limitations and normally the analysis is limited to two or three dimensions. Four or more dimensions render MDS virtually useless as a method of mak- ing complex data more accessible to the human mind. The degree of correspondence between the distances among points implied by MDS map and the input ma- trix is measured by a stress function. The general form of these functions is $\sqrt{\frac{\sum (f(x_{i,j})-d_{i,j})^2}{\sum (d_{i,j})^2}}$. In this equation $d_{i,j}$ refers to the Euclidean distance, across all di- mensions, between points i and j on the map, $f(x_{i,j})$ is some function of the input data. When the MDS map perfectly reproduces the input data, $f(x_{i,j}) - d_{i,j}$ is calculated for all i and j, so stress is zero. Thus, the smaller the stress, the better the representation. When looking at a map that has non-zero stress, one

must keep in mind that the distances among items are imperfect, distorted, representations of the relation- ships given by your data. From a mathematical stand- point, non-zero stress values occur for only one rea- son: insufficient dimensionality. That is, for any given dataset, it may be impossible to perfectly represent the input data in two or other small number of dimensions.

On the other hand, any dataset can be perfectly repre- sented using $n-1$ dimensions, where n is the number of items scaled. As the number of dimensions used

goes up, the stress must either comes down or stays the same. Of course, it is not necessary that an MDS map have zero stress in order to be useful. A certain amount of distortion is tolerable [18].

There are three important aspects to keep in mind when analyzing a MDS map: (i) The axes are, in them- selves, meaningless; (ii) the orientation of the picture is arbitrary; and (iii) the substantive dimensions or at- tributes under analysis do not need to correspond in number or direction to the mathematical dimensions (axes) which define the vector space (i.e., the MDS map). In relation to this last point, in fact the number of dimensions used to generate similarities may be much larger than the number of mathematical dimensions needed to reproduce the observed pattern. This is be- cause the mathematical dimensions are necessarily or- thogonal (perpendicular), and therefore maximally ef- ficient. In contrast, the human dimensions, while cog- nitively distinct, may be highly intercorrelated and, therefore, contain some redundant information.

3 Dynamics of stock market indexes

In this section we study numerically ten national stock market indexes for the period from 1 January 2000 to 31 December 2009.

The data consist of daily closing price $x_k(t)$, were $1 \le k \le 10$ identify the national stock market indexes, and the continuously compounded return $\ln \frac{x_k(t)}{x_k(t-1)}$ with data provided by the Yahoo Finance web site [1].

For each signal index data we analyze the fractional behavior from the viewpoint of FrFT with orders $0.001 \le a \le 1.0$ $(\alpha = a\frac{\pi}{2})$ and MDS.

3.1 Analysis of national stock market indexes from a continuously compounded return standpoint

In this section we calculate the daily continuously compounded rate of return on a stock for one day as $r_k(t) = \ln \left[\frac{x_k(t)}{x_k(t-1)}\right]$, where $x_k(t)$ is the close price of the day t and $x_k(t-1)$ is the close price of the day $t-1$, for the index labeled k, and ln represents the natural logarithm function.

Figure 1 depicts the evolution of daily continuously compounded return value $r_9(t)$ of the S&P500 index versus time. The spectrum resulting of the application of the linear transformation is approximately constant over a broad frequency band.

This is an analogy with a ergodic, or stochastic stationary signal, like the white noise. In this case there is no memory of and no correlation with past data; therefore, each new data value adds the same amount of new information.

This kind of signals is not the most suitable for revealing the dynamical relationships and therefore, it was decided to adopt the daily closing price $x_k(t)$.

3.2 Analysis of national stock market indexes from a daily closing price standpoint

Figure 2 depicts the time evolution, of daily closing price of the S&P500 index versus time with the well-know noisy and "chaotic-like" characteristics.

3.2.1 Fourier analysis

For each signal index $x_k(t)$ is calculated the corresponding FrFT and a power trendline approximation.

In order to examine the behavior of the signal spectrum, a power law trendline is superimposed to the FrFT signal, that is, we approximate the modulus of the FrFT amplitude through the power law (PL):

$$\left|\mathcal{F}\{x_k(t)\}\right| \approx p_k u^{-q_k}, \quad \{p_k, q_k\} \in \mathbb{R}^+, \; k = \{1, \ldots, 10\} \tag{7}$$

where F is the fractional Fourier operator, $x_k(t)$ represents the value of $1 \le k \le 10$ index, t is time, u is the frequency, p_k a positive constant that depends on the signal amplitude and q_k is the trendline slope [23].

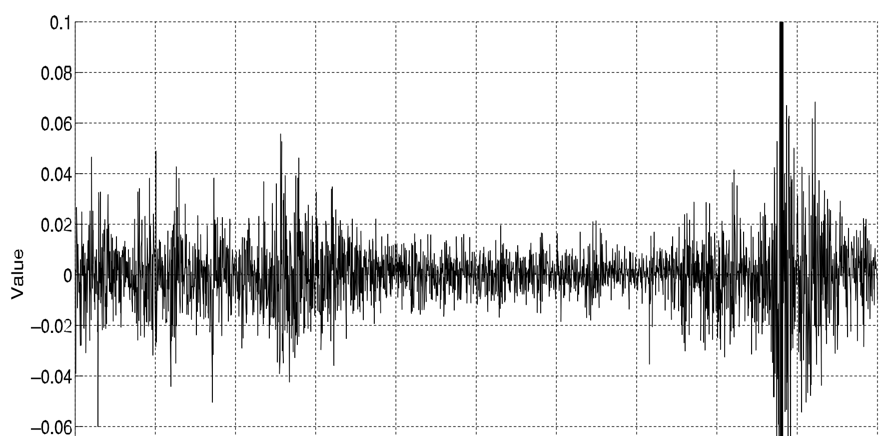
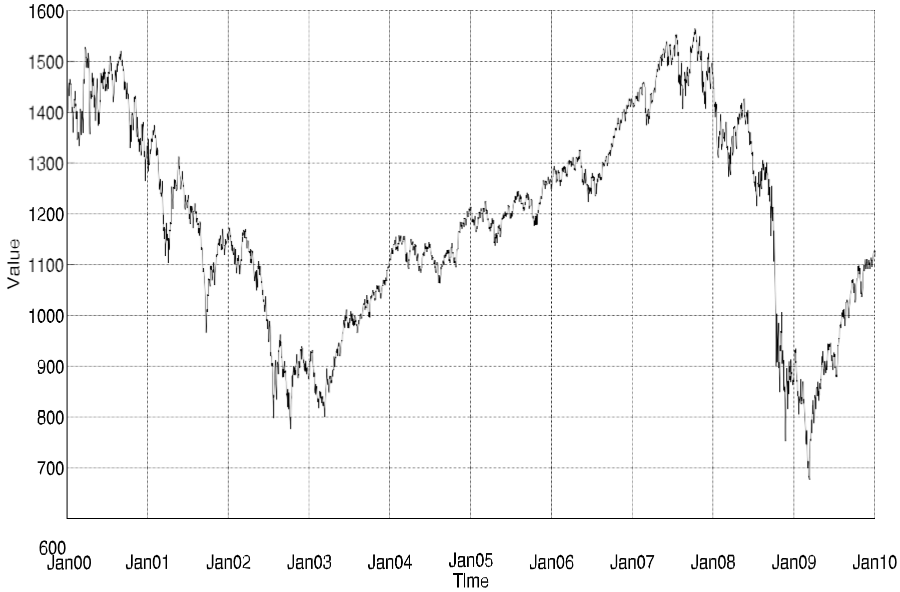


Fig. 1 The temporal evolution of the daily continuously compounded rate of return, $r_9(t)$, for the S&P500 index, from January 2000 to December 2009

Fig. 2 The temporal evolution of the daily closing value, $r_9(t)$, for the S&P500 index, from January 2000 to December 2009



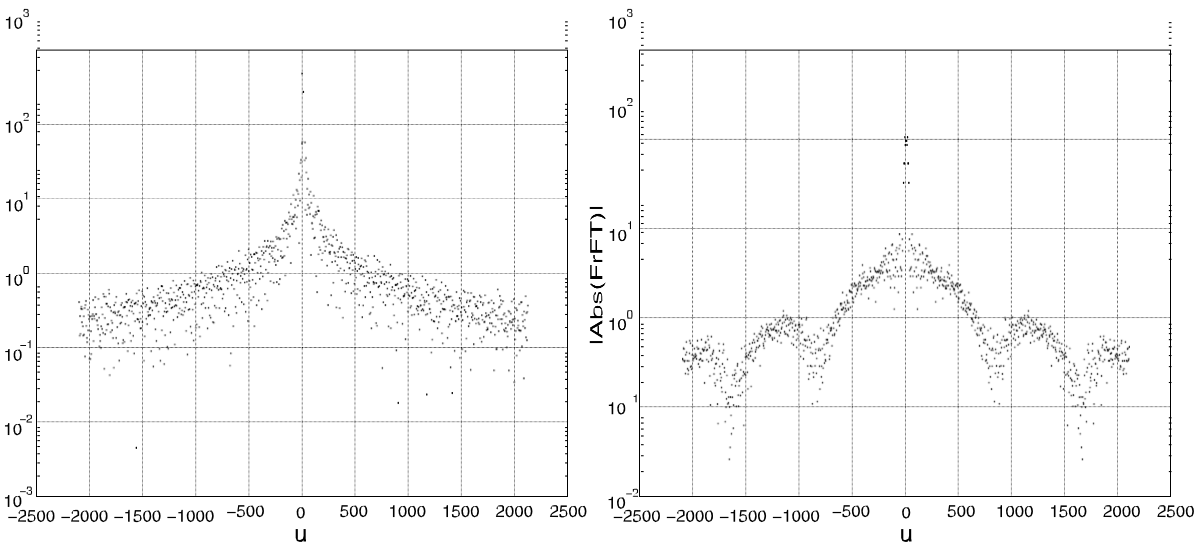


Fig. 3 $|\mathrm{FrFT}\{x_9(t)\}|$ for the daily closing value of the S&P500 signal index for orders $a = 0.5$ (*left*) and $a = 1.0$ (*right*)

According to the values of q_k the signals can exhibit either an integer or fractional order behavior.

Figures 3 and 4 show the amplitude of the FrFT and the power trendline of the S&P500 index, respectively for $a = \{0.5, 1.0\}$. For each of the ten signals a power trendline is calculated. Table 1 depicts the corresponding slope values, q_k, for all orders calculated and in Table 2 are the slope values for all ten indexes with $a = \{0.5, 1.0\}$. We verify that, in all the cases, we get a fractional spectrum in between the white ($q = 0.0$) and the Brownian ($q = 2.0$) noises, most near the so-called pink noise ($q = 1.0$) representing a considerable volatility but, still, with clear dynamical proprieties.

3.2.2 Multidimensional scaling

In order to reveal possible relationships between the signals the MDS technique is used and several dis- tance criteria are tested. The Sammon criterion [4, 8], which tries to optimize a cost function that describes how well the pairwise distances in a dataset are pre- served, revealed good results and is adopted in this work. For this purpose we adopt two “distance mea-

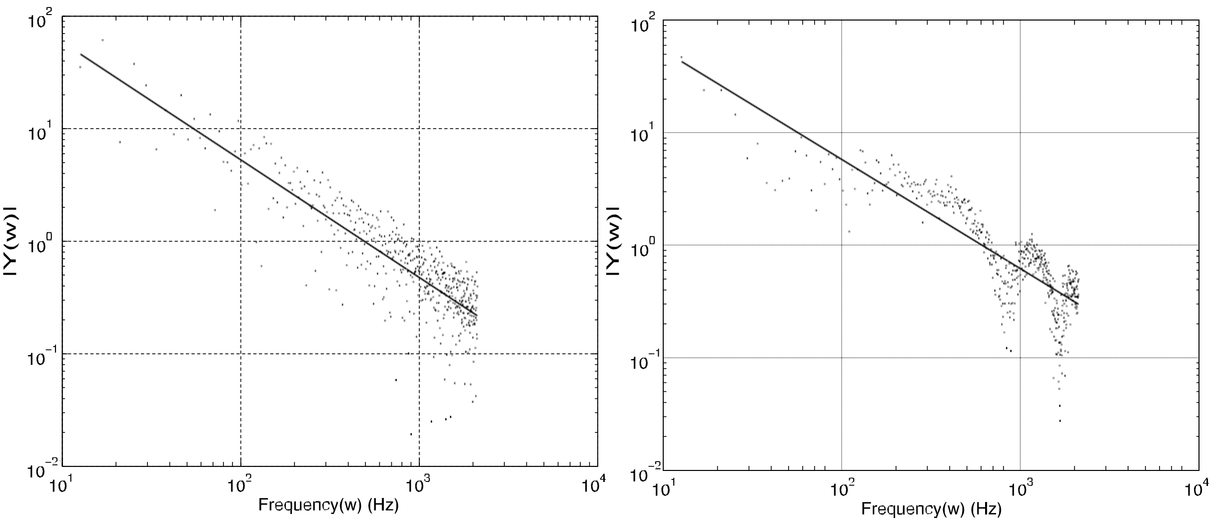


Fig. 4 |FrFT{$x_9(t)$}| and the power trendline for the daily closing value of the S&P500 signal index for orders $a = 0.5$ (*left*) and $a = 1.0$ (*right*)

Table 1 Parameters {p_9, q_9} of the power law trendline of FrFT with $0.001 \le a \le 1.0$ for the S&P500 index

Order *a*	p_9	q_9
0.001	186.88	1.937
0.005	274.95	1.662
0.01	269.53	1.419
0.05	246.13	1.099
0.1	330.49	1.074
0.15	373.71	1.056
0.20	388.98	1.037
0.30	523.48	1.052
0.40	641.53	1.062
0.50	653.44	1.046
0.60	674.54	1.040
0.70	716.87	1.041
0.80	843.58	1.059
0.90	658.37	1.019
1.0	500.49	0.970

sures" defined in the following equations:

$$d_1(i,j) = \exp\left[-(q_i - q_j)^2\right] \tag{8}$$

$$d_2(i,j) = \frac{\sum_u \left[(|\mathrm{Re}_i - \mathrm{Re}_j|^2 + |\mathrm{Im}_i - \mathrm{Im}_j|^2)^{\frac{1}{2}}\right]}{\sum_u \left[(|\mathrm{Re}_i + \mathrm{Re}_j|^2 + |\mathrm{Im}_i + \mathrm{Im}_j|^2)^{\frac{1}{2}}\right]} \tag{9}$$

where q_i and q_j are slopes of the power law trendlines,
$i, j = 1, \ldots, 10$, u is the frequency domain, Re_i, Im_i,

part of the FrFT for the indexes i and j with order $a = \{0.5, 1.0\}$.

For each a value we calculate a 10 × 10 matrix **M** based on $d_r(i,j)$, $r = \{1, 2\}$ defined in (8) and (9). In matrix **M** each cell represents the distance between a pair of indexes and is subjected to a MDS calculation with the following parameters:

Re_j and Im_j are the values of real part and imaginary

- Metric multidimensional scaling
- Uniform weighting

Table 2 Parameters $\{p_k, q_k\}$ of the power law trendline of FrFT for the ten indexes when $a = \{0.5, 1.0\}$

k	Index	$a = 0.5$		$a = 1.0$	
		p_k	q_k	p_k	q_k
1	CAC	2317.3	1.033	2927.2	1.040
2	CCSI	168.1	0.731	338.3	0.840
3	DAX	3327.4	1.054	4575.5	1.072
4	DJI	5453.3	1.042	4404.2	0.976
5	HSI	11 292.5	1.063	9190.5	0.995
6	KS11	907.7	1.077	746.5	1.010
7	Nikkei	12 130.2	1.119	9187.4	1.034
8	NYA	3631.5	1.045	2554.4	0.965
9	SP500	653.4	1.046	500.5	0.970
10	SSEC	1272.3	1.021	1508.1	1.015

- Absolute scaling model
- Stress method: Sammon
- Dimension of the representation space: 3
- Repetitions: 20
- Iterations: 200
- Convergence: 0.0001

Figure 5 shows the 3D locus of index positioning in the perspective of expressions (8) and (9) for $a = \{0.5, 1.0\}$.

The measures d_1 and d_2 were tested against several FrFTs calculated for distinct values of a. The FrFT reveals considerable changes for variations of a near zero, while, on the other hand, changes slowly for the rest of values of a. The variations, either abrupt or smooth, are reflected by d_1 and d_2 upon the MDS plots. Several tests did not produce any criteria for the optimal tuning of a and, therefore, this subject remains open. Bearing these facts in mind, for demonstrating the extra degree of freedom provided by the FrFT it was adopted $a = 0.5$ in the subsequent MDS charts.

Figure 6 depicts the stress as function of the dimension of the representation space, revealing that in some cases, a high dimensional space would probably describe slightly better the “map” of the 10 signal indexes. However, the three dimensional representations were adopted, as usual, because the graphical representation is easier to analyze while yielding a reasonable accuracy. Moreover, the resulting Shepard plots, represented in Fig. 7, show that a good distribution of points around the 45 degree line is obtained. In Figs. 6 and 7 the chart shows the situation for $a = 1.0$, but with $a = 0.5$ we have similar shapes.

Analyzing Fig. 5 we conclude that we can have dif- ferent grouping according with the distance measures and their parameters. When analyzing the shape cor- responding to d_1 it is visible that the indexes under study are distributed over a "wave", divided into the clusters:

- for $\alpha = 0.5$: {ssec, cac, dji, nya, SP500, dax, hsi, ks11}, {nikkei},{ ccsi}
- for $\alpha = 1.0$: {dax}, {cac, nikkei, ssec, ks11, hsi, dji, SP500, nya}, { ccsi}

In fact, the same pattern can be seen in Table 2 when grouping the indexes by the values of the frac- tional slope q_k.

In the d_2 charts the indexes are grouped into a higher number of clusters, namely:

- for $\alpha = 0.5$: {hsi, nikkei}, {dji, nya}, {dax, cac}, {ssec, ccsi}, {SP500, ks11}
- for $\alpha = 1.0$: {hsi, nikkei}, {dji, nya, dax, cac}, {ssec}, { SP500, ks11}, {ccsi}

We verify that, due to the different nature of the two "distance measures" in (8) and (9), we obtain different MDS shapes and groups. We should note again that when using MDS we should not take into considera- tion translation and rotation. MDS reveals essentially the clusters.

From the point of view of the MDS representation, d_1 seems "better" because leads easily to a good low- dimensional graphical representation, as proved by the Shepard and stress plots. We observe also that the d_2 measure seems to separate the indexes into smaller clusters. Therefore, the adoption of each specific case

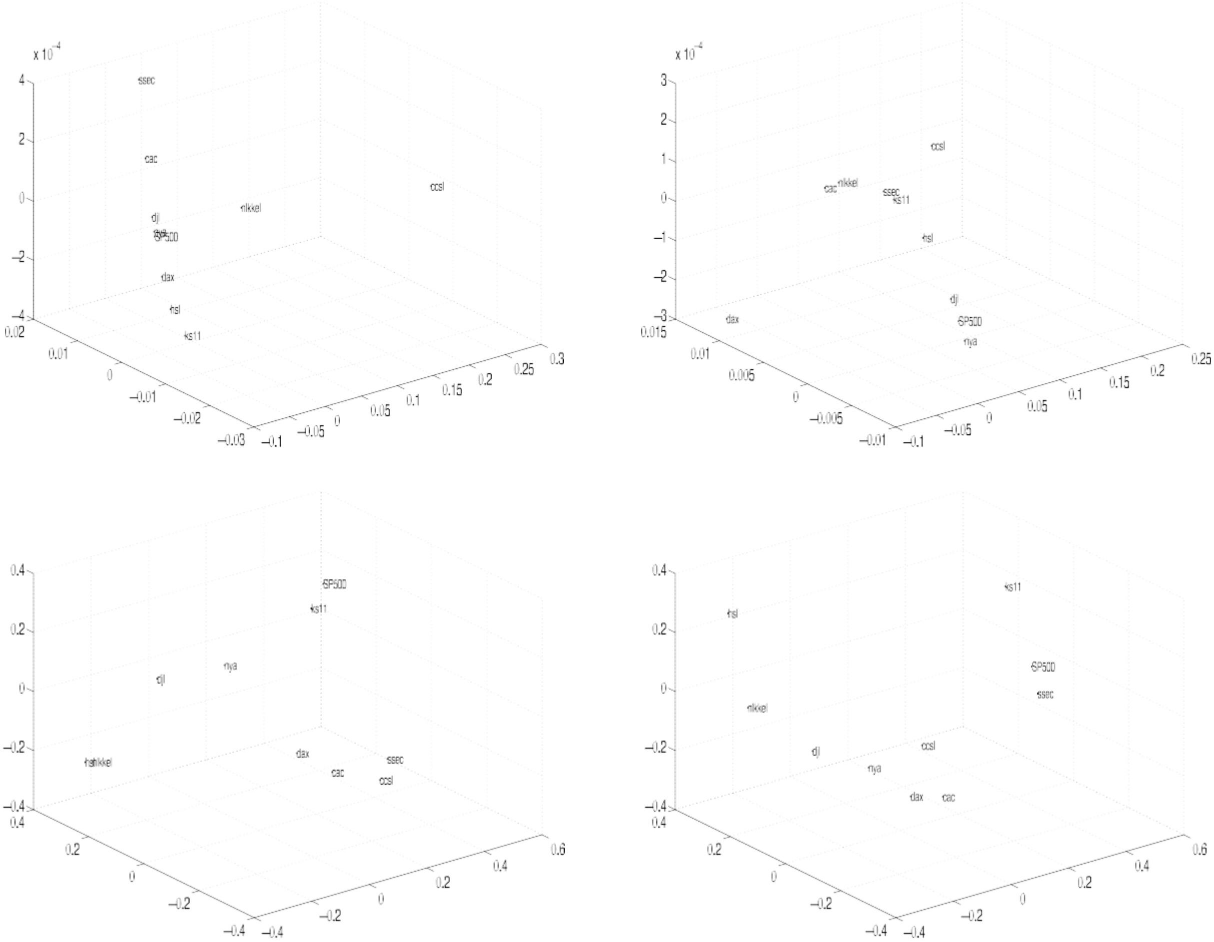


Fig. 5 Three dimensional MDS plots for the ten indexes for the distances d_1 (*top*) and d_2 (*bottom*) and $a = 0.5$ (*left*) and $a = 1.0$ (*right*)

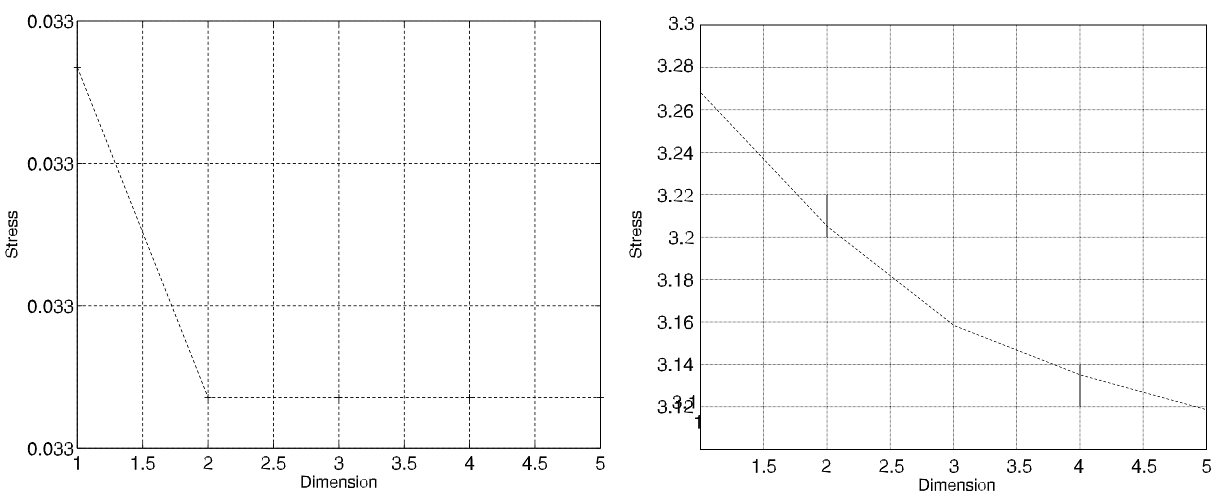


Fig. 6 Stress plot of MDS representation *vs.* number of dimension for the distances d_1 (*left*) and d_2 (*right*) with a = 1.0

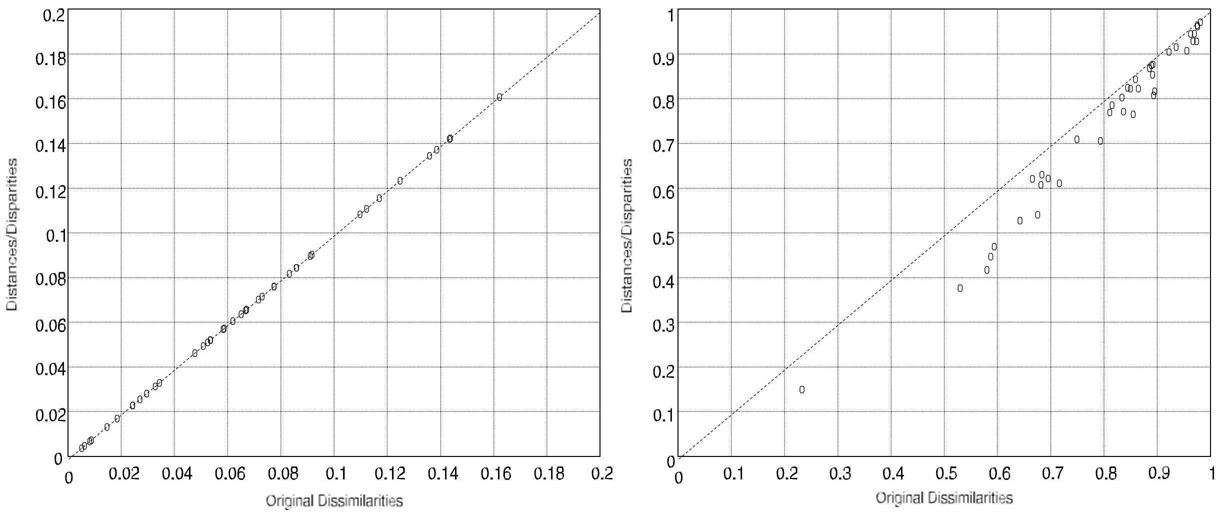


Fig. 7 Shepard plot for MDS representation based on the distances d_1 (*left*) and d_2 (*right*) with a =1.0

is still a matter of decision of the stock market han- dler.

In the authors' opinion d_1, using the fractional slope of the modulus of the FrFT, seems the more interesting when having in mind a dynamical analysis. In fact, d_1 inherently includes (i) the emergence of the fractional order behavior characterized by the power law pu^q approximation, (ii) the robustness against the magnitude of the values under comparison that affect only the parameter p while q remains only for the dy- namics, and (iii) the noise filtering with the trendline slope calculation.

In conclusion, the proposed method starts by ana- lyzing the dynamics through the FrFT. The informa- tion is then used by the MDS to reveal clusters and patterns. For that purpose are tested measures either with a intermediary stage of "filtering" by a power law approximation, or directly by comparing the real and imaginary components. While the usefulness of the FrFT over the FT needs still further research, the adoption of fractional order trendlines reveals to be a good strategy, diminishing the volatility effects and easing the MDS in the job of clustering the indexes.

4 Conclusions

In this paper a dynamical analysis was conducted to investigate possible relationships between several national stock market indexes.

The proposed tools, namely Fourier transform and multidimensional scaling, proved to be assertive meth- ods to analyze stock market indexes; the first for cap- turing the dynamics and the second for revealing the clusters. In future this approach should be applied for other market characteristics like the P/E ratio (price- to-earnings ratio). In this perspective, the replicated MDS technique can be used to analyze the respective matrices of dissimilarity.

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