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ON THE DESCRIBING FUNCTION METHOD AND THE PREDICTION OF LIMIT CYCLES IN NONLINEAR DYNAMICAL SYSTEMS

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This paper analyses the limit cycle characteristics of systems with nonlinear friction or dynamic backlash. The study is based on the describing function (DF) of nonlinear systems being the nonlinear blocks composed on a mass subjected to nonlinear friction or two masses subjected to backlash. The reliability of the method is analysed through the *extended* harmonic content of the systems DFs and using the Nyquist plot of the dynamic response used in the DFs calculations. The overall systems performance is tested using classical PID controllers.

Keywords: Describing function; harmonic analysis; backlash; friction

1. INTRODUCTION

The progress in computational systems made possible the intensive study of nonlinear systems through simulation and the development of computer-based control techniques [1, 2]. In this perspective, this paper investigates the dynamics of systems with friction and backlash through the describing function (DF) method. These nonlinear dynamic phenomena have been an active area of research but well established conclusions are still lacking. Dupont [3, 4] studied the effect of Coulomb friction in the existence and uniqueness of the solution of the direct dynamics. Dupont showed that the problems of

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existence and uniqueness occur even for a system with a single degree-of-freedom. Studies about nonlinear friction modelling can be found also in the references [5–15]. In these papers it has been investigated the position and velocity dependence of friction phenomena. A computer simulation of the stick-slip friction was developed by Karnopp [16], that presented an efficient algorithm for the problem. Several compensation schemes of the nonlinear friction are found in the articles [17–19]. In these studies it is employed the model of the dynamic nonlinear friction for the development of an efficient controller. Canudas de Wit *et al.* [20, 21] model the friction through bristles with good results. Nevertheless, in order to compare results with previous studies, in this article we adopt the classical Coulomb and viscous friction model, because of its simplicity.

The phenomenon of backlash is also found in many physical systems. Tao and Kokotovic [22, 23] considered this problem and developed an algorithm for the compensation of kinematic backlash based on an adaptive controller and an unknown backlash model. Also, in other works [24–26] it is studied the phenomenon of backlash with some simplifications in the dynamical models. Other cases of backlash compensation and control can be found in references [27–31].

In this line of thought, this article is organised as follows. In Section 2 we formulate the main aspects of the problem and we introduce the DF method. In Section 3 we study the DF of nonlinear blocks with energy storage (such as the friction and the dynamic backlash). Finally, in Section 4, we draw the main conclusions.

2. THE FUNDAMENTAL CONCEPTS TOWARDS THE DESCRIBING FUNCTION ANALYSIS

In this section we present a summary of the DF method and its application to the prediction of limit cycles in nonlinear systems. The purpose is to analyse the controller performance in the presence of systems with nonlinear friction and backlash. Due to the nonlinear nature of the problem a possible approach would be the simulation of all possible systems which, obviously, is a time consuming and fastidious task. Therefore, the strategy taken here is to study the DF evolution in the Nyquist diagram of each controller and plant. By this way, we can study the stability and we can predict approximately the occurrence and the characteristics of limit cycles.

It is a well-known fact that many relationships among physical quantities are not linear, although they are often approximated by linear equations, mainly for mathematical simplicity. This simplification may be satisfactory as long as the resulting solutions are in agreement with experimental results. In fact, Cox [32] demonstrated that this is the case with the approximation of nonlinear systems by a DF where limit cycles can be predicted with reasonable accuracy. It must be emphasised that the DF method is not the only one tractable to limit cycle prediction, being the most important others the harmonic balance and the amplitude dependent gain margin methods. Nevertheless, in the condition of limit cycle occurrence all of the methods are equivalent to the DF method [32]. Patra and Singh [33] developed a graphical method to limit cycle prediction, in specific multivariable nonlinear systems, that can be efficiently used in computer graphics programs. Motivated by these facts, in the sequel we will introduce the fundamental aspects of the DF method of analysis.

Suppose that the input to a nonlinear element is sinusoidal. The output of the nonlinear element is, in general, not sinusoidal. Assume that the output is periodic with the same period as the input, containing higher harmonics in addition to the fundamental harmonic component. In the DF analysis, we assume that only the fundamental harmonic component of the output is significant. Such assumption is often valid since the higher harmonics in the output of a nonlinear element are usually of smaller amplitude than the amplitude of the fundamental component. Moreover, most control systems are "low-pass filters" with the result that the higher harmonics are further attenuated.

The DF, or sinusoidal DF, of a nonlinear element is defined as the complex ratio of the fundamental harmonic component of the output $Y_1 \cos(\omega t + \Phi_1)$ and the input $a \cos(\omega t)$, that is:

$$N = \frac{Y_1}{a} e^{j\Phi_1} \tag{1}$$

where the symbol N represents the DF, a is the amplitude of the input sinusoid and Y_1 and Φ_1 are the amplitude and the phase shift of the fundamental harmonic component of the output, respectively. Several

DFs of standard nonlinear system elements can be found in the references [2].

In general, the DF can be computed evaluating the expression:

$$N(a,\omega) = \frac{2}{aT}\int_{t_1}^{T+t_1} y(\omega t)\, e^{-j\omega t} dt \tag{2}$$

where ω is the angular frequency of the input and output waveforms and $T = 2\pi/\omega$.

Once calculated, the DF can be used for the approximate stability analysis of a nonlinear control system. Let us consider the standard control system shown in Figure 1 where the block N denotes the DF of the nonlinear element.

If the higher harmonics are sufficiently attenuated, N can be treated as a real or complex variable gain and the closed-loop frequency response becomes:

$$\frac{C(j\omega)}{R(j\omega)} = \frac{NG(j\omega)}{1 + NG(j\omega)} \tag{3}$$

The characteristic equation is:

$$1 + NG(j\omega) = 0 \tag{4}$$

or

$$G(j\omega) = -\frac{1}{N} \tag{5}$$

If Eq. (5) is satisfied, then the system will exhibit a limit cycle which may be found to be stable or unstable through a graphical, numerical or a mathematical analysis.

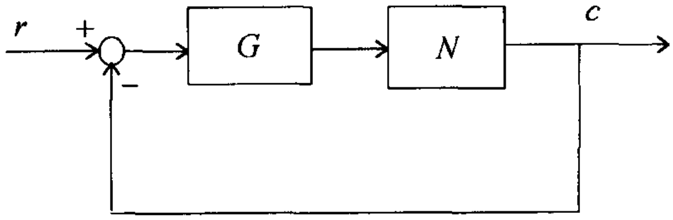


FIGURE 1 Nonlinear control system.

3. THE DESCRIBING FUNCTION OF ELEMENTS WITH ENERGY STORAGE

In this section we analyse the dynamic properties of nonlinear elements with energy storage through the DF method. In this line of thought, we start by pointing out the major differences between the DFs of nonlinear elements with and without energy storage. Then, we study, through the DF method, mechanical systems with nonlinear friction and dynamic backlash.

3.1. The Differences of the DFs for Dynamic and Kinematic Nonlinearities

For nonlinear systems that do not involve energy storage, the DF is merely amplitude-dependent, that is $N = N(a)$. When dealing with nonlinear elements that store energy, the DF method is both amplitude and frequency dependent, that is, $N = N(a, \omega)$. In this case, to determine the DF usually we have a numerical approach rather than a symbolic one because, in general, it is impossible to find a closed-form solution for the differential equations that model the nonlinear element. Nevertheless, it is possible to calculate the approximate analytical expressions for such DFs, namely with the aid of computer algebra packages. As we will see in the next two sub-sections, for systems with nonlinear friction and dynamic backlash the corresponding DF must be obtained through numerical calculations, because of the unexistence of a symbolic closed-form solution.

3.2. Systems with Nonlinear Friction

In this section we calculate the DF of a dinamical system with nonlinear firction and we study its properties.

Let us consider a system with a mass M, moving on a horizontal plane, under the effect of a Coulomb (K) plus a viscous friction (B). This type of system is depicted in Figure 2.

The steady-state response to a sinusoidal input force $F = a\cos(\omega t)$ becomes:

$$x(t) = \begin{cases} \alpha_1 \sin(\omega t + \phi) + k_1 + k_2 e^{-\frac{B}{M}t} - \frac{K}{B}t, & \dot{x} > 0 \\ \alpha_1 \sin(\omega t + \phi) + k_3 + k_4 e^{-\frac{B}{M}t} + \frac{K}{B}t, & \dot{x} < 0 \end{cases} \tag{6}$$

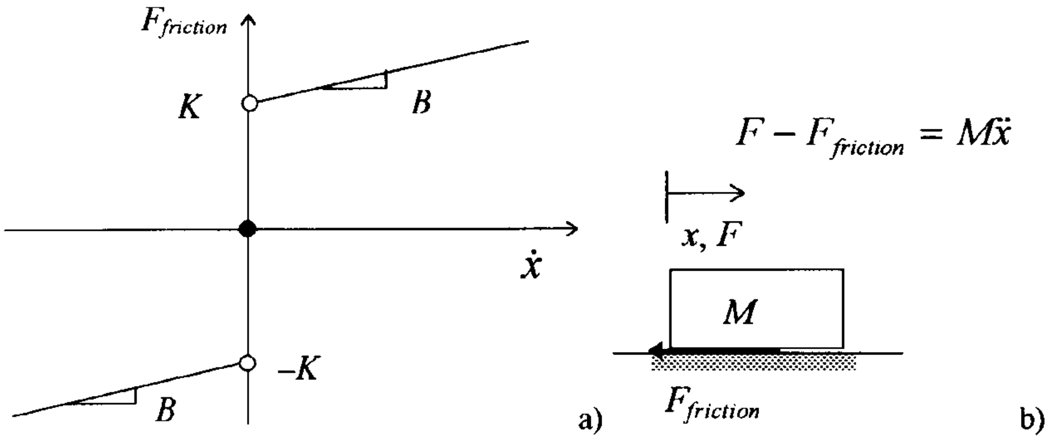


FIGURE 2 a) Model of a Coulomb and viscous friction; (b) System with a mass plus nonlinear friction.

where the parameters ϕ, k_1, k_2, k_3 and k_4, that depend both on the input signal and initial conditions, cannot be determined in closed-form. Therefore, the DF must be determined numerically. Figure 3 shows the function $-1/N(a, \omega)$ for a system with $M=9$ Kg, $K=5$ N and $B=0.5$ Ns/m.

In Figure 4, it is depicted the harmonic content of the output signal $x(t)$ of this system when the input $F(t)$ is a sinusoid. The graphic shows the ratio between the amplitude of each harmonic of the output X_i and the input amplitude a for the first five harmonics.

From this chart we conclude that the output signal has a half-wave symmetry, because the harmonics of even order are negligible. Moreover, the fundamental component of the output signal is the

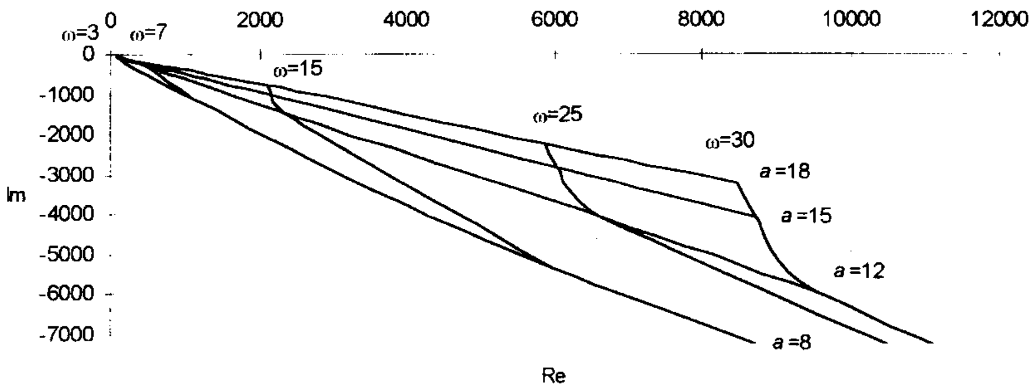


FIGURE 3 The function $-1/N(a, \omega)$ for a system with a mass subjected to Coulomb an viscous friction.

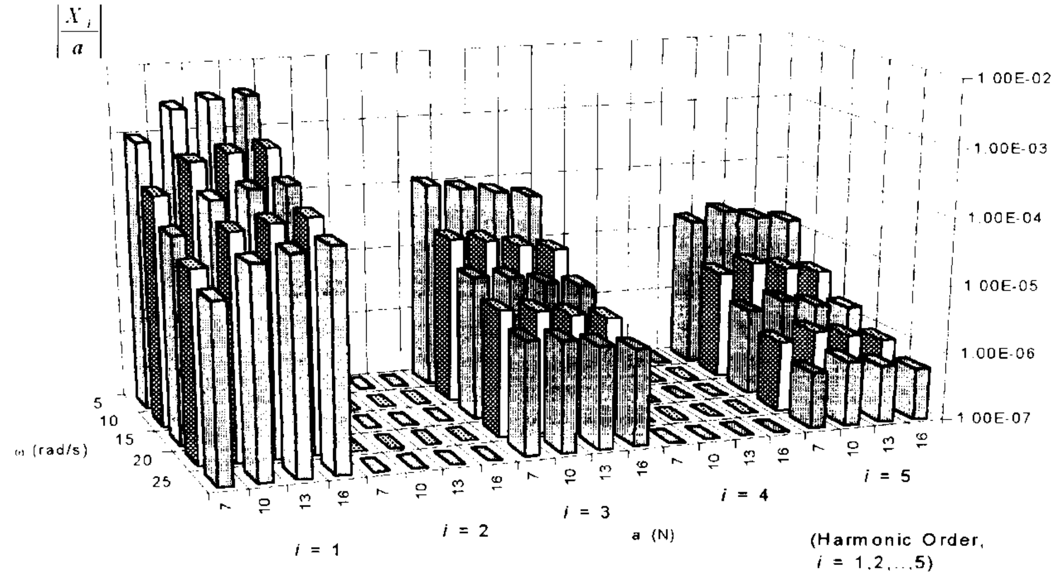


FIGURE 4 Harmonic content of the output system signal when the input $F(t)$ is a sinusoid.

most important, because the amplitude of the high order harmonics decays significantly. Therefore, we can conclude that the DF method leads to a good approximation. If we introduce the stiction [16] (static friction) in the friction model, the results become almost similar, except for high frequencies (ω) or small amplitudes (a), where the harmonic content becomes zero because the mass sticks. By other words, the DF method is also reliable for that case, though in the neighbourhood of zero velocity the system is more difficult to analyse, because there is a certain error difficult to quantify.

Employing a classical PID controller:

$$F(t) = F_{\text{PID}}(t) = K_P e(t) + K_D e(t) + K_I \int e(t)dt \tag{7}$$

$$\frac{F_{\text{PID}}(j\omega)}{E(j\omega)} = K_P + j\left(\omega K_D - \frac{K_{\text{I}}}{\omega}\right) \tag{8}$$

in the closed-loop system we obtain a limit cycle for the parameters $K_P = 2{,}000$, $K_I = 44{,}250$ and $K_D = 130$ (Fig. 5).

Comparing the results of Figure 5 with those obtained through Eq. (5) we conclude that the frequency of oscillation ω is accurately predicted with the DF method, while the amplitude a is less amenable to check from this method. This stems from the fact that the

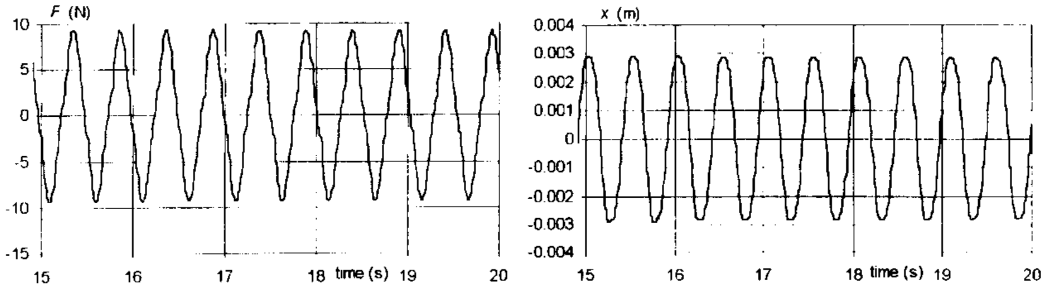


FIGURE 5 Time response of the system with nonlinear friction (without stiction): a) Actuation force; b) Output displacement.

intersection in the Nyquist diagram of the controller frequency response with the $-1/N$ function of the plant is nearly perpendicular for the frequencies, while it is nearly tangential for the amplitudes [2].

3.3. Systems with Dynamic Backlash

In this sub-section we use the DF method to analyse systems with dynamic backlash and its control requirements.

The standard approach to the backlash study is based on the adoption of a kinematic model [2] that neglects the dynamic phenomena involved in the impact process. Due to that reason often real results differ significantly from those "predicted" by that model. Therefore, we analyse this problem considering a system consisting of two masses subjected to dynamic backlash (Fig. 6).

A collision between the masses M_1 and M_2 occurs when $x_1 = x_2$ or $x_2 = h_1 + x_1$. In this case we can compute the velocities of masses M_1

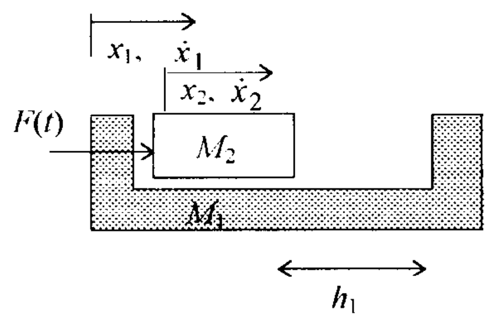


FIGURE 6 System with two masses subjected to dynamic backlash.

and M_2 after the impact $\dot{x}'_1$ and $\dot{x}'_2$, respectively by applying the Newton's law:

$$\dot{x}'_{12} = -\varepsilon \dot{x}_{12}, \quad 0 \leq \varepsilon \leq 1 \tag{9}$$

where $x_{12} = x_1 - x_2$. The coefficient of resitution ε varies in the interval $0 < \varepsilon < 1$ being $\varepsilon = 0$ in fully plastic materials and $\varepsilon = 1$ in the elastic cases. On the other hand, by the principle of conservation of momentum it comes:

$$M_1 \dot{x}'_1 + M_2 \dot{x}'_2 = M_1 \dot{x}_1 + M_2 \dot{x}_2 \tag{10}$$

From Eqs. (9) and (10) we obtain:

$$\begin{cases} \dot{x}'_1 = \frac{\dot{x}_1 (M_1 - \varepsilon M_2) + \dot{x}_2 (1+\varepsilon) M_2}{M_1 + M_2} \\ \dot{x}'_2 = \frac{M_1 (1+\varepsilon) \dot{x}_1 + (M_2 - \varepsilon M_1) \dot{x}_2}{M_1 + M_2} \end{cases} \tag{13}$$

For this system we calculated $-1/N(a, \omega)$ via numeric simulations. The input sinusoidal force was applied to mass M_2 and the output position x_1 monitored. For example, the DF for a system with parameters $M_1 = 8$Kg, $M_2 = 1$Kg, $\varepsilon = 0.2$ and $\varepsilon = 0.8$, $h_1 = 18$ mm, under the action of an input force $F(t) = a\cos(\omega t)$ is presented in Figure 7.

We conclude [34–36] that an intersection between $-1/N$ and G can occur in the first quadrant, making this system prone to limit cycle generation under the control of a PID algorithm.

In Figure 8 it is presented the harmonic content of the output signal of this sytem when the input is a sinusoid. The graphic shows the ratio between the amplitude of each harmonic of the output X_i and the input amplitude a for the first five harmonics.

We conclude that the system with the higher coefficient of restitution leads to some errors, due to significant high-order harmonic content. So, the DF approach for this system will lead to higher approximation errors the higher value of ε.

Employing a PID controller (7)–(8) in the closed-loop for the system with dynamic backlash and $\varepsilon = 0.2$, we obtain a limit cycle for the parameters $K_P = 1, 800$, $K_I = 19{,}500$ and $K_D = 100$ (Fig. 9).

Analysing Eq. (5) we conclude that the frequency of oscisllation ω is accurately predicted with the DF method, while the amplitude a is less amenable to check from this method.

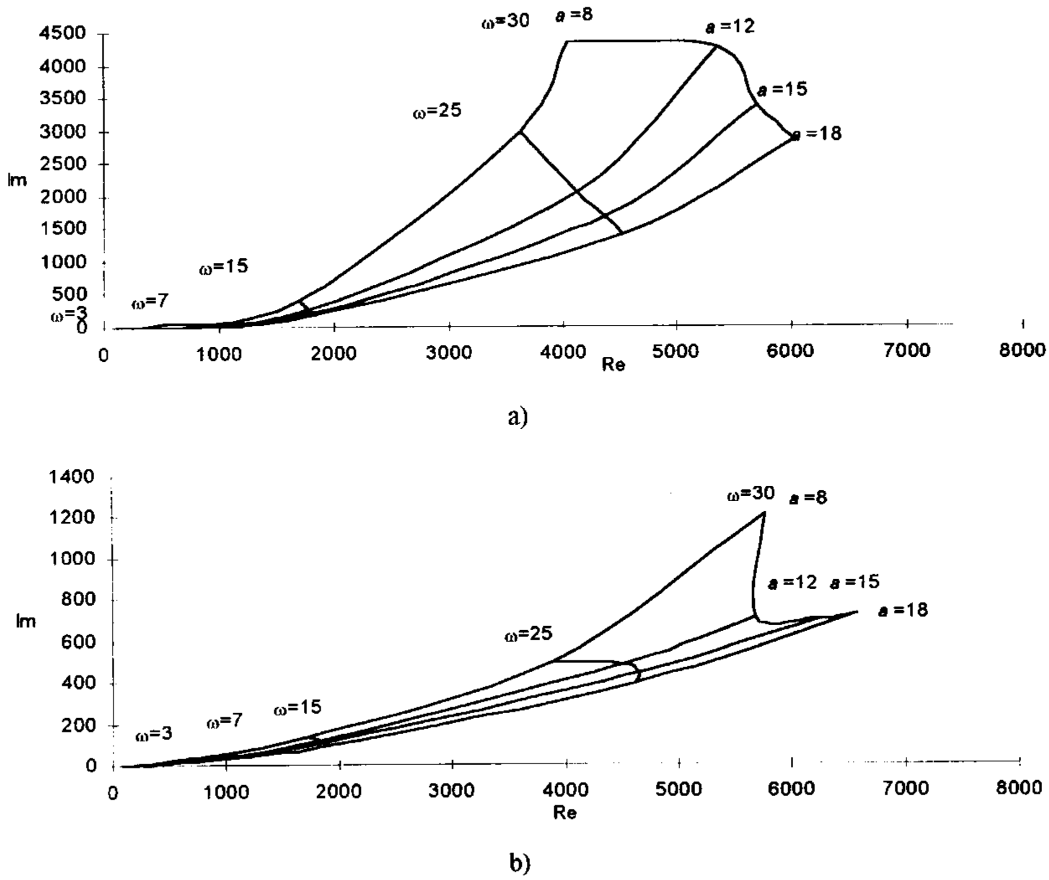
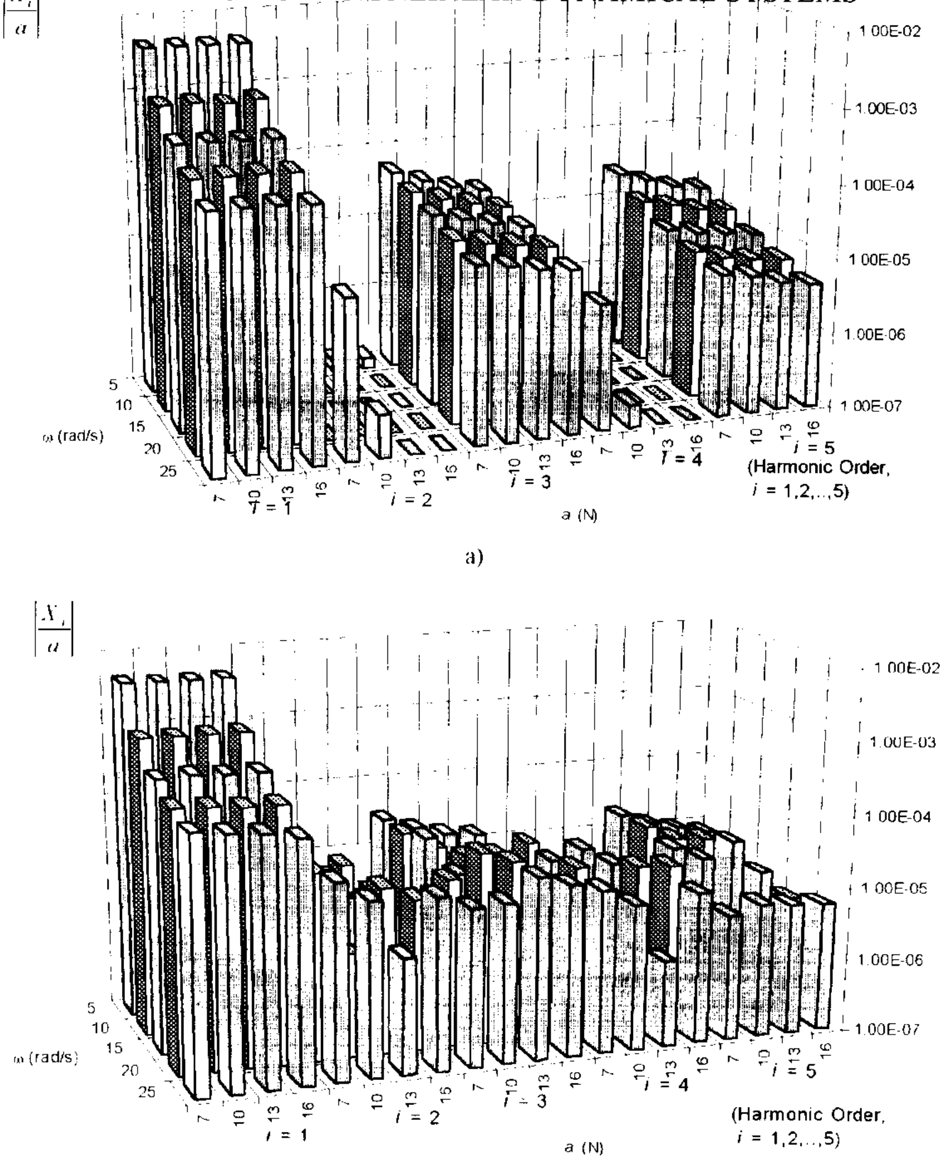


FIGURE 7 Plot of $-1/N(a, \omega)$ for a system with backlash: a) $\varepsilon = 0.2$, b) $\varepsilon = 0.8$.

In a similar experiment, with $\varepsilon = 0.8$ and a PID controller having $K_P = 1{,}800$, $K_I = 19{,}500$ and $K_D = 100$ we get the limit cycle depicted in Figure 10. In this case, the DF prediction leads to higher errors namely for a, due to the almost "chaotic" behaviour of the system output.

4. CONCLUSIONS

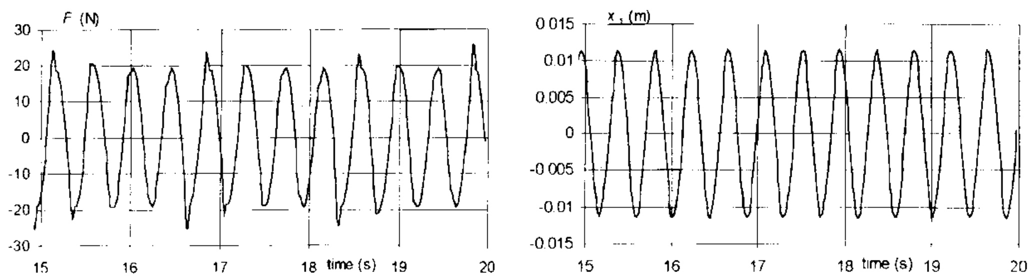
This paper studied, through the DF method, the dynamical properties of systems with nonlinear friction or dynamic backlash. In terms of controllability and DF analysis, the worst system, is the one with dynamic backlash because it is more sensitive to approximation errors. The DF method of predicting limit cylces has shown a very good accuracy in terms of the frequency of the oscillation, even in the cases of intrinsically high nonlinear characteristics and, therefore, difficult to analyse through other approaches. Also, this methodology allows to



a)

b)

FIGURE 8 Harmonic content of the system output $x_1(t)$ when the input $F(t)$ is a sinusoid a) $\varepsilon = 0.2$ b) $\varepsilon = 0.8$.



a) b)

FIGURE 9 Time response of the system with dynamic backlash ($\varepsilon = 0.2$): a) Actuation force; b) Output displacement.

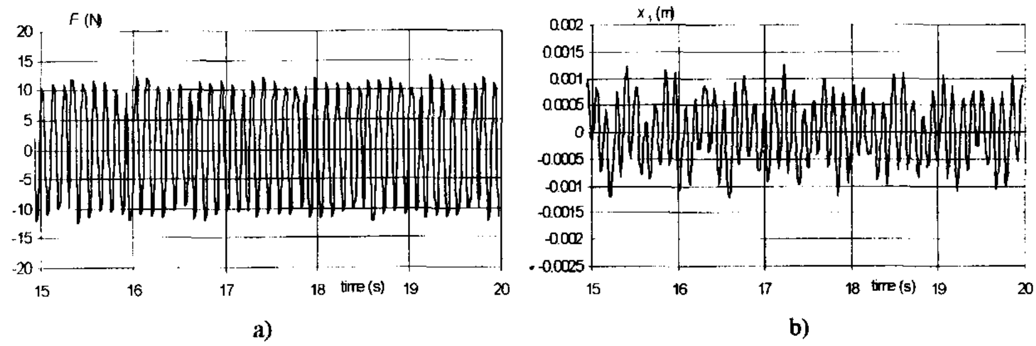


FIGURE 10 Time response of the sytem with dynamic backlash ($\varepsilon = 0.8$): a) Actuation force; b) Output displacement.

design efficient controllers for nonlinear systems, because one can methodology allows to design efficient controllers for nonlinear systems, because one can predict accurately the system overall stability. This fact was confirmed by several experiments that lead to system limit cycles corresponding to the prediction based on the DF method.

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