



Fractional Calculus & Applied Analysis

An International Journal for Theory and Applications

VOLUME 19, NUMBER 5 (2016)

(Print) ISSN 1311-0454
(Electronic) ISSN 1314-2224

ROUND TABLE DISCUSSION

FRACTIONAL CALCULUS: D'OÙ VENONS-NOUS? QUE SOMMES-NOUS? OÙ ALLONS-NOUS?

(Contributions to Round Table Discussion held at ICFDA 2016)

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Abstract

Discussions under this title, inspired by the famous Gauguin painting, were held during a Round Table in frames of the International Conference “*Fractional Differentiation and Applications*” (ICFDA '16) held in Novi Sad (Serbia), 18-20 July 2016, <http://www.icfda16.com/public/>; see details at Announcements (21 September Closure):

<http://icfda16.com/public/announcements.php>; and Report on ICFDA '16 in Editorial Note in this journal's issue, pp. 1059–1060.

Along with the presentations made during this Round Table, we include here some contributions by the participants sent afterwards and also by few colleagues planning but failed to attend. The intention of these discussions was to continue the useful traditions from the first conferences on Fractional Calculus (FC) topics, to pose open problems, challenging hypotheses and questions “where to go”, “how to save and improve the prestige of FC”, to share opinions and try to find ways to resolve them.

MSC 2010: Primary 26A33, 01A67; Secondary 34A08, 35R11, 60G22

Key Words and Phrases: fractional calculus – open problems, progress and trends in its development, fractional order differential equations, fractional order mathematical models, International Conferences “Fractional Differentiation and Application” (FDA), perspectives

1. Introduction

The International Conference on Fractional Differentiation and its Applications 2016 (ICFDA '16) took place at Novi Sad, Serbia, July 18-20, 2016, as the 8th successful issue of the periodical FDA events. It brought together 120 participants, with presented papers authored and co-authored by 225 researchers who like to differentiate and integrate of arbitrary order and to enjoy the beauty and usefulness of Fractional Calculus.

The title of the discussion was inspired from the famous Gauguin painting represented in Figure 1, see more details at Wikipedia [61]. It was aimed as a natural continuation of the previous discussion “Fractional Calculus: Quo Vadimus?” (Where are we going?) from ICFDA '14 (Catania, Con-venor: F. Mainardi), see notes [31] published in FCAA, **18**, No 2 (2015), <http://www.degruyter.com/view/j/fca.2015.18.issue-2/fca-2015-0031/fca-2015-0031.xml?format=INT>.

The idea was to continue the tradition of Open Problems sessions at earlier FC Conferences and periodical FDA, but shifting the focus on the trends of recent development of Fractional Calculus (FC), its problems and perspectives. Since the time of the first specialized conferences when this topic was still assumed as an interesting but paradoxical theoretical extension of the classical Calculus, currently FC has become an unavoidable tool of mathematical modeling and to deal with produced a phenomenon of a real boom. There is a flood of studies and exploitations, to an extend that could threaten the FC prestige achieved by serious efforts and contributions for more than three centuries. So, where we need and can going next?

During ICFDA '16, there were on schedule two Round Table discussions, the other one initiated and conducted by YangQuan Chen, entitled “*How to Improve Image and Impact of Fractional Calculus Research Community*”. There, also many participants took place and stressed on related questions, and practically these two discussions were merged, and became continuations and complements each to another.



Fig. 1: The painting (1897-1898) entitled “Where Do We Come From? What Are We? Where Are We Going?” by French artist Paul Gauguin.

2. Researchers' Contributions

In this section are listed, by alphabetical order of family name, the opinions of several researchers concerning the “D’où venons-nous? Que sommes-nous? Où allons-nous?” query about the future progress of Fractional Calculus.

2.1. The point of view of Teodor Atanacković. Fractional calculus of real and complex order are established as important methods in the analysis of various Mathematical, Physical and Engineering problems. For the case of variable order derivatives there is not even agreement what should be taken as definition. Namely there are several, not equivalent, definitions, see for example [27].

There are many numerical procedures developed for specific differential equations with variable order derivatives. What is needed is the agreement about the definition and development of analytical results concerning the fractional derivatives of variable order. This may promote the use of variable order derivatives in new areas of Physics. It seems that the potential of application in this area is not explored (see, [2, 3]). I believe that in the process of fractionalization of equations of the mathematical physics, the order of fractional derivative must be treated as a constitutive quantity and that it must satisfy the restrictions that are usually imposed on constitutive equations in Continuum mechanics.

2.2. The point of view of Dumitru Baleanu. Based mainly on the fact that the anomalous diffusion can be better described by fractional differential equations we can suggest two main questions:

1) *Can mathematical models with fractional space and/or time derivative describe a large types of complex phenomena except the ones possessing power law effect ?*

2) *How can fractional models based on nonsingular and non-local kernels can be observed experimentally ?*

Despite of the fact that in many areas of science and engineering new and interesting results provided by the fractional models were reported and proved experimentally [1, 62, 4], still we are a little bit far from a solid answer to the above mentioned questions.

Keeping in mind the spirit of the fractional calculus we suggest to apply in the future new type of fractional derivatives and integrals, e.g. the ones based on the nonsingular non-local kernels of Mittag-Leffler type.

During the last few years some new directions started to emerge within fractional calculus:

- discrete fractional calculus and its applications

- fuzzy fractional differential equations and their applications
- fractional derivatives with Mittag-Leffler kernel and their applications
- generalized fractional Riemann-Liouville and Caputo like derivative for functions defined on fractal sets
- regular fractional dissipative boundary value problems

2.3. The point of view of YangQuan Chen. As Bruce J. West patiently explained in his kind-hearted book [59], our worlds are complex. The inherent uncertainty, inequality and unfairness can be better appreciated using distributions with heavy-tailedness or algebraic tail instead of exponential tail like in Gaussian distribution. The connection to fractional calculus (FC) was made much clearer and more explicit in [2] in such a way that fractional calculus should be regarded as “the language of complexity.” I suggest everyone should embrace FC as a way of thinking in face of our complex worlds. In my view, FC is a very application oriented subject because we always wish to better sense, characterize, understand, and change (if changeable) complex system behaviors.

We are not in an urgent need of yet another generalization of fractional operator unless we have a compelling reason to cope with the complex problem at hand. I would like to suggest more efforts should be put in STEM (Science, Technology, Engineering and Mathematics) in the following manner:

- (1) Science. FC should be used as an enabler only to enable new sciences in the sense of [60]. All reported anomalous behaviors or phenomena should be revisited using FC as a tool.
- (2) Technology. FC should be used as a differentiator to enhance productivity, marketability and sustainability.
- (3) Engineering. FC should be used to show case or to pursue “more optimal” results with better performance than the best achievable by using integer-order methods under fairness comparison.
- (4) Mathematics. FC should be introduced only when driven by real world needs although the needs could be in distant future. Any ad hoc, recreational introduction of FC re-writing of existing math objects should be avoided if not forbidden.

I firmly believe FC has a bright future since FC can help us to better understand our complex worlds and make our world a better place [8].

2.4. The point of view of a CRONE team: Rachid Malti, Pierre Melchior, Patrick Lanusse, Stéphane Victor, and Alain Oustaloup. Fractional systems are long memory systems with a polynomial decay which makes them very useful for modeling some kind of long memory processes such as diffusive and/or infinite dimensional ones. Consequently, a single

derivative fractional differential system *can definitely not be initialized with a single initial condition* (neither of integer nor of fractional type), [47].

The future behavior of such fractional systems depends on their whole past, according to the *initialization function approach* [26]. This dependence can also be expressed in terms of all frequencies at the initial time, according to the *continuous frequency approach* [56]. Both approaches give comparable results [18] and the missing terms, when considering Caputo and/or Riemann-Liouville (RL) derivatives, are explicitly given in [56].

So, how long should the past be considered and/or how to deal with all frequencies? Since fractional systems decay polynomially, the past may be truncated and yet have a negligible truncation error regarding the free response (due to non-zero initial conditions). Alternatively, in the frequency domain the contribution of low and high frequencies decay as well as a truncation can generate negligible errors. When initial conditions are properly taken into account, by considering the missing terms [56], fractional differential systems with Caputo and RL derivatives exhibit identical behaviors. A good news! *Because a (fractional) physical system does not choose whether to respond in a Caputo or in a RL way.*

Although these results have been established five to ten years ago, a lot of researchers carry on using the Caputo derivative, without the missing terms, because it allows handling initial conditions in *a simple way*. Simple but unfortunately erroneous!

Furthermore, generalization of some results of rational (or integer-) order systems should be done with a lot of care. There are many areas of development of fractional order tools in modeling, system identification and control system design, [34, 58, 24]. Moreover, it would be particularly interesting for the researchers to apply their theoretical developments on real technological and/or industrial problems.

2.5. The point of view of Kai Diethelm.

- Fractional Calculus in Applications. **What FC needs:**
 - Validation of existing or newly developed FC-based mathematical models with experimental data
(probably requires much more experimental measurements than currently available)
 - Convincing (non-mathematical) explanations why FC models work better than classical models
 - Resolution of starting point issue
 - A repository of current (proven!) applications that can serve as benchmarks for demonstrating the validity of FC as a modeling tool
(possibly in the form of a web site, augmented by a book discussing selected special cases)

- A “killer application”
- Fractional Calculus in Teaching. **What FC needs:**
 - Some agreement on a “canonical” curriculum for a basic course
 - * possibly depending on the target group — mathematicians, engineers, physicists, ...
 - * different approaches and definitions of fractional operators need to be taken into account *in a constructive manner*
 - Good textbooks for the different target groups
 - * possibly separate textbooks for
 - “classical” analytical aspects
 - (ordinary and partial) differential and integral equations
 - numerical methods
 - applications
 - ...
- Foundations for Fractional Calculus Research. **What FC needs:**
 - Up-to-date monographs on analytical and on numerical aspects
 - * starting with work of survey type
 - * followed by more detailed and specialized publications
 (FC is in a mature state in a number of respects but results tend to be scattered over a large set of sources, and some important proofs seem to be not easily accessible, missing or incorrect)
 - A “Bourbaki type” group to lead the way?
 - * not in the sense of Bourbaki’s style of writing
 - * but in the sense of writing up a comprehensive and mathematically rigorous foundation of the state of the art for others to build upon

2.6. **The point of view of Clara Ionescu.** Given my past expertise in modelling biological tissue, I believe the next thing unmapped in this quest is to apply FC to model diffusion mechanisms in the lung [20, 19]. This could be embossed in a mathematical framework of porous tissue mechanics whereas properties such as granulation, porosity and viscoelasticity change as disease progresses. Specifically, the fractional order impedance model may be employed either in its electrical, or its mechanical equivalent.

A similar problem may be found in drug diffusion dynamics, whereas FC may be also applied to obtain fractional order differential equations to model various degree of diffusion between the compartments used to model the major types of tissue in the body: blood, muscles and fat. Obviously, each of them have various properties and diffusion may occur at different rates between them. These different rates provide the biological framework to introduce fractional order derivatives with various orders to capture drug assimilation and clearance. In this topic, the sponge theorem may be used to characterize time-varying fractional order derivatives. Specifically, once

the tissue assimilates drug into it, the next quantity of drug will be assimilated at a slower or faster rate than the previous; i.e. the clearance rate may be different from the assimilation rate.

2.7. The point of view of Roberto Garrappa and Marina Popolizio.

Problem: Evaluating Mittag-Leffler functions with matrix arguments: why, when and how?

The importance of the Mittag-Leffler (ML) function

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}$$

in fractional calculus is widely known: it is indeed the eigenfunction of the fractional derivative and plays a fundamental role in the analysis of stability of the solution of fractional differential equations (FDEs).

The computation of $E_{\alpha,\beta}(z)$ has been largely considered over the years, especially for complex arguments $z \in \mathbb{C}$ which represent a very challenging (and sometimes prohibitive) task; however nowadays a number of methods are available for the computation of this function (e.g., see [13, 15]).

The use of the ML function for the numerical solution of FDEs and systems of FDEs still however remains a rather unexplored field. Given a matrix $A \in \mathbb{R}^{N \times N}$, it is indeed possible to represent the solution of a system ${}_0D_t^\alpha Y(t) = AY(t)$, for $0 < \alpha < 1$, directly in terms of the ML function as $Y(t) = E_{\alpha,1}(t^\alpha A)Y(0)$. This way of solving FDEs has the irrefutable advantage of reducing the computational effort in particular when the solution is requested just at a given time t and not along the whole interval $[0, t]$. With minor changes, approaches of this type can be applied to higher order problems and to semilinear systems [14].

Using the ML function to solve systems of FDEs is obviously possible (and computationally affordable) only when robust, accurate and fast methods for evaluating matrix ML functions are available.

This problem has been studied mainly for the exponential function with applications to differential equations of integer order. It is possible to find more than 500 papers in Scopus discussing the evaluation of exponential matrices and several codes are freely available: see, for instance, `expm` (in Matlab), `MatrixExp` (in Mathematica), `ExpoKit` (in Fortran 99) and others (a catalogue of codes has been published in 2014 by Higham and Deadman).

On the contrary very few works are dedicated to the computation of the ML function with matrix arguments (we are just able to cite [9, 35]) and no codes are available in software libraries. We think that it is time to fill-in this gap and to dedicate to the computation of matrix ML functions the same attention dedicated in the past to matrix exponentials: the development of efficient methods can indeed represent an important enhancement in the field of the numerical treatment of FDEs.

2.8. The point of view of Reyad El-Khazali. Real and Complex-order Fractal Elements Fabrication

Where do we come from? As FDA community, we embrace and follow the fundamental question that was raised by Leibniz 300 years ago, which triggered the theory of fractional calculus and its applications. Today we can say with certainty that both scientists and engineers have widely recognized the need to use fractal theory along with the theory of fractional integral-differential operators to solve a variety of problems that emerge in various fields of modern science and technology. There are many open problems to tackle, from the geometry and definition of complex fractal elements, to the physical fabrication and implementation of these elements.

Rashid Sh. Nigmatullin [57] was the first to physically implement fractional integration and differentiation real electromechanical diode which has input impedance $Z(j\omega) = \frac{1}{A\sqrt{\omega}}e^{-j\frac{\pi}{4}}$.

The fabrication process had also evolved to make solid electrolyte electrochemical signal converter. The silver electrode is the only reversible substances like RbAg4I5, that facilitate silver ion conductivity. Experiments proved that platinum electrodes could be used as polarizable electrodes. One should emphasize that polymer components can be made of one or more dimensional nanostructure structure material using partial oxidation of metallic complexes.

A new continuous phase element (CPE) denoted by Fractor was proposed by G. Bohannan [6, 5] using fractal geometry properties of the electrode-electrolyte interface. It was used to design controllers in robot applications.

Where are we? The fractal elements can be fabricated not only by a single layer but from a multilayer configuration with homogeneous or heterogeneous geometry. Obviously, the structure, the composition, and the number of layers yield fractal elements that could exhibit frequency response of complex orders [16, 46].

Where are we going? The fabrication of the future fractal devices will be directed toward two and third dimensional devices to model fractal devices of both complex structure and complex orders [45].

Figure 2 shows a structure of a heterogeneous medium with the corresponding frequency phase response. The images of the RC-DP topologies indicate that the light squares symbolize the removed resistive layer (0-layer), the dark ones represent the removed conducting layer (R-layer), while the grey ones define the R-C-G-0-layered structure.

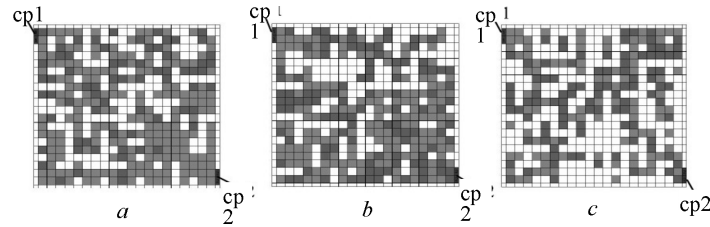


Fig. 2: Topology of two-dimensional heterogeneous medium of RC-DP synthesized to model input impedances with phases: a) $\varphi_{Z_c} = -36^\circ \pm 1^\circ$; b) $\varphi_{Z_c} = -45^\circ \pm 1^\circ$; c) $\varphi_{Z_c} = -53^\circ \pm 1^\circ$.

Figure 3 shows a heterogeneous multilayer RC-DP fractal device with its corresponding equivalent circuit. Clearly, it shows that the future fabrication of fractal devices enjoy no limit since it could follow infinitely many geometrical shapes using many different compositions.

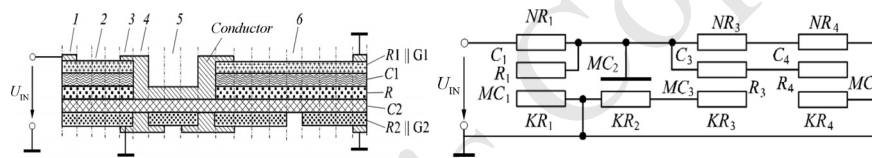


Fig. 3: One dimensional heterogeneous RC-DP fractal device and its equivalent circuit: 1) 0-G1-C1-R-C2-R2-layered Final Element (FE); 2) R1-C1-R-C2-R2-layered FE; 3) 0-G1-C1-R-C2-G2-0-layered FE; 4) 0-C2-0-layered FE; 5) 0-C2-G2-R2-layered FE; 6) R1-G1-C1-R-layered FE.

In conclusion, fabrication of fractal elements is gradually evolving from using electrochemical devices to solid-state ones that have different shapes and geometry. Nanotechnology is used to fabricate such devices. It allows one to design fractal elements that modes RC distributed and lumped circuits of different real and complex orders. The aim is to minimize the size of these devices so they can be used in IC-circuits design to model fractional-order systems and to process unconventional signals and systems.

2.9. The point of view of Virginia Kiryakova. *A similar title, also inspired by the famous painting, I found (after the initiative for this Round Table was already announced by JAT Machado) in a CERN Preprint of J. Ellis [11], where the 3 Gauguin's questions are discussed when concerning particle physics (surely interesting to read). Google provides also several other weblinks to sites and personal blogs where the meanings of the stuff on the picture are interpreted from various, including philosophical/ theological points of view.*

First, *let us say few words on Gauguin's painting*, some information available at Wikipedia [61] and by other online sources. The French artist

(Eugene Henri) *Paul Gauguin* (P.G.), born 7 June 1848 - died 8 May 1903, was under the influence of impressionists Monet and Manet, but also of Van Gogh, and is characterized as a post-impressionist and an important figure in the Symbolist movement. He had a teacher who lead the school-boys towards spiritual reflections on the nature of life. The 3 fundamental questions in his theory (catechism) were: "Where does the humanity come from?", "Where is it going to?", "How does humanity proceed?" - obviously lodged in Gauguin's mind, and "where" became the key question he asked in his art. Same is in the discussed painting (1897-1898) from his Tahiti period, oil on canvas, exhibited in Museum of Fine Arts - Boston.

The author P.G. indicated that *the painting should be read from right to left*. The 3 major figure groups illustrate the stages of the life (from childhood to old age) and the questions posed in the title. He considered this picture as a masterpiece and culmination of his thoughts. In that time he was in despair, caused by serious life and health problems. He painted this picture in a kind of fever, and planned to kill himself on finishing it - making subsequently an unsuccessful attempt to suicide with an overdose of arsenic. This should explain a bit of pessimistic impression given by the whole scene: - Enclosed in a kind of blue bubble, limited by the upper corners of the bright yellow displaying left the table title and right the painter's signature; - The horizon is blocked? This cold blue bubble surprises in a Polynesian landscape but could be interpreted as the cave or the original matrix of Mother Earth; - Most of the figures are not communicating with the viewer - with except for two women in middle; the trees have no leafs, the soil is rocky without any grass. The baby (origin of the life) sounds as abandoned by his mother, lying on her back side... The overall landscape show Tahitians sitting on the edge of the sea, near a river in a forest, a peaceful scene which combines humans and animals resting in the nature. The 3 women with a baby represent the beginning of the life; one woman's head is turned to the right (place of origin), while a black dog symbolizes the day Before, as watchdog of the underground? The middle group illustrates the daily experience of the young adulthood; in the final group - an old woman approaching the death appears reconciled and resigned to her thoughts, at her feet a strange white bird... (death?). The light source is natural (Sun), is off-screen and comes from to the left (side of death), which illuminates the old woman and those in the foreground, as if the light (conscious) illuminates the end of life and broadcasts on live. The blue idol in religious gesture in the background apparently represents what P.G. described as the "Beyond" - Beyond our thoughts, beyond our will, this is what God planned... - the unsolved problem of our origin and our future. Note also a couple resembling Adam and Eve (the girl eating

an apple and eager for knowledge)... Gauguin painted some paradoxical exotic paradise. The table can be understood both as a celebration of the natural man and the harmony of nature. But some worst aspects show the limits, rather than revisit myths and beliefs to show the cycle of life. So it sounds as a combination of the joyful and painful aspects of life, and the mysterious side, sometimes disturbing the spiritual future of humanity.

A mathematician may observe lines of opposing forces: horizontal vs. vertical, giving imbalance. The horizontal figures prevail on the painting (mainly Women) with a vertically standing Man in the middle, occupying the entire height of the table, with hands up - picking the Apple (the fruit of Knowledge). He may resemble somehow also the Jesus on the Christ..., or... *the peak*, the unit-impulse *Dirac delta-function*?

Let us try to focus on the 3 questions on the development of the FC?



Fig. 4: Time scale (read from right to left):

$$t \rightarrow \infty, \quad t = T, \quad t = x, \quad t = t_0 (t = 0), \quad t \rightarrow -\infty$$

One can interpret this time scale, as in a Liouville fractional integral, say:

$$I^\alpha f(t) = \int_{-\infty}^{\infty} \dots = \int_{-\infty}^{t_0} \dots + \int_{t_0}^x \dots + \int_x^T \dots + \int_T^{\infty} \dots := I_1 + I_2 + I_3 + I_4,$$

$$I_1 = I_+^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x (x-t)^{\alpha-1} f(t) dt \quad (\text{once upon a time...}),$$

$$I_2 = \frac{1}{\Gamma(\alpha)} \int_{t_0}^x (x-t)^{\alpha-1} f(t) dt, \quad I_3 = \frac{1}{\Gamma(\alpha)} \int_x^T (t-x)^{\alpha-1} f(t) dt$$

(today is yesterday's tomorrow...),

$$I_4 = I_-^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_T^{\infty} (t-x)^{\alpha-1} f(t) dt$$

(whatever will be, will be (que sera, sera...)).

Return to the Questions: Where we need and can go next with FC? And how, first task – to save the prestigious image of FC, and then – possibly to improve it and its impact? The destination is the Key. Let us remind a talk from *Alice's Adventures in Wonderland*, Lewis Carol, where the Cheshire Cat (C.) forces Alice (A.) to make a cautious choice: – A. “Would you please tell me which way to go from here?”, – C. “That depends on where you want to get to”, – A. “I don't much care where ...”, – C. “Then it doesn't matter which way you go”, – A. “So long as I get *somewhere...*”, – C. “Oh, you're sure to do that”, said the Cat, “if you only walk long enough”. That is, publish, and publish, more and more papers and books on FC, claiming for its applications in what possibly diverse and strange areas...? Everything can and need to be fractalized?

Some observed dangers:

– D1. Authors of too many recent works “forgot” (or have no knowledge on) the well-known notions and results by classicists of FC, and “invent” them as new notions, often called after their own names. *Recipe:* first read the good old books (a lot of “forgotten” stuff there), then rediscover bicycle. And let the ages to estimate your contributions and associate your names to the worthy ones.

– D2. Plagiarism. In the era of online access, many authors recently practice the techniques “copy + paste”. It is not the problem of self-plagiarism, but of assigning to themselves notions and results by other contemporary colleagues, whose works they surely know - as sometimes referring to them, sometimes neglecting. *Recipe:* although losing time unfruitfully, it sounds necessary we start to claim for fairness by public actions.

– D3. Multiple co-authorship, well known trick: the authors A_1, A_2, \dots, A_n (often $n \geq 3$) submit in short period many “joint” papers as P_1, P_2, \dots, P_m . The numerical expression of this collaboration, often rather artificial, is that each of them has $N = n \times m$ publications (dozens per year). One can observe such teams having publications on various topics, sometimes rather far each from other. In other cases, such co-authors never met and discussed the paper's matter, some “great names” are just added to the list in hope to ensure acceptance. *Recipe:* let the authors be warned that in *FCAA* we try to avoid this bad practice.

– D4. Multiplication of wrong or not adequately related to FC theories and results. Once published, other authors read, trust and then put in the base of their own next publications. *Recipe:* Start and encourage publications pointing out to such faults and popularize them, to avoid spreading.

Many journals, even by prestigious publishers, accept and publish papers with shortcomings as above D1–D4, since they are not specialized on the topic of FC and thus, the reviewers they ask are not experts in the area.

– D5: A great number of mathematical topics could be formally “extended” somehow to fractional analogues... Sometimes, indeed only formally. *Recipe*: Avoid generalization and extension for itself (just to produce publications), if no proved or indicated applications (even in theoretical aspect).

As Editor of *FCAA* and reviewer for other journals, I could prolong the list of recently appearing problems that could spoil the prestige of FC achieved by so many efforts for more than 3 centuries.

Let us try to do our best to avoid some pessimistic perspectives and limitations, inspired both by the P.G. painting, and by the dangers and critics mentioned also in the contributions by other participants in these notes as well as in previous ones (RT of ICFDA '14), [31].

2.10. The point of view of Changpin Li. In this modern era, we often hear and/or talk about big data. Why is the data so big? I think that there are at least two respects which produce big data. One is the long-term accumulation. The other is the long-range interaction. The former means history dependence which can be characterized by the time-fractional derivatives, for example, Caputo derivatives. The latter indicates spatial correlation which can be described by the space-fractional derivatives, for example, Riemann-Liouville derivatives, or Riesz derivatives. Based on the above two reasons, time-space-fractional partial differential equations are likely the best mathematical models for reflecting the evolution of big data. Generally speaking, if we slightly accurately reflect the natural phenomenon, we have to include nonlinearity into our mathematical modelling. On the other hand, uncertain factors ubiquitously exist in the real world which can not be ignored. So in order to more precisely model big data, we had better establish nonlinear stochastic time-space-fractional partial differential equations to unveil the myth of big data.

Once the mathematical models for big data are available, the next duty is to solve some equations which is by no means a facile problem. Similar to the integer-order partial differential equations, we can choose finite difference methods, finite element methods, and spectral methods to solve them [25]. From the numerical experiments, we need check whether or not the established mathematical models really and truly reflect the behaviors of big data.

2.11. The point of view of Yuri Luchko. While dealing with a particular anomalous diffusion process, it is often difficult to *a priori* decide

which of the available models in form of the time-fractional diffusion equations could be suitable for its mathematical description. Thus a general framework is desirable that should include not only known types of the time-fractional diffusion equations (e.g., single-, multi-term, or distributed order equations) but also some new objects of this kind.

• **General Fractional Derivatives and Integrals**

General Fractional Derivatives

The general fractional derivative of the Caputo type is defined in the form

$$(\mathbb{D}_{(k)}^C f)(t) = \int_0^t k(t - \tau) f'(\tau) d\tau \tag{2.1}$$

and the general fractional derivative of the Riemann-Liouville type is defined in the form

$$(\mathbb{D}_{(k)}^{RL} f)(t) = \frac{d}{dt} \int_0^t k(t - \tau) f(\tau) d\tau, \tag{2.2}$$

where k is a nonnegative locally integrable function.

Particular Cases

Setting

$$k(\tau) = \frac{\tau^{-\alpha}}{\Gamma(1 - \alpha)}, \quad 0 < \alpha < 1, \tag{2.3}$$

in the formulas (2.1) and (2.2), we have the conventional Caputo and Riemann-Liouville fractional derivatives, respectively. Other important particular cases of (2.1) and (2.2) are given by

$$k(\tau) = \sum_{k=1}^n a_k \frac{\tau^{-\alpha_k}}{\Gamma(1 - \alpha_k)}, \quad 0 < \alpha_1 < \dots < \alpha_n < 1 \tag{2.4}$$

and

$$k(\tau) = \int_0^1 \frac{\tau^{-\alpha}}{\Gamma(1 - \alpha)} d\rho(\alpha), \tag{2.5}$$

where ρ is a Borel measure on $[0, 1]$. They correspond to the multi-term derivatives and derivatives of the distributed order, respectively.

General Fractional Integrals

Under some conditions on the kernel function k there exists a function $\kappa = \kappa(t)$ with the property

$$(k * \kappa)(t) = \int_0^t k(t - \tau) \kappa(\tau) d\tau \equiv 1, \quad t > 0. \tag{2.6}$$

For $f \in L_1^{loc}(\mathbb{R}_+)$, the relation

$$(\mathbb{D}_{(k)}^C \mathcal{I}_{(k)} f)(t) = f(t) \tag{2.7}$$

holds true, where the general fractional integral $\mathcal{I}_{(k)}$ is defined by the formula

$$(\mathcal{I}_{(k)}f)(t) = \int_0^t \kappa(t-\tau)f(\tau) d\tau. \quad (2.8)$$

It follows from the last formula and a relation between the general Caputo and Riemann-Liouville type derivatives that

$$(\mathbb{D}_{(k)}^{RL}\mathcal{I}_{(k)}f)(t) = f(t), \quad (2.9)$$

too.

• General Fractional Differential Equations

General Ordinary Fractional Differential Equations

Under some conditions on the kernel function k we have the statements: For any $\lambda > 0$, the initial value problem

$$(\mathbb{D}_{(k)}^C u)(t) = -\lambda u(t), \quad t > 0, \quad (2.10)$$

$$u(0) = 1, \quad (2.11)$$

has a unique solution $u_\lambda = u_\lambda(t)$ that belongs to the class $C^\infty(\mathbb{R}_+)$ and is a completely monotone function, i.e.,

$$(-1)^n u_\lambda^{(n)}(t) \geq 0, \quad t > 0, \quad n = 0, 1, 2, \dots \quad (2.12)$$

General Partial Fractional Differential Equations

Results already obtained:

- Analysis of the Cauchy problems for the general time-fractional diffusion equation
- Weak maximum principle for the initial-boundary-value problems for the general time-fractional diffusion equation
- Uniqueness of the solution to the initial-boundary-value problem for the general time-fractional diffusion equation
- Existence of a suitably defined generalized solution to the initial-boundary-value problems for the general time-fractional diffusion equation

Open Problems:

- Asymptotics of solutions to the general fractional relaxation equation as well as of solutions to the Cauchy problem and to the initial-boundary-value problem for the general fractional diffusion equation
- General fractional differential equations in other spaces of functions, say, in L_p or in the fractional Sobolev spaces
- Algorithms for numerical evaluation of solutions to the initial-boundary-value problems for the time-fractional diffusion equation with the general fractional derivative

- Inverse problems for the initial-boundary-value problems for the general time-fractional diffusion equation
- etc. ...

Relevant Publications: [21, 28].

2.12. The point of view of J. Tenreiro Machado. Presently fractional calculus is a well established research topic, but future progress will depend on the synergies between applied sciences and mathematics. In fact, new areas of application such as economy, finance, biology, geosciences, art and many others, can open new perspectives and pose new challenges that will also stimulate the development of more formal approaches.

We should also have in mind that classical sciences address mainly the natural world. Therefore, we can direction fractional calculus research towards the unexplored field of artificial phenomena using big data available for computer processing [29].

Some initial heuristic approaches in these new fields, even if not adopting the formal tools of fractional calculus, should be cherished, since they may stimulate the scientific community and extend the scope of present day applications of fractional calculus.

We have now novel proposals for the definitions of fractional derivatives that seem not fitting some mathematical principles [10]. But, are they just wrong, or do they represent some new perspective, or even a different type of calculus?

The fractional calculus community should have a rigorous, but flexible position, since new ideas are emerging and we are not able to have a global view of all paradigm. We can recall the famous Indian fable “**The Blind Men and the Elephant**” (Fig. 5) that was put in a poem by John Godfrey Saxe (1816-1887).

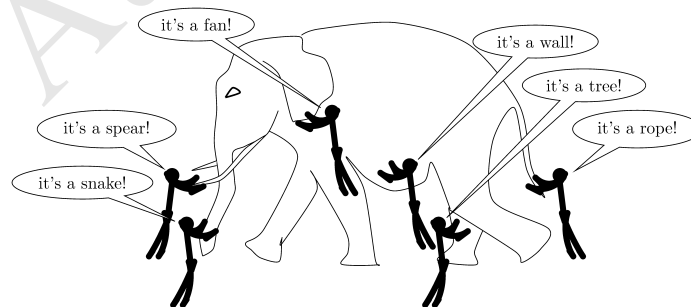


Fig. 5: The parable “The six blind men and the elephant” originated in the Indian subcontinent.

In short, it is the story of six blind scholars, eager for learning, that touch distinct parts of an elephant. Each one creates a different version of reality based on his limited sensory input. One man touched his leg and believes it is a tree. Another man touched the tail and claims that it is a rope. The third blind man puts his hand on the belly of the elephant and thinks it is a wall. The fourth touched the ear of the elephant, felt the moving air and, therefore, believes it is a big fan. The fifth touched the tusk and says that it is like a spear. Finally, the last one touched the elephant's trunk and says it is a snake. The poem is interpreted as a metaphor in many disciplines, warning for our limited understanding on the nature of things.

Another important aspect to consider is education. We have presently a large number of books addressing distinct topics in fractional calculus that students can use to start their research. However, we need books with simple concepts and exercises that focus fractional calculus without requiring complex mathematical tools. In that perspective we can embed fractional calculus in present day undergraduate courses [30] and have future generations of students familiar with those ideas.

2.13. The point of view of Francesco Mainardi. Since several years the number of articles concerning fractional calculus is growing more than any expected measure. However the quality is not always worth for publication. Furthermore the number of special issues in well established journals as well the number of new journals devoted to fractional calculus is still increasing. It seems that many researchers have suddenly discovered that the use of integro-differential equations and non-local operators is more suitable than the use of conventional ordinary and partial differential equations in order to model physical, biological, economic and social phenomena. As in the past the known repulsion against the use of fractional operators was dominant, nowadays we assist at the opposite phenomenon; both of these are in my opinion to be criticized. The so-called fractionalization of many processes, also with new and strange fractional derivatives, should be limited only to the strict necessary.

A field of application of Fractional Calculus that in my opinion should be surely pursued for the benefit of science is concerning image processing, because it appears quite promising and useful, over all in medical physics. This is my invitation for the younger generation of scientists interested in the theory and applications of the fractional calculus, including the treatment of the fractional Laplacian.

2.14. The point of view of Raoul Nigmatullin.

- How to obtain the accurate relationship between the procedure of averaging a smooth function over 1D fractal set?

- As an example we consider the Cantor bar and its modification containing M bars.
- The accurate results obtained for this case allow to generalize the conventional relationship valid for one recap element containing one-power exponent.
- From the results shown in this presentation one can see how to generalize the conventional result for the situation containing many self-similar elements forming an additive combination and giving finally multiple combination of power-law exponents.
- These results will help to understand more complex cases as 2D and 3D fractal structures which can be used for receiving the spacial fractional integrals of different types.

One of the basic problems that did *not* accurately solved yet in the fractional calculus community is the finding of the justified and accurate relationships between the smoothed functions averaged over fractal objects and fractional operators.

This problem was solved partly for the time-dependent functions averaged over Cantor sets in monograph [33] and paper [38], where the influence of unknown log-periodic function (leading finally to the understanding of the meaning of the fractional integral with the complex-conjugated power-law exponents) was taken into account. the basic reason that serves as a specific mathematical *obstacle* in accurate establishing of the desired relationship between the fractal object and the corresponding fractional integral is the *absence* of the 2D- and 3D-Laplace transformations.

Therefore, the basic problem that will be considered in this talk can be formulated as: What accurate form of the fractional operator is generated in the results of the averaging procedure of a smoothed function over the given 1D fractal set if we want to realize this procedure without *any* approximations?

The accurate relationships between fractals and fractional integrals remain a “hot” spot for many researches working in the fractional calculus and fractal geometry field. From the results shown in this presentation one can notice that the complex-conjugated part figuring in the fractional power-law exponent plays a *crucial* role. It is necessary to say that the power-law exponent with complex additive are appeared in some papers but the physical/geometrical origin of this additive was not clear. From the results obtained above one can say that the complex-conjugated part is tightly associated with the *discrete* structure of the fractal process and should be taken into account in the fractional and kinetic equations that pretend on description of self-similar processes in time-domain, at least.

As for the spatial fractional integral the finding of the accurate relationship for the given fractal in space remains an open problem. This problem can be divided at least on two parts:

- The finding of the proper fractional integral based on the given fractal structure
- To find a proper fractal for the given fractional integral that is chosen for description of the self-similar process in space.

From our point of view, the general and accurate solution of this complex problem is absent because each fractal in space can generate a specific fractional integral [37] but any efforts of researches actively working in this interesting field are very welcome [7, 32]. For many researchers it is very important to find the physical meaning of the fractional integral of the given type and any attempt to establish the accurate relationship between the procedures of averaging of a smooth function over fractal sets of different types and their relationships with fractional integrals of different modifications are becoming urgent and important.

2.15. The point of view of Manuel D. Ortigueira.

- *What FC is not/what is not FC*

The word “fractional” became like a fashion. It is interesting to use it close to some kind of operator. It serves to “sell” papers. It is clear that the answer to the question

Is Fractional Calculus (FC) \Leftrightarrow Fractional something?

must be negative. This means that we should remove from our special issues and conferences some kinds of “outliers” that regularly appear. See some of them:

- “*Fractional Fourier Transform*” – FrFT

It is defined by:

$$F_{\alpha}(f)(\omega) = \sqrt{\frac{1 - i \cot(\alpha)}{2\pi}} e^{i \frac{\cot(\alpha)\omega^2}{2}} \int_{-\infty}^{\infty} f(t) e^{-i \frac{\cos(\alpha)\omega t + \cot(\alpha)t^2}{2}} dt.$$

It was called “Angular Fourier Transform”, but for unclear reasons its name was changed. It states *only* a rotation of the frequency axis in a time-frequency plane.

- *Jumarie’s derivative*

Jumarie’s derivative constitutes a derivative definition and sequence of results. It is mathematically incorrect as shown in the papers [50, 52, 39].

- “*Derivative without singular kernel*”

This operator is defined by $D_t^{\alpha} f(t) = \int_a^t f'(x) e^{-\alpha \frac{t-x}{1-\alpha}} dx \Rightarrow \mathcal{L}[D_t^{\alpha} f(t)] =$

$\mathcal{L}f(t) \cdot \frac{s}{s+1-\alpha}$ The operator $H(s) = \frac{s}{s+1-\alpha} \Rightarrow \frac{df(t)}{dt} + \frac{\alpha}{1-\alpha} f(t) = \frac{dg(t)}{dt}$ is an integer order highpass filter: neither fractional nor derivative.

- “Conformable fractional derivative”

This is another operator that claims to be “fractional”. From its definition $T_\alpha f(t) = \lim_{\epsilon \rightarrow 0} \frac{f(t+\epsilon t^{1-\alpha}) - f(t)}{\epsilon} = t^{1-\alpha} f'(t)$, it is clear that it is *not* fractional.

To conclude: **These operators are NOT FC operators!**

- *On the actual situation: some questions*

- Several formulations: which one is more suitable for Science and Engineering?

Most known practical manifestations of fractional behaviour can be seen in frequency domains like spectra or Bode plots. This means that we are dealing with frequency responses which trails two important consequences: a) we must use derivatives for which the derivative of a sinusoid is a sinusoid [42], and b) the observation intervals must be high. So it is natural, to assume that the correct fractional derivative definitions are those defined on \mathbb{R} , see [40, 41].

- How can we define fractional *line and surface* integrals?

Although FC dates back to Leibniz and after many published books and papers there are not definitions of line and surface integrals definitions. They are very important since they are needed to generalize classic theorems useful in Sciences and Engineering.

- What about Gauss, Green and Stokes theorems?

Similarly to the previous item, these theorems do not have undoubtful generalizations. Some were proposed but raise reserves.

- *Other directions*

- Quantum derivative

The quantum derivative did not attracted the attention of many people but can be useful in dealing with problems depending on scale, not on time or space.

- Discrete-time differential systems

Many papers on discrete-time differential systems were published in recent years. However, some definitions create difficulties, since the domain of definition of the derivatives is not the same of the functions. This restricts the applications. To maintain a close relation and similarity with continuous-time formulations the derivatives and original functions must be defined on the same set.

– FC on time scales

It is possible to define fractional derivative on any time scale, continuous, discrete uniform or nonuniform, and mixed. For the general discrete nonuniform the definition of fractional derivative is given by [43],

$$\delta_{\nabla}^{(\alpha)}(t) = -\frac{1}{2\pi i} \oint_{\gamma} s^{\alpha} e_{\nabla}(t + \mu(t), t_0; s) ds \Rightarrow x^{(\alpha)}(t) = \delta_{\nabla}^{(\alpha)}(t) * x(t).$$

In particular, we obtain two definitions for:

* Uniform time scales – constant graininess

$$\delta_{\nabla}^{(\alpha)}(t) = h^{-\alpha} \frac{(-\alpha)_n}{n!} \epsilon(nh),$$

* Nonuniform time scales – different graininess values

$$\delta_{\nabla}^{(\alpha)}(t) = (-1)^n \sum_{k=1}^{n+1} \mu_k(t_0)^{-\alpha+n} \prod_{m=1; m \neq k}^{n+1} \left[\frac{1}{\mu_m(t_0) - \mu_k(t_0)} \right] \epsilon(t).$$

This is suitable for fractal scales.

2.16. The point of view of Dragan Spasić. Let me add some short comments related to the other RT discussion at ICFDA '16, “*How to Improve Image and Impact of Fractional Calculus Research Community*”, conducted by YQ Chen, V. Tarasov, D. Baleanu and myself. This panel discussion was motivated by the following statement of Paul Halmos [17]: “The major part of every meaningful life is the solution of problems; a considerable part of the professional life of technicians, engineers, scientists, etc., is the solution of mathematical problems.” Therefore, it is very easy to declare that in the heart of Fractional Calculus (FC) is a strategy of *problem-posing* and *problem-solving*. Considering FC as a part of mathematics one may say that it is a language of science.

Along these lines, as well as a reference to the discussion of Professor Kai Diethelm on fractional pharmacokinetics presented in ICFDA14 Round Table held in Catania (see in [31]), during this panel discussion I myself presented a MEDLEM project: “Cost-effective microfluidic electronic devices for optimal drug administration based on fractional pharmacokinetics for leukemia treatments”. This project has received funding from the European Union’s Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement. It comprises fractional calculus, optimal control, microelectronics and clinical pharmacology. Based on real data corresponding to low and high drug doses, the purpose of FC is to predict state dependent optimal chemotherapy treatments in adults and children. In doing so, as a first part experiments on animals, are performed.

2.17. **The point of view of Bogoljub Stanković.** *A problem which can be treated in the future:*

**Integral Transforms of Functions
and Distributions on Bounded Domains**

Integral transforms are powerful mathematical theory very useful in practice. But they have at least one important shortcoming: It calls for some growth conditions at ∞ of the treated elements. To overcome this difficulty mathematicians gave different ideas. We mention only the idea of H. Komatsu [22, 23]. Defining the Laplace transform of hyperfunctions, he defined the Laplace transform of hyperfunctions on a bounded domain. But this definition can not be extended, for example, for distributions. The reason is in the fact that hyper functions are a flabby sheaf. This structure property gives that each hyperfunction on any open domain can be extended to a hyperfunction on the entire space \mathbb{R}^n . Distributions also form a sheaf but it is not flabby. Also many sets of classical functions have not this structure. In such a way the posed mathematical problem has the properties:

- It is of large scale because we have different integral transforms. Every integral transform has its properties.
- It request theoretical investigations about the flabbnes of some sheaf of functions and distributions.
- Any solution of this problem can have a lot of practical applications.
- There is a sense to begin work in classical analysis.

One idea how to begin solving the problem:

To give one idea how to begin with the posed problem in classical analysis I published the paper [51]. In this paper it is used the property that a function belonging to $L [0, b_1]$ can be extended to $L [0, b_2]$ for every $b_1 < b_2$. Consequently, one should know how to define the Laplace transform for functions belonging to $L [0, b]$, $b > 0$. This part is elaborated in the cited paper. Then it is now easy to define the Laplace transform for any locally integrable function. This part also with applications will be elaborated in the next paper of the same journal, which is preparation for publication.

2.18. **The point of view of Vasily Tarasov.** *Some important open questions of fractional calculus:*

1. What is fractional derivative from algebraic point of view?

It is very important to have strict mathematical theorems, which will be fractional calculus analogous of the well-known theorem: If a linear operator D_x^α on the function space $C^\infty(U)$, where $U \subset \mathbb{R}$ be a neighborhood of the point x_0 , satisfies the Leibniz rule

$$D_x^\alpha (f(x) g(x)) = (D_x^\alpha f(x)) g(x) + f(x) (D_x^\alpha g(x)) \quad (2.13)$$

for all $f(x), g(x) \in C^\infty(U)$, and all $x \in U$, then D_x^α is the derivative of first order. Equation (2.13) is a characteristic property of the first order derivatives. In fractional calculus it is important to have a “fractional” characteristic property (FCP) in the form of the theorem: If FCP holds for linear operator, then this operator is fractional derivative. This theorem allows us to use algebraic definitions of derivation on algebras. For quantum theory, it is important to have a notion of the fractional derivations on operator algebras.

2. What is fractional differences? What is an exact discrete analog of fractional derivative?

We assume that we can use the following principle of algebraic correspondence [53, 54]: *The exact finite-difference operators of integer and non-integer orders should satisfy the same characteristic algebraic relations as the corresponding differential operators on a function space.*

For integer orders, this principle means: *The finite-difference operator ${}^T\Delta_h^k$ is an exact finite-difference of integer order $k > 0$ if the equality $D_x^k f(x) = g(x)$ (for all $x \in \mathbb{R}$) leads to the equation ${}^T\Delta_h^k f(n) = g(n)$ (for all $n \in \mathbb{Z}$) for all entire functions $f(x)$.*

We assume that the exact fractional-order differences can be defined [53, 54] by the equation

$${}^T\Delta_h^\alpha f(x) := \sum_{m=-\infty}^{+\infty} K_\alpha(m) f(x - mh) \quad (\alpha > -1), \quad (2.14)$$

where the kernel $K_\alpha(m)$ has the form

$$\begin{aligned} K_\alpha(m) = & \cos\left(\frac{\pi\alpha}{2}\right) \frac{\pi^\alpha}{\alpha+1} {}_1F_2\left(\frac{\alpha+1}{2}; \frac{1}{2}, \frac{\alpha+3}{2}; -\frac{\pi^2 m^2}{4}\right) \\ & - \sin\left(\frac{\pi\alpha}{2}\right) \frac{\pi^{\alpha+1} m}{\alpha+2} {}_1F_2\left(\frac{\alpha+2}{2}; \frac{3}{2}, \frac{\alpha+4}{2}; -\frac{\pi^2 m^2}{4}\right), \end{aligned} \quad (2.15)$$

where ${}_1F_2(a; b, c; z)$ is the generalized hypergeometric function. For integer values $\alpha = k \in \mathbb{N}$, the suggested correspondence principle holds on the space of entire functions [53, 54].

For non-integer orders, we have an algebraic correspondence of difference and derivatives in the form: “The exact fractional differences ${}^T\Delta^\alpha$ should satisfy the same algebraic characteristic rules as the fractional derivatives ${}^T D^\alpha$ on some function space $F_\alpha(\mathbb{R})$ ”. However, the following questions are open. What is algebraic characteristic rule of fractional derivatives? What is function space $F_\alpha(\mathbb{R})$? Whether the space $F_\alpha(\mathbb{R})$ is a space of entire functions for integer values of α ? In addition, the following identity holds ([53, 54]) for integer orders $\alpha > 0$,

$$\lim_{h \rightarrow 0} \frac{{}^T\Delta_h^\alpha f(x)}{h^\alpha} = {}^T D^\alpha f(x)$$

for entire functions $f(x)$. Is it true identity for non-integer orders $\alpha > 0$?

3. Is it possible a fractional generalization of the differential geometry?

Let us give an explanation of this open problem and let us note possible ways to solve it.

The main problem of building a self-consistent formulation of a fractional generalization of the differential geometry is a description admissible coordinate transformations. For the simplest case the coordinate transformations are based on the chain rule for the first order derivative

$$D_x^1 f(g(x)) = (D_g^1 f(g))_{g=g(x)} D_x^1 g(x), \tag{2.16}$$

where D_x^1 is the standard derivative of first order. The standard chain rule (2.16) can be considered as a simplest form of coordinate transformation for one-dimensional case. At the same time, the chain rule for the derivative of integer order $n \in \mathbb{N}$ has the form

$$D_x^n f(g(x)) = n! \sum_{m=1}^n (D_g^m f(g))_{g=g(x)} \sum \prod_{r=1}^n \frac{1}{a_r!} \left(\frac{D_x^r g(x)}{r!} \right)^{a_r}, \tag{2.17}$$

where \sum extends over all combinations of non-negative integer values of a_1, a_2, \dots, a_k such that $\sum_{r=1}^k r a_r = k$ and $\sum_r a_r = m$. Equation (2.17) is the Faá di Bruno's formula [12]. For analytic functions $f(x)$ on the interval (a, b) , which can be represented as a convergent power series on (a, b) , the left-sided Riemann-Liouville fractional derivative of order $\alpha > 0$ can be represented (see Lemma 15.3 of [48]) in the form

$${}^{RL}D_{a+}^\alpha f(x) = \sum_{n=0}^\infty \binom{\alpha}{n} \frac{(x-a)^{n-\alpha}}{\Gamma(n+1-\alpha)} D_x^n f(x), \quad x \in (a, b), \tag{2.18}$$

where $D_x^n f(x)$ is the standard derivative of order $n \in \mathbb{N}$ and $D_x^0 f(x) = f(x)$. Substitution of (2.17) into (2.18) gives the the chain rule for fractional derivative (see also Eq. 2.209 in Sect. 2.7.3 of [44]). As a result, we have the following correspondence principle with integer case: *For integer orders, the allowable coordinate transformations of fractional differential geometry should give transformations that are described by the Bruno formula (2.17)*. Because there is the precise fractional chain rule, we assume that it is possible to determine the allowable coordinate transformation and build a new kind of geometry ("fractional differential geometry"), which is based on this rule.

In my opinion, a self-consistent and correct fractional generalization of the differential geometry should be given on the basis of the geometry of infinite jet bundles [49, 36]. It can be formulated by using the concept of the infinite jets of functions. Fractional differential geometry can be considered as a special type of the geometry of infinite jets, [55].

3. Editors' Note

We have reported the points of view of a number of researchers that have been submitted for the ICFDA16 Round Table on “Fractional Calculus: D'où venons-nous? Que sommes-nous? Où allons-nous?”, without any our judgment on them. The points of view (perspectives) are referring to only those received by the organizers at the time of ICFDA16 or someday later. We are sorry for any forgotten item, without our willing.

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Acknowledgments

Special thanks are due to Professor Dragan Spasić, *Chair of the National Organizing Committee*, and his team, for the hard work and efficient organization of the conference and making the participants' stay fruitful and pleasant.

We express participants' thanks to the *local co-organizers*: University of Novi Sad, Faculty of Technical Sciences, Department of Mechanics; Serbian Society of Mechanics; Serbian Academy of Sciences and Arts – Branch in Novi Sad; and Ministry of Education, Science and Technological Development of the Republic of Serbia, for hosting and organization of ICFDA 2016, and to the *international organizations supporting the conference*: International Federation of Automatic Control (IFAC), International Union of Theoretical and Applied Mechanics (IUTAM) and Institute of Electrical and Electronics Engineers (IEEE) - Branch of Serbia and Montenegro.

For the 3rd and 4th authors V. Kiryakova and T. Atanacković, this work is part of the 2016' working program on the Bilateral Research Project: "Mathematical Modeling by Means of Integral Transform Methods, Partial Differential Equations, Special and Generalized Functions, Numerical Analysis" between Bulgarian Academy of Sciences (BAS) and Serbian Academy of Sciences and Arts (SASA).

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Received: September 4, 2016

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Please cite to this paper as published in:

Fract. Calc. Appl. Anal., Vol. **19**, No 5 (2016), pp. 1074–1104,
DOI: 10.1515/fca-2016-0059