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Review Article

A Review of Definitions for Fractional Derivatives and Integral

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This paper presents a review of definitions of fractional order derivatives and integrals that appear in mathematics, physics, and engineering.

1. Introduction

In 1695, l'Hôpital sent a letter to Leibniz. In his message, an important question about the order of the derivative emerged: What might be a derivative of order 1/2? In a prophetic answer, Leibniz foresees the beginning of the area that nowadays is named fractional calculus (FC). In fact, FC is as old as the traditional calculus proposed independently by Newton and Leibniz [1–4].

In the classical calculus, the derivative has an important geometric interpretation; namely, it is associated with the concept of tangent, in opposition to what occurs in the case of FC. This difference can be seen as a problem for the slow progress of FC up to 1900. After Leibniz, it was Euler (1738) [3] that noticed the problem for a derivative of noninteger order. Fourier (1822) [3, 5] suggested an integral representation in order to define the derivative, and his version can be considered the first definition for the derivative of arbitrary (positive) order. Abel (1826) [3, 5] solved an integral equation associated with the tautochrone problem, which is considered to be the first application of FC. Liouville (1832) [3, 5] suggested a definition based on the formula for differentiating the exponential function. This expression is known as the first Liouville definition. The second definition formulated by Liouville is presented in terms of an integral and is now called the version by Liouville for the integration of noninteger order. After a series of works by Liouville, the most important paper was published by Riemann [6], ten years after his death. We also note that both Liouville and Riemann formulations carry with them the so-called complementary function, a problem to be solved. Grünwald [7] and Letnikov [8], independently, developed an approach to noninteger order derivatives in terms of a convenient convergent series, conversely to the Riemann-Liouville approach, that is given by an integral. Letnikov showed that his definition coincides with the versions formulated by Liouville, for particular values of the order, and by Riemann, under a convenient interpretation of the so-called noninteger order difference. Hadamard (1892) [5] published a paper where the noninteger order derivative of an analytical function must be done in terms of its Taylor series.

After 1900, the FC experiences a fast development and, in an attempt to formulate particular problems, other definitions were proposed. We mention some of them. Weyl [9] introduced a derivative in order to circumvent a problem involving a particular class of functions, the periodic functions. Riesz [10, 11] proved the mean value theorem for fractional integrals and introduced another formulation that is associated with the Fourier transform. Marchaud (1927) [3, 5] introduced a new definition for noninteger order of derivatives. This definition coincides with the Liouville version for "sufficiently good" functions. Erdélyi-Kober (1940) [3, 5] presented a distinct definition for noninteger order of integration that is useful in applications involving integral and differential equations. Caputo (1967) [12] formulated a definition, more restrictive than the Riemann-Liouville but more appropriate to discuss problems involving a fractional differential equation with initial conditions [13–21].

Due to the importance of the Caputo version, we will compare this approach with the Riemann-Liouville formulation. The definition as proposed by Caputo inverts the order of integral and derivative operators with the noninteger order derivative of the Riemann-Liouville. We summarize the difference between these two formulations. In the Caputo: first the calculate derivative of integer order and after calculate the integral of noninteger order. In the Riemann-Liouville: first calculate the integral of noninteger order and after calculate the derivative of integer order. It is important to cite that the Caputo derivative is useful to affront problems where initial conditions are done in the function and in the respective derivatives of integer order.

After the first congress at the University of New Haven, in 1974, FC has developed and several applications emerged in many areas of scientific knowledge. As a consequence, distinct approaches to solve problems involving the derivative were proposed and distinct definitions of the fractional derivative are available in the literature. This paper presents in a systematic form the existing formulations of fractional derivatives and integrals. We should mention also that we can have several alternative expressions for the same definition. Therefore, we present only those more representative and we cite particular papers [22–32] and books [33–40] that we believe are the most relevant. Furthermore, the paper does not focus on the pros and cons of each definition and does not address the support of the function that is to be differentiated or integrated.

The paper is organized as follows. Section 2 presents the adopted notation. Sections 3 and 4 list the proposed definitions of fractional derivatives and integrals, respectively. Finally, Section 5 outlines some brief remarks.

2. Notation

The following remarks clarify the notation used in the sequel in Sections 3 and 4.

(i) Let $\alpha \in \mathbb{C} : \Re(\alpha) \in (n-1, n]$, $n \in \mathbb{N}$, where $\Re(\cdot)$ denotes the real part of complex number.

(ii) Let $[a, b]$ be a finite interval in $\mathbb{R}$, $k \in \mathbb{N}$, $\nu > 0$, and $\mathtt{f}(0) \equiv f(0^+) - f(0^-)$.

(iii) The floor function, denoted by $\lfloor\cdot\rfloor$, is defined as $\lfloor x\rfloor = \max\{z \in \mathbb{Z} : z \leq x\}$.

(iv) $[\alpha]$ is the integer part of number α and $\{\alpha\}$ the fractional part, $0 \leq \{\alpha\} < 1$, so that $\alpha = [\alpha] + \{\alpha\}$.

(v) $\Delta^{\alpha}[f(x) - f(x_0)] \simeq \Gamma(1+\alpha)\Delta[f(x) - f(x_0)]$.

(vi) $\alpha(\cdot,\cdot)$ is the variable fractional order with $0 < \alpha(x, t) < 1$ and $(x, t) \in [a, b]$. $\alpha(x)$ is a continuous function on $(0, 1]$.

(vii) $\mathscr{C}(a, z^+)$ is a closed contour, in the complex plane, starting at $\xi = a$, encircling $\xi = z$ once in the positive sense, and returning to $\xi = a$. $\mu, \nu \in \mathbb{R}/0$, with $0 < \mu < 1$ and $0 \leq \nu \leq 1$.

(viii) Consider $z \in \mathbb{C}$ and $k \in \mathbb{R}$. The so-called k-gamma function, denoted by $\Gamma_k(z)$, is related to the classical gamma function by means of $\Gamma_k(z) = k^{z/k-1}\Gamma(z/k)$.

(ix) The so-called k-Pochhammer symbol yields $(z)_{n,k} = \Gamma_k(x + nk)/\Gamma_k(x)$.

(x) The k-fractional Hilfer derivative recovers, as particular cases, the fractional Riemann-Liouville derivative if $\nu = 0$ and $k = 1$ and the fractional Caputo derivative if $\nu = 1 = k$ [41].

3. Definitions of Fractional Derivatives

Liouville derivative:

$$\mathrm{D}^{\alpha}[f(x)] = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}x}\int_{-\infty}^{x}(x-\xi)^{-\alpha}f(\xi)\,\mathrm{d}\xi, \quad -\infty < x < +\infty. \tag{1}$$

Liouville left-sided derivative:

$$\mathrm{D}_{0^+}^{\alpha}[f(x)] = \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_{0}^{x}(x-\xi)^{-\alpha+n-1}f(\xi)\,\mathrm{d}\xi, \quad x > 0. \tag{2}$$

Liouville right-sided derivative:

$$\mathrm{D}_{-}^{\alpha}[f(x)] = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_{x}^{\infty}(x-\xi)^{-\alpha+n-1}f(\xi)\,\mathrm{d}\xi, \quad x < \infty. \tag{3}$$

Riemann-Liouville left-sided derivative:

$${}^{\mathrm{RL}}\mathrm{D}_{a^+}^{\alpha}[f(x)] = \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_{a}^{x}(x-\xi)^{n-\alpha-1}f(\xi)\,\mathrm{d}\xi, \quad x \geq a. \tag{4}$$

Riemann-Liouville right-sided derivative:

$${}^{\mathrm{RL}}\mathrm{D}_{b^-}^{\alpha}[f(x)] = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_{x}^{b}(\xi-x)^{n-\alpha-1}f(\xi)\,\mathrm{d}\xi, \quad x \leq b. \tag{5}$$

Caputo left-sided derivative:

$${}_{*}\mathrm{D}_{a^+}^{\alpha}[f(x)] = \frac{1}{\Gamma(n-\alpha)}\int_{a}^{x}(x-\xi)^{n-\alpha-1}\frac{\mathrm{d}^n}{\mathrm{d}\xi^n}[f(\xi)]\,\mathrm{d}\xi, \quad x \geq a. \tag{6}$$

Caputo right-sided derivative:

$${}_{*}\mathrm{D}_{b^-}^{\alpha}[f(x)] = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_{x}^{b}(\xi-x)^{n-\alpha-1}\frac{\mathrm{d}^n}{\mathrm{d}\xi^n}[f(\xi)]\,\mathrm{d}\xi, \quad x \leq b. \tag{7}$$

Grünwald-Letnikov left-sided derivative:

$$ {}^{\mathrm{GL}}\mathrm{D}_{a^{+}}^{\alpha}\left[f(x)\right] = \lim_{h\to 0}\frac{1}{h^{\alpha}}\sum_{k=0}^{\lfloor n\rfloor}(-1)^{k}\frac{\Gamma(\alpha+1)\,f(x-kh)}{\Gamma(k+1)\,\Gamma(\alpha-k+1)}, \quad nh = x-a. \tag{8} $$

Grünwald-Letnikov right-sided derivative:

$$ {}^{\mathrm{GL}}\mathrm{D}_{b^{-}}^{\alpha}\left[f(x)\right] = \lim_{h\to 0}\frac{1}{h^{\alpha}}\sum_{k=0}^{\lfloor n\rfloor}(-1)^{k}\frac{\Gamma(\alpha+1)\,f(x+kh)}{\Gamma(k+1)\,\Gamma(\alpha-k+1)}, \quad nh = b-x. \tag{9} $$

Weyl derivative:

$$ {}_{x}\mathrm{D}_{\infty}^{\alpha}\left[f(x)\right] = \mathrm{D}_{-}^{\alpha}\left[f(x)\right] = (-1)^{m}\left(\frac{\mathrm{d}}{\mathrm{d}\xi}\right)^{n}\left[{}_{x}\mathrm{W}_{\infty}^{\alpha}\left[f(x)\right]\right]. \tag{10} $$

Marchaud derivative:

$$ \mathrm{D}_{+}^{\alpha}\left[f(x)\right] = \frac{\alpha}{\Gamma(1-\alpha)}\int_{-\infty}^{x}\frac{f(x)-f(\xi)}{(x-\xi)^{1+\alpha}}\mathrm{d}\xi. \tag{11} $$

Marchaud left-sided derivative:

$$ \mathrm{D}_{+}^{\alpha}\left[f(x)\right] = \frac{\alpha}{\Gamma(1-\alpha)}\int_{0}^{\infty}\frac{f(x)-f(x-\xi)}{\xi^{1+\alpha}}\mathrm{d}\xi. \tag{12} $$

Marchaud right-sided derivative:

$$ \mathrm{D}_{-}^{\alpha}\left[f(x)\right] = \frac{\alpha}{\Gamma(1-\alpha)}\int_{0}^{\infty}\frac{f(x)-f(x+\xi)}{\xi^{1+\alpha}}\mathrm{d}\xi. \tag{13} $$

Hadamard derivative [42]:

$$ \mathrm{D}_{+}^{\alpha}\left[f(x)\right] = \frac{\alpha}{\Gamma(1-\alpha)}\int_{0}^{x}\frac{f(x)-f(\xi)}{\left[\ln(x/\xi)\right]^{1+\alpha}}\frac{\mathrm{d}\xi}{\xi}. \tag{14} $$

Chen left-sided derivative:

$$ \mathrm{D}_{\mathtt{c}}^{\alpha}\left[f(x)\right] = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}x}\int_{\mathtt{c}}^{x}(x-\xi)^{-\alpha}f(\xi)\,\mathrm{d}\xi, \quad x > \mathtt{c}. \tag{15} $$

Chen right-sided derivative:

$$ \mathrm{D}_{\mathtt{c}}^{\alpha}\left[f(x)\right] = -\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}x}\int_{x}^{\mathtt{c}}(\xi-x)^{-\alpha}f(\xi)\,\mathrm{d}\xi, \quad x < \mathtt{c}. \tag{16} $$

Davidson-Essex derivative [15]:

$$ \mathrm{D}_{0}^{\alpha}\left[f(x)\right] = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}^{n+1-k}}{\mathrm{d}x^{n+1-k}} \times\int_{0}^{x}(x-\xi)^{-\alpha}\frac{\mathrm{d}^{k}}{\mathrm{d}\xi^{k}}\left[f(\xi)\right]\mathrm{d}\xi. \tag{17} $$

Coimbra derivative [43–45]:

$$ \mathrm{D}_{0}^{\alpha(x)}\left[f(x)\right] = \frac{1}{\Gamma(1-\alpha(x))} \times\left\{\int_{0}^{x}(x-\xi)^{-\alpha(x)}\frac{\mathrm{d}}{\mathrm{d}\xi}\left[f(\xi)\right]\mathrm{d}\xi + \mathtt{f}(0)\,x^{-\alpha(x)}\right\}. \tag{18} $$

Canavati derivative:

$$ {}_{a}\mathrm{D}_{x}^{\nu}\left[f(x)\right] = \frac{1}{\Gamma(1-\mu)}\frac{\mathrm{d}}{\mathrm{d}x}\int_{0}^{x}(x-\xi)^{\mu}\frac{\mathrm{d}^{n}}{\mathrm{d}\xi^{n}}\left[f(\xi)\right]\mathrm{d}\xi, \quad n = \lfloor\nu\rfloor, \quad \mu = n-\nu. \tag{19} $$

Jumarie derivative, $n = 1$:

$$ \mathrm{D}_{x}^{\alpha}\left[f(x)\right] = \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^{n}}{\mathrm{d}x^{n}} \times\int_{0}^{x}(x-\xi)^{n-\alpha-1}\left[f(\xi)-f(0)\right]\mathrm{d}\xi. \tag{20} $$

Riesz derivative:

$$ \mathrm{D}_{x}^{\alpha}\left[f(x)\right] = -\frac{1}{2\cos(\alpha\pi/2)}\frac{1}{\Gamma(\alpha)}\frac{\mathrm{d}^{n}}{\mathrm{d}x^{n}} \cdot\left\{\int_{-\infty}^{x}(x-\xi)^{n-\alpha-1}f(\xi)\,\mathrm{d}\xi + \int_{x}^{\infty}(\xi-x)^{n-\alpha-1}f(\xi)\,\mathrm{d}\xi\right\}. \tag{21} $$

Cossar derivative:

$$ \mathrm{D}_{-}^{\alpha}\left[f(x)\right] = -\frac{1}{\Gamma(1-\alpha)}\lim_{N\to\infty}\frac{\mathrm{d}}{\mathrm{d}x}\int_{x}^{N}(\xi-x)^{-\alpha}f(\xi)\,\mathrm{d}\xi. \tag{22} $$

Local fractional Yang derivative [40]:

$$ \mathrm{D}_{-}^{\alpha}\left[f(x)\right]\Big|_{x=x_{0}} = \lim_{x\to x_{0}}\frac{\Delta^{\alpha}\left[f(x)-f(x_{0})\right]}{(x-x_{0})^{\alpha}}. \tag{23} $$

Left Riemann-Liouville derivative of variable fractional order:

$$ {}_{a}\mathrm{D}_{x}^{\alpha(\cdot,\cdot)}\left[f(x)\right] = \frac{\mathrm{d}}{\mathrm{d}x}\int_{a}^{x}(x-\xi)^{-\alpha(\xi,x)}f(\xi)\,\frac{\mathrm{d}\xi}{\Gamma\left[1-\alpha(\xi,x)\right]}. \tag{24} $$

Right Riemann-Liouville derivative of variable fractional order:

$$ {}_{x}\mathrm{D}_{b}^{\alpha(\cdot,\cdot)}\left[f(x)\right] = \frac{\mathrm{d}}{\mathrm{d}x}\int_{x}^{b}(\xi-x)^{-\alpha(\xi,x)}f(\xi)\,\frac{\mathrm{d}\xi}{\Gamma\left[1-\alpha(\xi,x)\right]}. \tag{25} $$

Left Caputo derivative of variable fractional order:

$$ {}_{a}\mathrm{D}_{x}^{\alpha(\cdot,\cdot)}\left[f(x)\right] = \int_{a}^{x}(x-\xi)^{-\alpha(\xi,x)}\frac{\mathrm{d}}{\mathrm{d}\xi}f(\xi)\,\frac{\mathrm{d}\xi}{\Gamma\left[1-\alpha(\xi,x)\right]}. \tag{26} $$

Right Caputo derivative of variable fractional order:

$$ {}_{x}\mathrm{D}_{b}^{\alpha(\cdot,\cdot)}[f(x)] = \int_{x}^{b} (\xi - x)^{-\alpha(\xi,x)} \frac{\mathrm{d}}{\mathrm{d}\xi} f(\xi) \frac{\mathrm{d}\xi}{\Gamma[1-\alpha(\xi,x)]}. \tag{27} $$

Caputo derivative of variable fractional order:

$$ {}_{*}\mathrm{D}_{x}^{\alpha(x)}[f(x)] = \frac{1}{\Gamma(1-\alpha(x))} \int_{0}^{x} (x-\xi)^{-\alpha(\xi,x)} \frac{\mathrm{d}}{\mathrm{d}\xi} f(\xi)\,\mathrm{d}\xi. \tag{28} $$

Modified Riemann-Liouville fractional derivative:

$$ \mathrm{D}^{\alpha}[f(x)] = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \int_{0}^{x} (x-\xi)^{-\alpha}[f(\xi) - f(0)]\,\mathrm{d}\xi. \tag{29} $$

Osler fractional derivative [46]:

$$ {}_{a}\mathrm{D}_{z}^{\alpha} f(z) = \frac{\Gamma(\alpha+1)}{2\pi i} \int_{\mathscr{C}(a,z^{+})} \frac{f(\xi)}{(\xi - z)^{1+\alpha}} \mathrm{d}\xi. \tag{30} $$

k-fractional Hilfer derivative [41]:

$$ {}^{k}\mathrm{D}^{\mu,\nu} f(x) = \mathrm{I}_{k}^{\nu(1-\mu)} \frac{\mathrm{d}}{\mathrm{d}x} \mathrm{I}_{k}^{(1-\mu)(1-\nu)} f(x). \tag{31} $$

4. Definitions of Fractional Integrals

Riemann-Liouville left-sided integral:

$$ {}^{\mathrm{RL}}\mathrm{I}_{a^{+}}^{\alpha}[f(x)] = \frac{1}{\Gamma(\alpha)} \int_{a}^{x} (x-\xi)^{\alpha-1} f(\xi)\,\mathrm{d}\xi, \quad x \geq a. \tag{32} $$

Riemann-Liouville right-sided integral:

$$ {}^{\mathrm{RL}}\mathrm{I}_{b^{-}}^{\alpha}[f(x)] = \frac{1}{\Gamma(\alpha)} \int_{x}^{b} (\xi - x)^{\alpha-1} f(\xi)\,\mathrm{d}\xi, \quad x \leq b. \tag{33} $$

Hadamard integral:

$$ \mathrm{I}_{+}^{\alpha}[f(x)] = \frac{1}{\Gamma(\alpha)} \int_{0}^{x} \frac{f(\xi)}{[\ln(\xi/x)]^{1-\alpha}} \cdot \frac{\mathrm{d}\xi}{\xi}, \quad x > 0,\ \alpha > 0. \tag{34} $$

Weyl integral:

$$ {}_{x}\mathrm{W}_{\infty}^{\alpha}[f(x)] = \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} (\xi - x)^{\alpha-1} f(\xi)\,\mathrm{d}\xi. \tag{35} $$

Chen left-sided integral:

$$ \mathrm{I}_{\mathrm{c}}^{\alpha}[f(x)] = \frac{1}{\Gamma(\alpha)} \int_{\mathrm{c}}^{x} (x-\xi)^{\alpha-1} f(\xi)\,\mathrm{d}\xi, \quad x > \mathrm{c}. \tag{36} $$

Chen right-sided integral:

$$ \mathrm{I}_{\mathrm{c}}^{\alpha}[f(x)] = \frac{1}{\Gamma(\alpha)} \int_{x}^{\mathrm{c}} (\xi - x)^{\alpha-1} f(\xi)\,\mathrm{d}\xi, \quad x < \mathrm{c}. \tag{37} $$

Cossar integral [47]:

$$ \mathrm{I}_{\mathrm{c}}^{\alpha}[f(x)] = \frac{1}{\Gamma(\alpha)} \int_{\mathrm{c}}^{x} (x-\xi)^{\alpha-1} f(\xi)\,\mathrm{d}\xi, \quad x > \mathrm{c}. \tag{38} $$

Erdélyi (left-sided) integral:

$$ \mathrm{I}_{\sigma,\eta}^{\alpha}[f(x)] = \frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \int_{0}^{x} (x^{\sigma} - \xi^{\sigma})^{\alpha-1} \xi^{\sigma\eta+\sigma-1} f(\xi)\,\mathrm{d}\xi. \tag{39} $$

Erdélyi (right-sided) integral:

$$ \mathrm{I}_{\sigma,\eta}^{\alpha}[f(x)] = \frac{\sigma x^{\sigma\alpha}}{\Gamma(\alpha)} \int_{x}^{\infty} (\xi^{\sigma} - x^{\sigma})^{\alpha-1} \xi^{\sigma(1-\alpha-\eta)-1} f(\xi)\,\mathrm{d}\xi. \tag{40} $$

Kober (left-sided) integral:

$$ \mathrm{I}_{1,\eta}^{\alpha}[f(x)] = \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)} \int_{0}^{x} (x-\xi)^{\alpha-1} \xi^{\eta} f(\xi)\,\mathrm{d}\xi. \tag{41} $$

Kober (right-sided) integral:

$$ \mathrm{I}_{1,\eta}^{\alpha}[f(x)] = \frac{x^{\eta}}{\Gamma(\alpha)} \int_{x}^{\infty} (\xi - x)^{\alpha-1} \xi^{-\alpha-\eta} f(\xi)\,\mathrm{d}\xi. \tag{42} $$

Local fractional Yang integral:

$$ {}_{a}\mathrm{I}_{b}^{\alpha}[f(x)] = \frac{1}{\Gamma(1+\alpha)} \int_{a}^{b} f(\xi)(\mathrm{d}\xi)^{\alpha}. \tag{43} $$

Left Riemann-Liouville integral of variable fractional order:

$$ {}_{a}\mathrm{I}_{x}^{\alpha(\cdot,\cdot)}[f(x)] = \int_{a}^{x} (\xi - x)^{\alpha(\xi,x)-1} f(\xi) \frac{\mathrm{d}\xi}{\Gamma[\alpha(\xi,x)]}. \tag{44} $$

Right Riemann-Liouville integral of variable fractional order:

$$ {}_{x}\mathrm{I}_{b}^{\alpha(\cdot,\cdot)}[f(x)] = \int_{x}^{b} (x-\xi)^{\alpha(\xi,x)-1} f(\xi) \frac{\mathrm{d}\xi}{\Gamma[\alpha(\xi,x)]}. \tag{45} $$

k-fractional Hilfer integral:

$$ \mathrm{I}_{k}^{\alpha} f(x) = \frac{1}{k\Gamma_{k}(\alpha)} \int_{0}^{x} (x-\xi)^{\alpha/k-1} f(\xi)\,\mathrm{d}\xi. \tag{46} $$

5. Some Remarks

Remark 1. If D^{α} is any fractional derivative, the Miller-Ross sequential derivative of order $k\alpha, k \in \mathbb{Z}$, is given by [3]

$$ \mathscr{D}^{\alpha} = D^{\alpha}, \qquad \mathscr{D}^{k\alpha} = D^{\alpha}\mathscr{D}^{(k-1)\alpha}. \tag{47} $$

Remark 2. Whatever the definition employed, $\mathrm{I}^{0} f(x) = \mathrm{D}^{0} f(x) = f(x)$.

Remark 3. Some authors do not distinguish the definition employed by means of a superscript (GL, RL, C, and L) but use different fonts for the operator instead (D, D, $\mathbf{D}$, $\mathfrak{D}$, and $\mathscr{D}$). The particular correspondence between fonts and definitions varies. Very often no indication at all is given, save perhaps in the accompanying text, and the reader is presumed to understand from the context which particular definition is intended.

Remark 4. In the literature, several alternative notations for operator D may be found:

$$\begin{aligned}
\mathrm{D}_{a+}^{\alpha} f(x) &= \left(\mathrm{D}_{a+}^{\alpha} f\right)(x) = {}_{a}\mathrm{D}_{x}^{\alpha} f(x) = {}_{a}\mathrm{I}_{x}^{-\alpha} f(x) \\
&= \mathrm{D}_{x-a}^{\alpha} f(x) = \frac{\mathrm{d}^{\alpha} f(x)}{\mathrm{d}(x-a)^{\alpha}}, \\
\mathrm{D}_{b-}^{\alpha} f(x) &= \left(\mathrm{D}_{b-}^{\alpha} f\right)(x) = {}_{x}\mathrm{D}_{b}^{\alpha} f(x) = {}_{x}\mathrm{I}_{b}^{-\alpha} f(x) \\
&= \mathrm{D}_{b-x}^{\alpha} f(x) = \frac{\mathrm{d}^{\alpha} f(x)}{\mathrm{d}(b-x)^{\alpha}}.
\end{aligned} \tag{48}$$

Only one of the two operators I and D needs to be used, since it is all a matter of changing the sign of α. In practice, D is the one more often used.

Remark 5. In the expressions for the right and left Liouville fractional derivatives (2) and (3), respectively, some authors have a slight distinct expression, instead of 0^+ just + and at the lower limit $-\infty$.

Remark 6. We can mention the "difference of fractional order," discussed by Bosanquet [48], and the "Ruscheweyh Derivative," presented in [42, 49–51].

Remark 7. The authors' intention is not to discuss pros and cons of the list of definitions of fractional derivatives and integrals in Sections 3 and 4. Having in mind that the reader can find benefits in applying the correct definition for his/her specific research interest, it can be said that the most used definitions are the Riemann-Liouville (e.g., in calculus), the Caputo (e.g., in physics and numerical integration), and the Grünwald-Letnikov (e.g., in signal processing, engineering, and control). The problem of initialization plays an important role in applied sciences and, consequently, various definitions are occasionally adopted within the scope of specific topics, but the overall problem remains to be clarified.

Remark 8. The paper does not focus on particular relations involving explicit parameters, intervals, or constants, associated with the distinct derivatives. For example, we can mention that, for $\Re(\alpha) = 0$, with $\alpha \neq 0$, the Liouville fractional derivatives are of purely imaginary order. Also, for $\alpha = n \in \mathbb{N}$, we recover the derivative of integer order. For example, $\mathrm{D}_{+}^{n}[f(x)] = f^{(n)}(x)$ and $\mathrm{D}_{-}^{n}[f(x)] = (-1)^{n} f^{(n)}(x)$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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