



A Publication of the International Association  
of Science and Technology for Development

550

Proceedings of the 26th IASTED International Conference on

# Modelling, Identification, and Control



Editor: L. Bruzzone

February 12 – 14, 2007  
Innsbruck, Austria

ISBN: 978-0-88986-635-5



## FRACTIONAL DYNAMICS AND CONTROL OF HEAT DIFFUSION SYSTEMS

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### ABSTRACT

In this paper we study a heat diffusion system on a fractional calculus perspective. Bearing these ideas in mind, several fractional PID tuning methodologies are investigated and compared. The simulations demonstrate the good performance of the proposed fractional-order algorithm.

### KEY WORDS

Fractional calculus, Control, Diffusion Systems.

### 1. Introduction

Fractional calculus (FC) is a generalization of integration and differentiation to a non-integer order  $\alpha \in \mathbb{C}$ , being the fundamental operator  ${}_a D_t^\alpha$ , where  $a$  and  $t$  are the limits of the operation [1], [2].

In the last years, FC has been used increasingly to model the constitutive behavior of materials and physical systems exhibiting hereditary and memory properties. This is the main advantage of fractional derivatives in comparison with classical integer models, where these effects are simply neglected. It is well-known that the fractional-order operator  $s^{0.5}$  appears in several types of problems. The transmission lines, heat flow or the diffusion of neutrons in a nuclear reactor are examples where the half-operator is the fundamental element. On the other hand, diffusion is one of the three fundamental partial differential equations of mathematical physics [3].

In this paper we investigate the heat diffusion system in the perspective of applying the FC theory. A fractional-order PID algorithm is presented and compared with the classical scheme. The fractional-order  $PI^\alpha D^\beta$  controller involves an integrator of order  $\alpha \in \mathbb{R}^+$  and a differentiator of order  $\beta \in \mathbb{R}^+$ .

Bearing these ideas in mind, the paper is organized as follows. Section 2 gives the fundamentals of fractional-order control systems. Section 3 introduces the heat diffusion system and describes its simulation. Section 4 points out several control strategies for the heat system and discusses the results. Finally, section 5 draws the main

conclusions and addresses perspectives towards future developments.

### 2. Fractional-Order Control Systems

Fractional controllers are characterized by differential equations that have, in the dynamical system and/or in the control algorithm, an integral and/or a derivative of fractional-order. Due to the fact that these operators are defined by irrational continuous transfer functions, in the Laplace domain, or infinite dimensional discrete transfer functions, in the  $Z$  domain, we often encounter evaluation problems in the simulations. Therefore, when analyzing fractional systems, we usually adopt continuous or discrete integer-order approximations of fractional-order operators.

The mathematical definition of a fractional derivative and integral has been the subject of several different approaches [1], [2]. One commonly used definition is given by the Riemann-Liouville expression ( $\alpha > 0$ ):

$${}_a D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \quad n-1 < \alpha < n \quad (1)$$

where  $f(t)$  is the applied function and  $\Gamma(x)$  is the Gamma function of  $x$ . Another widely used definition is given by the Grünwald-Letnikov approach ( $\alpha \in \mathbb{R}$ ):

$${}_a D_t^\alpha f(t) = \lim_{h \rightarrow 0} \frac{1}{h^\alpha} \sum_{k=0}^{\lfloor \frac{t-a}{h} \rfloor} (-1)^k \binom{\alpha}{k} f(t-kh) \quad (2)$$

where  $h$  is the time increment and  $[x]$  means the integer part of  $x$ .

The “memory” effect of these operators is demonstrated by (1) and (2), where the convolution integral in (1) and the infinite series in (2), reveal the unlimited memory of these operators, ideal for modeling hereditary and memory properties in physical systems and materials.

An alternative definition to (1) and (2), which reveals useful for the analysis of fractional-order control systems, is given by the Laplace transform method. Considering vanishing initial conditions, the fractional *differintegration* is defined in the Laplace domain,  $F(s) = L\{f(t)\}$ , as:

$$L\left\{{}_a D_t^\alpha f(t)\right\} = s^\alpha F(s), \quad \alpha \in \mathfrak{R} \quad (3)$$

An important aspect of fractional-order algorithms can be illustrated through the elemental control system, with open-loop transfer function  $G(s) = Ks^{-\alpha}$  ( $1 < \alpha < 2$ ) in the forward path. The open-loop Bode diagrams of amplitude and phase have correspondingly a slope of  $-20\alpha$  dB/dec and a constant phase of  $-\alpha\pi/2$  rad over the entire frequency domain. Therefore, the closed-loop system has a constant phase margin of  $\text{PM} = \pi(1 - \alpha/2)$  rad, that is independent of the system gain  $K$ , and the closed-loop system is robust against gain variations exhibiting step responses with an iso-damping property [4].

In this paper we adopt discrete integer-order approximations to the fundamental element  $s^\alpha$  ( $\alpha \in \mathfrak{R}$ ) of a fractional-order control (FOC) strategy. The usual approach for obtaining discrete equivalents of continuous operators of type  $s^\alpha$  adopts the Euler, Tustin and Al-Alaoui generating functions.

It is well known that rational-type approximations frequently converge faster than polynomial-type approximations and have a wider domain of convergence in the complex domain. Thus, by using the Euler operator  $w(z^{-1}) = (1 - z^{-1})/T$ , and performing a power series expansion of  $[w(z^{-1})]^\alpha = [(1 - z^{-1})/T]^\alpha$  gives the discretization formula corresponding to the Grünwald-Letnikov definition (2):

$$D^\alpha(z^{-1}) = \left(\frac{1 - z^{-1}}{T}\right)^\alpha = \sum_{k=0}^{\infty} h^\alpha(k) z^{-k} \quad (4)$$

$$h^\alpha(k) = \left(\frac{1}{T}\right)^\alpha \binom{k - \alpha - 1}{k} \quad (5)$$

A rational-type approximation can be obtained by applying the Padé approximation method to the impulse response sequence (5)  $h^\alpha(k)$ , yielding the discrete transfer function:

$$H(z^{-1}) = \frac{b_0 + b_1 z^{-1} + \dots + b_m z^{-m}}{1 + a_1 z^{-1} + \dots + a_n z^{-n}} = \sum_{k=0}^{\infty} h(k) z^{-k} \quad (6)$$

where  $m \leq n$  and the coefficients  $a_k$  and  $b_k$  are determined by fitting the first  $m+n+1$  values of  $h^\alpha(k)$  into the impulse response  $h(k)$  of the desired approximation  $H(z^{-1})$ . Thus, we obtain an approximation that has a perfect match to the desired impulse response  $h^\alpha(k)$  for the first  $m+n+1$  values of  $k$ . Note that the above Padé approximation is obtained by considering the Euler operator but the determination process will be exactly the same for other types of discretization schemes.

### 3. Heat Diffusion

The heat diffusion is governed by a linear unidimensional partial differential equation (PDE) of the form:

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \quad (7)$$

where  $k$  is the diffusivity,  $t$  is the time,  $u$  is the temperature and  $x$  is the space coordinate. The system (7) involves the solution of a PDE of parabolic type for which the standard theory guarantees the existence of a unique solution [5].

For the case of a planar perfectly isolated surface we usually apply a constant temperature  $U_0$  at  $x=0$  and analyzes the heat diffusion along the horizontal coordinate  $x$ . Under these conditions, the heat diffusion phenomenon is described by a non-integer order model:

$$U(x, s) = \frac{U_0}{s} G(s), \quad G(s) = e^{-x\sqrt{\frac{s}{k}}} \quad (8)$$

where  $x$  is the space coordinate,  $U_0$  is the boundary condition and  $G(s)$  is the system transfer function.

In our study, the simulation of the heat diffusion is performed by adopting the Crank-Nicholson implicit numerical integration based on the discrete approximation to differentiation as [6]:

$$\begin{aligned} -ru[j+1, i+1] + (2+r)u[j+1, i] - ru[j+1, i-1] = \\ = ru[j, i+1] + (2-r)u[j, i] + u[j, i-1] \end{aligned} \quad (9)$$

where  $r = k\Delta t(\Delta x^2)^{-1}$ ,  $\{\Delta x, \Delta t\}$  and  $\{i, j\}$  are the increments and the integration indices for space and time, respectively [7].

### 4. Control Strategies

The generalized PID controller  $G_c(s)$  has a transfer function of the form:

$$G_c(s) = K \left[ 1 + \frac{1}{T_i s^\alpha} + T_d s^\beta \right] \quad (10)$$

where  $\alpha$  and  $\beta$  are the orders of the fractional integrator and differentiator, respectively. The constants  $K$ ,  $T_i$  and  $T_d$  are correspondingly the proportional gain, the integral time constant and the derivative time constant.

Clearly, taking  $(\alpha, \beta) = \{(1, 1), (1, 0), (0, 1), (0, 0)\}$  we get the classical {PID, PI, PD, P} controllers, respectively. The  $\text{PI}^\alpha \text{D}^\beta$  controller is more flexible and gives the possibility of adjusting more carefully the closed-loop system characteristics.

In the next two sub-sections, we analyze the system of Fig. 1 by adopting the classical integer-order PID and a fractional  $\text{PID}^\beta$ , respectively.

#### 4.1 PID Tuning Using the Ziegler-Nichols Rule

In this sub-section we analyze the closed-loop system with a conventional PID controller given by the transfer function (10) with  $\alpha = \beta = 1$ . Usually, the PID parameters ( $K$ ,  $T_i$ ,  $T_d$ ) are tuned by using the so-called Ziegler-Nichols open loop (ZNOL) method [8]. The ZNOL heuristics are based on the approximate first-order plus dead-time model:

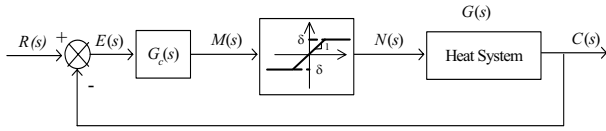


Figure 1 - Closed-loop system with PID controller  $G_c(s)$ .

$$\hat{G}(s) = \frac{K_p}{\tau s + 1} e^{-sT} \quad (11)$$

For the heat system, the resulting parameters are  $\{K_p, \tau, T\} = \{0.52, 162, 28\}$  leading to the PID constants  $\{K, T_i, T_d\} = \{18.07, 34.0, 8.5\}$ .

A step input is applied at  $x = 0.0$  m and the closed-loop response  $c(t)$  is analyzed for  $x = 3.0$  m, without actuator saturation (Fig. 2). We verify that the system with a PID controller, tuned through the ZNOL heuristics, does not produce satisfactory results giving a significant overshoot  $ov$ , a large settling time  $t_s$  and a time delay  $t_d$ , namely  $\{t_s, t_p, t_r, ov(\%), t_d\} \equiv \{27.5, 44.8, 12.0, 68.56, 3.0\}$ , where  $t_p$  represents the peak time and  $t_r$  the rise time. We analyze two indices that measure the response error, namely the integral square error (ISE) and the integral time square error (ITSE) criteria defined as:

$$ISE = \int_0^{\infty} [r(t) - c(t)]^2 dt \quad (12)$$

$$ITSE = \int_0^{\infty} t [r(t) - c(t)]^2 dt \quad (13)$$

We can use other performance criteria such as the integral absolute error (IAE) or the integral time absolute error (ITAE); however, in the present case the ISE and the ITSE criteria have produced the best results and are adopted in the study.

Another performance index consists on the energy  $E_m$  at the PID controller output  $m(t)$  given by the expression:

$$E_m = \int_0^{T_e} m^2(t) dt \quad (14)$$

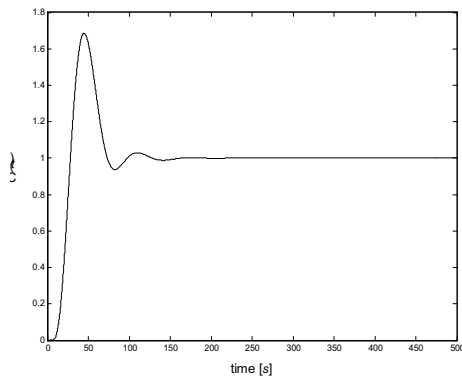


Figure 2 - Step responses of the closed-loop system for the PID controller and  $x = 3.0$  m.

where  $T_e$  is the time window needed to stabilize the systems output  $c(t)$ . In this case, the PID reveals the following values for parameters  $(ISE, ITSE, E_m) = (27.53, 613.97, 2.52 \times 10^5)$ .

The poor results indicate again that the method of tuning may not be the most adequate for the control of the heat system. In fact, the inherent fractional dynamics of the system lead us to consider other configurations. In this perspective, we propose the use of fractional controllers tuned by the minimization of the indices ISE and ITSE.

## 4.2 PID<sup>β</sup> Tuning Using Optimization Indices

In this sub-section we analyze the closed-loop system under the action of the PID<sup>β</sup> controller given by the transfer function (10) with  $\alpha = 1$  and  $0 \leq \beta \leq 1$ . The fractional derivative term  $T_d s^\beta$  in (10) is implemented through a 4<sup>th</sup>-order Padé discrete rational transfer function of type (6). It is used a sampling period of  $T = 0.1$  s.

The PID<sup>β</sup> controller is tuned by the minimization of an integral performance index. For that purpose, we adopt the ISE and ITSE criteria.

A step reference input  $R(s) = 1/s$  is applied at  $x = 0.0$  m and the output  $c(t)$  is analyzed for  $x = 3.0$  m, without actuator saturation. The heat system is simulated for 3000 seconds. Fig. 3 illustrates the variation of the fractional PID parameters  $(K, T_i, T_d)$  as function of the order's derivative  $\beta$ , for the ISE and the ITSE criteria. The dots represent the values corresponding to the classical PID addressed in the previous section.

The curves reveal that for  $\beta < 0.4$  the parameters  $(K, T_i, T_d)$  are slightly different, for the two ISE and ITSE criteria, while for  $\beta \geq 0.4$  they lead to almost similar values. This fact indicates a large influence of a weak order derivative on system's dynamics.

To further illustrate the performance of the fractional-order controllers a saturation nonlinearity is included in the closed-loop system of Fig. 1 and inserted in series with the output of the controller  $G_c(s)$ . The saturation element is defined as:

$$n(m) = \begin{cases} m, & |m| < \delta \\ \delta \text{ sign}(m), & |m| \geq \delta \end{cases} \quad (15)$$

The controller performance is evaluated for  $\delta = \{20, \dots, 100\}$  and  $\delta = \infty$  which corresponds to a system without saturation. We use the same fractional-PID parameters obtained without considering the saturation nonlinearity.

Figures 4 and 5 show the step responses of the closed-loop system and the corresponding controller output, for the PID<sup>β</sup> tuned in the ISE and ITSE perspectives for  $\delta = 10$  and  $\delta = \infty$ , respectively. The controller parameters  $\{K, T_i, T_d, \beta\}$  correspond to the minimization of those indices leading to the values ISE:  $\{K, T_i, T_d, \beta\} \equiv \{3, 23, 90.6, 0.875\}$  and ITSE:  $\{K, T_i, T_d, \beta\} \equiv \{1.8, 17.6, 103.6, 0.85\}$ .

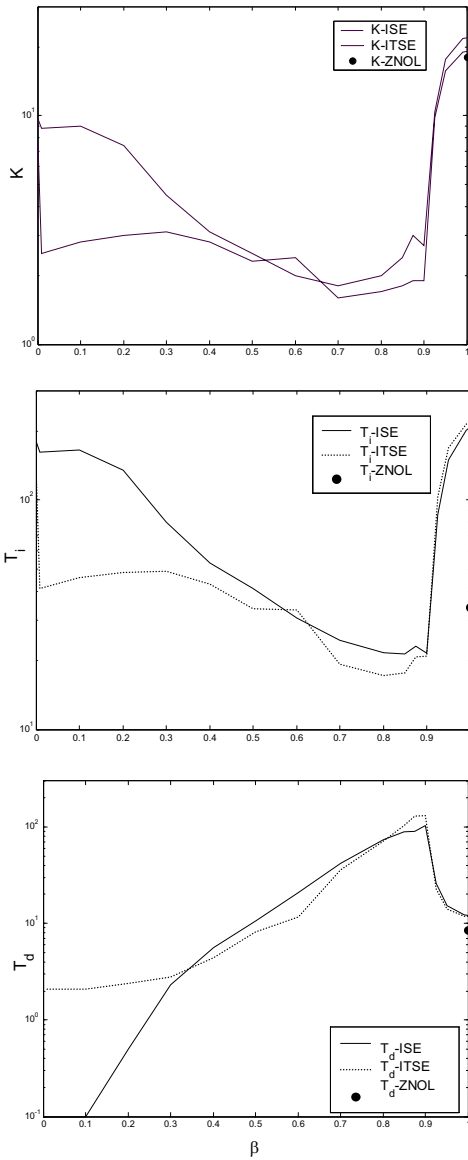


Figure 3 - The  $PID^\beta$  parameters ( $K$ ,  $T_i$ ,  $T_d$ ) versus  $\beta$  for the ISE and ITSE criteria. The dot represents the PID-ZNOL.

The step responses reveal a large diminishing of the overshoot and the rise time when compared with the integer PID, showing a good transient response and a zero steady-state error.

The  $PID^\beta$  leads to better results than the classical PID controller tuned through the ZNOL rule. These results demonstrate the effectiveness of the fractional algorithms when used for the control of fractional-order systems. The step response and the controller output are also improved when the saturation level  $\delta$  is diminished.

Figure 6 depicts the ISE and ITSE indices for  $0 \leq \beta \leq 1$ , when  $\delta = \{20, \dots, 100\}$  and  $\delta = \infty$ . We verify the existence of a minimum for  $\beta = 0.875$  and  $\beta = 0.85$  for the ISE and ITSE cases, respectively. Furthermore, the higher the  $\delta$  the lower the value of the index.

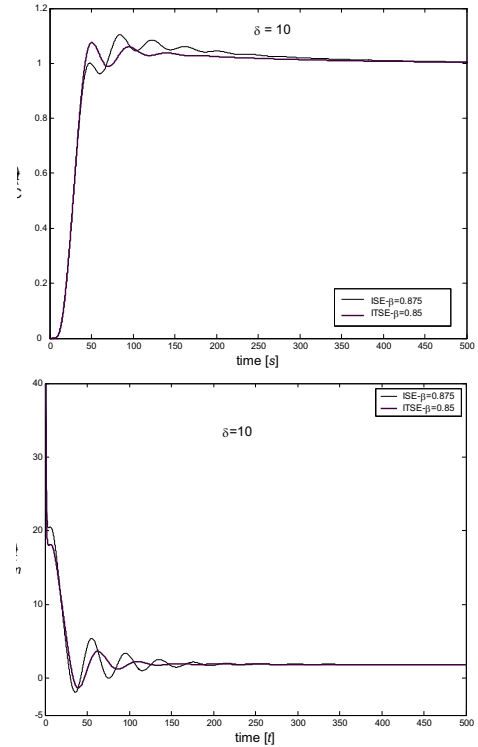


Figure 4 - Step responses of the closed-loop system and the controller output for the ISE and the ITSE indices, with a  $PID^\beta$  controller,  $\delta = 10$  and  $x = 3.0$  m.

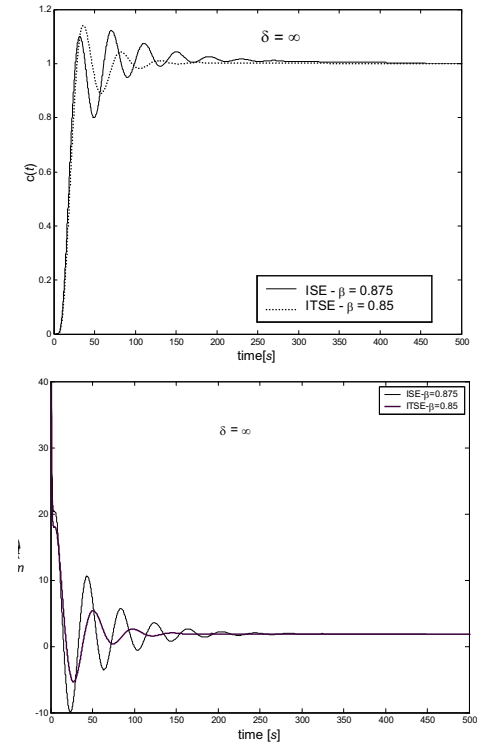


Figure 5 - Step responses of the closed-loop system and the controller output for the ISE and the ITSE indices, with a  $PID^\beta$  controller,  $\delta = \infty$  and  $x = 3.0$  m.

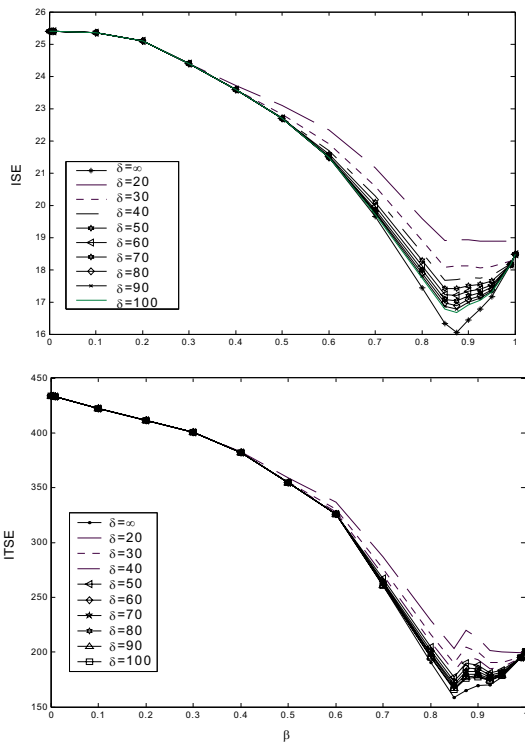


Figure 6 - ISE and ITSE versus  $0 \leq \beta \leq 1$  for  $\delta = \{20, \dots, 100\}$  and  $\delta = \infty$ .

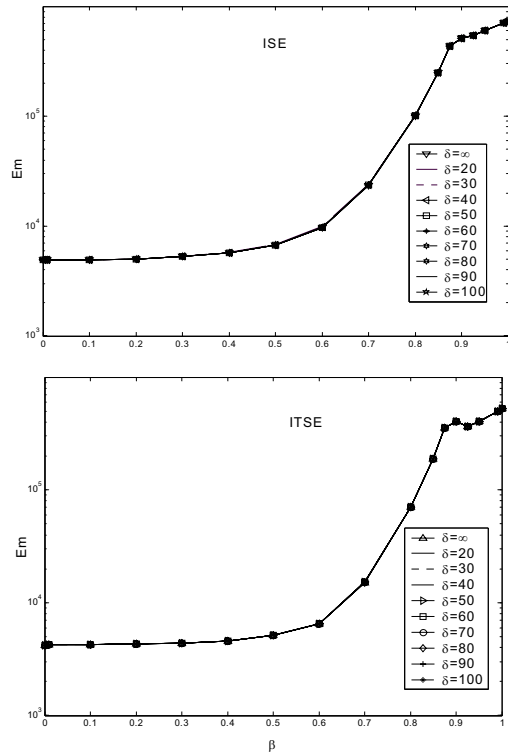


Figure 7 - Control action energy  $E_m$  for the ISE and ITSE indices versus  $0 \leq \beta \leq 1$ , when  $\delta = \{20, \dots, 100\}$  and  $\delta = \infty$ .

Figure 7 depicts the energy of the control action  $E_m$  as function of the ISE and the ITSE indices when  $0 \leq \beta \leq 1$ , for  $\delta = \{20, \dots, 100\}$  and  $\delta = \infty$ . As can be seen, the energy for ISE increases rapidly for  $0 \leq \beta \leq 0.875$ , while for  $\beta > 0.875$  the energy increases smoothly. In the ITSE case the same conclusions can be outlined for  $\beta = 0.85$ .

Figures 8 and 9 show the variation of the settling time  $t_s$ , the peak time  $t_p$ , the rise time  $t_r$ , and the percent overshoot  $ov(\%)$ , for the closed-loop response tuned through the minimization of the ISE and the ITSE indices, respectively. In the ISE case  $t_s$ ,  $t_p$  e  $t_r$  diminish rapidly for  $0 \leq \beta \leq 0.875$ , while for  $\beta > 0.875$  the parameters increase smoothly. For the ITSE we verify the same behavior for  $\beta = 0.85$ . On the other hand,  $ov(\%)$  increases smoothly for  $0 \leq \beta \leq 0.7$ , while for  $\beta > 0.7$  it decreases very quickly, both for the ISE and the ITSE indices.

In conclusion, for  $0.85 \leq \beta \leq 0.875$  we get the best controller tuning, superior to the performance revealed by the classical integer-order scheme.

## 5. Conclusion

This paper presented the fundamental aspects of the FC theory. We demonstrated that FC is a paradigm allowing a deeper understanding of physical phenomena than traditional methodologies. In this perspective, we studied the heat diffusion system, and its control using classical and fractional PID schemes. The results reveal the superior performance of the FC based algorithm.

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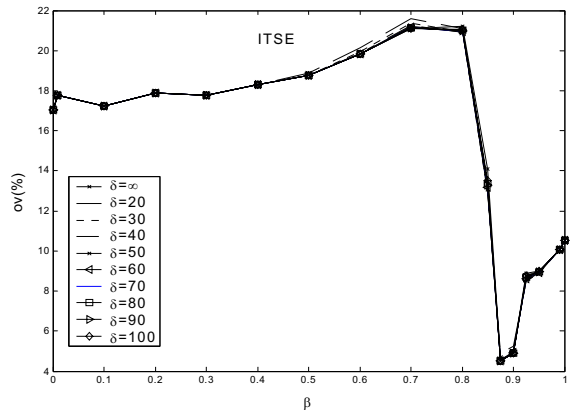
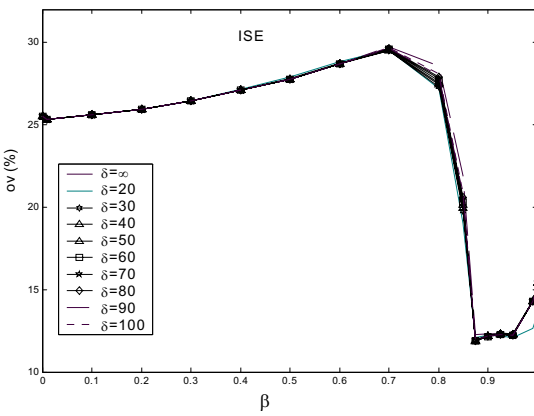
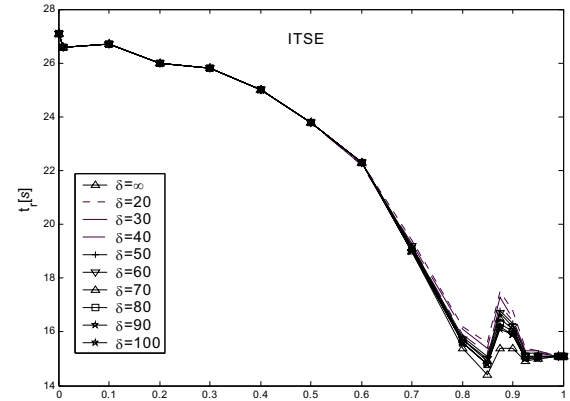
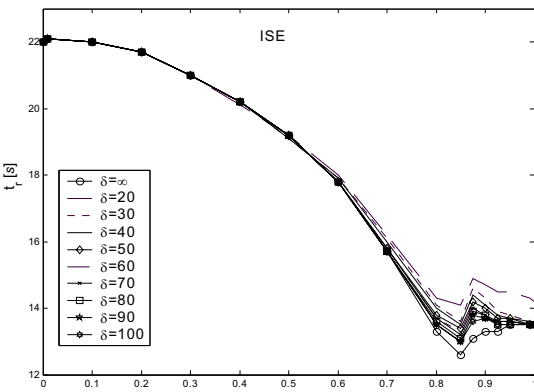
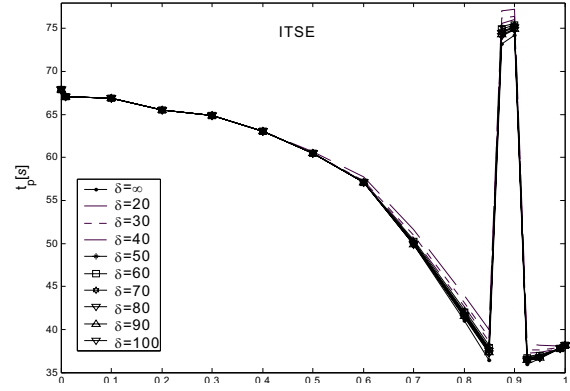
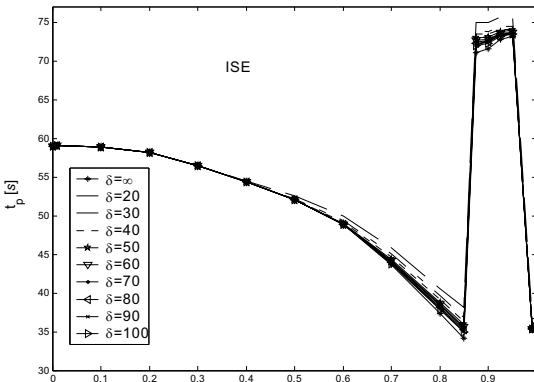
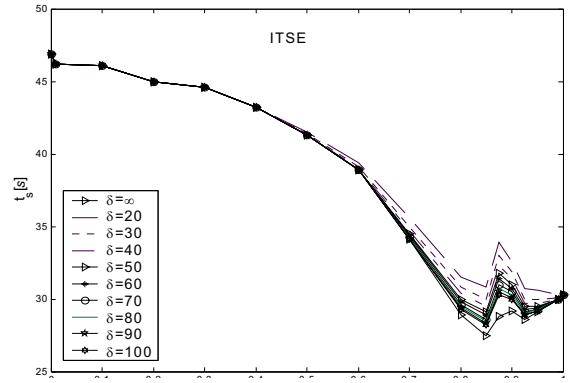
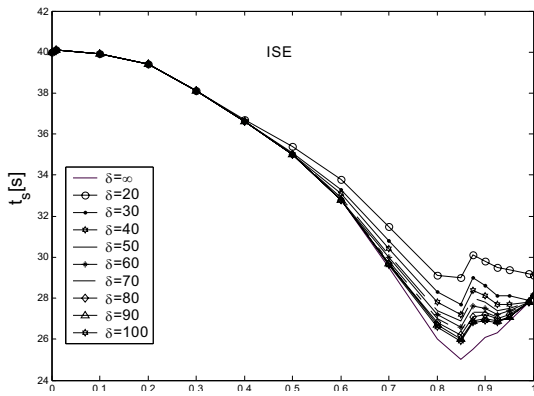


Figure 8 - Parameters  $t_s$ ,  $t_p$ ,  $t_r$ ,  $ov(\%)$  for the step responses of the closed-loop system for the ISE indice, with a  $PID^\beta$  controller, when  $\delta = \{20, \dots, 100\}$  and  $\delta = \infty$ ,  $x = 3.0$  m.

Figure 9 - Parameters  $t_s$ ,  $t_p$ ,  $t_r$ ,  $ov(\%)$  for the step responses of the closed-loop system for the ITSE indice, with a  $PID^\beta$  controller, when  $\delta = \{20, \dots, 100\}$  and  $\delta = \infty$ ,  $x = 3.0$  m.