Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations 243 (2007) 168–178
Journal of Differential Equations
www.elsevier.com/locate/jde

Hausdorff dimension bounds for smoothness of holonomies for codimension 1 hyperbolic dynamics

A.A. Pinto [a,*], D.A. Rand [b], F. Ferreira [c]

[a] *Faculdade de Ciências, Universidade do Porto, Rua do Campo Alegre, 687, 4169-007 Porto, Portugal*
[b] *Mathematics Institute, University of Warwick, Coventry CV4 7AL, UK*
[c] *E.S.E.I.G., Instituto Politécnico do Porto, R. D. Sancho I, 981, 4480-876 Vila do Conde, Portugal*

Received 7 September 2006; revised 24 February 2007

Abstract

We prove that the stable holonomies of a proper codimension 1 attractor Λ, for a C^r diffeomorphism f of a surface, are not $C^{1+\theta}$ for θ greater than the Hausdorff dimension of the stable leaves of f intersected with Λ. To prove this result we show that there are no diffeomorphisms of surfaces, with a proper codimension 1 attractor, that are affine on a neighbourhood of the attractor and have affine stable holonomies on the attractor.

1. Introduction

In this paper, we introduce the notion of a twinned pair of leaves for a diffeomorphism f of a surface with a basic set Λ. In Theorem 1, we prove that every proper codimension 1 attractor Λ contains a twinned pair of leaves. The relevance of the existence of a twinned pair of leaves is that these basic sets do not have affine models (see Theorem 2), i.e. an affine set of charts that cover Λ and in which f and the holonomies are affine. Hence, if Λ is a proper codimension 1 attractor then there are no affine models for f on Λ.

In [20], it is proved that the stable and unstable holonomies of a basic set Λ are $C^{1+\alpha}$ local diffeomorphisms, for some $\alpha(f) > 0$. We prove, in Theorem 3, the existence of an upper bound

---

[*] Corresponding author.
E-mail addresses: aapinto@fc.up.pt (A.A. Pinto), david_rand@mac.com (D.A. Rand), flavioferreira@eseig.ipp.pt (F. Ferreira).

doi:10.1016/j.jde.2007.02.013

on the degree of smoothness for the holonomies of a basic set Λ, for a diffeomorphism f, with a twinned pair of leaves. To prove this result, rather than consider all holonomies, it is enough to consider a complete set of holonomies as introduced in [19]. As a corollary, we obtain that for a proper codimension 1 attractor Λ, the stable holonomies do not have $C^{1+\alpha}$ extensions for $\alpha > HD^s$, where HD^s is the Hausdorff dimension of the stable leaves intersected with the attractor Λ.

The works of Masur [14], Penner [17], Thurston [28,29] and Veech [30] show a strong link between affine interval exchange maps and Anosov and pseudo-Anosov maps. In [24], it is developed a smooth version of the above link proving that every $C^{1+\alpha}$ diffeomorphism f on a surface, with a codimension 1 hyperbolic attractor, induces a $C^{1+\alpha}$ Cantor exchange system Φ_f, for some $\alpha > 0$. E. Ghys and D. Sullivan (see E. Cawley [3]) observed that Anosov diffeomorphisms on the torus determine circle diffeomorphisms whose $C^{1+\alpha}$ conjugacy classes are $C^{1+\alpha}$ fixed points of a renormalization operator, for some $\alpha > 0$. In the same direction, in [24], it is proved that every $C^{1+\alpha}$ diffeomorphism f of a surface, with a codimension 1 hyperbolic attractor, determines a renormalization operator acting on the topological conjugacy class $[\Phi_f]_{C^0}$ of Φ_f. Furthermore, it is proved that every $C^{1+\alpha}$ Cantor exchange system $\Phi \in [\Phi_f]_{C^0}$, that is a $C^{1+\alpha}$ fixed point of renormalization $[R_f\Phi]_{C^{1+\alpha}} = [\Phi]_{C^{1+\alpha}}$, determines a unique $C^{1+\alpha}$ diffeomorphism g, topologically conjugate to f, with an invariant measure absolutely continuous with respect to the Hausdorff measure on its invariant set. Using the results of this paper, it is shown in [24] that there is no $C^{1+\alpha}$ Cantor exchange system $\Psi \in [\Phi_f]_{C^0}$, with bounded geometry, that is a $C^{1+\alpha}$ fixed point of renormalization with α greater than the Hausdorff dimension of the Cantor invariant set of Ψ. This result is linked with the conjecture of J. Harrison (see [9]) that there are no $C^{1+\gamma}$ Denjoy maps [4] with $\gamma > HD$. This conjecture has been proved, partially, by A. Norton in [16] and by B. Kra and J. Schmeling in [12].

1.1. Twinned pair of leaves

Throughout this paper, we consider a C^r diffeomorphism f, with $r > 1$, of a compact surface S with a basic set Λ, i.e. Λ is a topologically transitive hyperbolic invariant subset, with local product structure (see Appendix A.2).

Let $W^s(x)$ (respectively $W^u(x)$) denote the stable (respectively unstable) immersed manifold passing through $x \in \Lambda$ (see Appendix A.1). We will use, from now on, ι to denote an element of the set $\{s, u\}$ of the stable and unstable superscripts and ι' to denote the element of $\{s, u\}$ that is not ι. For $\iota \in \{s, u\}$, a *full ι-leaf segment I* (or, equivalently, a local ι-leaf) is defined as a connected subset of $W^\iota(x)$, and an *ι-leaf segment* is the intersection with Λ of a full ι-leaf segment. The *endpoints* of an ι-leaf segment I are the endpoints of the minimal closed full ι-leaf segment containing I (see Appendix A.1).

Definition 1.1. A *twinned pair of u-leaves* (I, J) in a basic set Λ consists of a pair of u-leaf segments I and J with the following properties (see Fig. 1):

(i) an endpoint p of I and an endpoint q of J are periodic points under f;
(ii) $(I \setminus \{p\}) \cap (J \setminus \{q\}) = \emptyset$;
(iii) for all $z \in I \setminus \{p\}$ there is a full s-leaf segment γ_z in the stable manifold through z which has endpoints z and z' such that $z' \in J \setminus \{q\}$ and $\gamma_z \cap \Lambda = \{z, z'\}$.

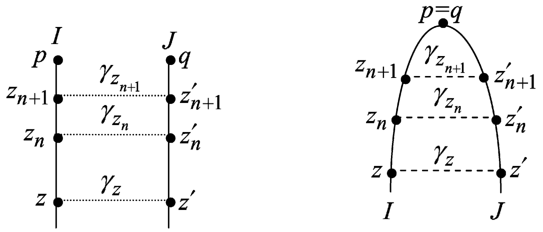

Fig. 1. An illustration of twinned pair of u-leaves.

It follows from this that if a sequence $z_n \in I \setminus \{p\}$ converges to p then the corresponding sequence $z'_n \in J \cap \gamma_{z_n}$ converges to q. Also, it follows that the periodic points p and q must have the same period. A twinned pair of s-leaves in a basic set Λ is similarly defined.

Remark 1. In the previous definition we allow the points p and q to coincide. However, if p is different from q then there is no stable leaf containing both p and q (otherwise they would converge under iteration by f which is absurd).

The set $\Lambda \subset M$ is an *attractor* for f if there is an open set $U \subset M$ such that $\Lambda = \bigcap_{i=0}^{\infty} f^i(U)$. We say that Λ is a *proper codimension* 1 *attractor* if Λ is an attractor basic set, the Hausdorff dimension of the unstable leaf segments is one, and the Hausdorff dimension of the stable leaf segments is strictly less than one (see [1] and [31]).

Theorem 1. *If Λ is a proper codimension* 1 *attractor then Λ contains a twinned pair of u-leaves.*

1.2. Affine models

The *stable basic holonomies* are maps defined between s-leaf segments defined by travelling along the unstable manifolds (see Appendix A.4). In [20], it is proved that these maps have $C^{1+\alpha}$ extensions to the full s-leaf segments for some $\alpha > 0$ with respect to the full s-leaf charts. Therefore, all s-leaf segments have the same Hausdorff dimension which we denote by HD^s. Unstable basic holonomies and HD^u are similarly defined.

Definition 1.2. A *hyperbolic affine model for f on Λ* is an atlas $\mathcal{A}$ with the following properties:

(i) the union of the domains U of the charts $i : U \to \mathbb{R}^2$ of $\mathcal{A}$ (which are open sets of M) cover Λ;
(ii) any two charts $i : U \to \mathbb{R}^2$ and $j : V \to \mathbb{R}^2$ in $\mathcal{A}$ have overlap maps $j \circ i^{-1} : i(U \cap V) \to \mathbb{R}^2$ with affine extensions to $\mathbb{R}^2$;
(iii) f is affine with respect to the charts in $\mathcal{A}$;
(iv) Λ is a basic hyperbolic set;
(v) the images of the stable and unstable local leaves under the charts in $\mathcal{A}$ are contained in horizontal and vertical lines; and
(vi) the basic holonomies have affine extensions to the stable and unstable leaves with respect to the charts in $\mathcal{A}$.

Theorem 2. *If a basic set Λ contains a twinned pair of ι-leaves then there are no affine models for f on Λ.*

Putting together Theorems 1 and 2 we obtain the following corollary:

Corollary 1. *If Λ is a proper codimension* 1 *attractor then there are no affine models for f on Λ.*

1.3. $C^{1,HD}$ complete set of holonomies

Before constructing a $C^{1,HD}$ complete set of holonomies, we have to introduce the definition of $C^{1,\alpha}$ classes of smooth regularities for homeomorphisms on the real line, for $0 < \alpha < 1$.

Definition 1.3. Let $h : I \subset \mathbb{R} \to J \subset \mathbb{R}$ be a homeomorphism between two open intervals I and J. If $0 < \alpha < 1$, then h is said to be $C^{1,\alpha}$ if it is differentiable and for all points $x, y \in I$

$$\left|h'(y) - h'(x)\right| \leqslant \chi_h\bigl(|y - x|\bigr) \tag{1}$$

where the positive function $\chi_h(t)$ is $o(t^\alpha)$, i.e. $\lim_{t\to 0} \chi_h(t)/t^\alpha = 0$.

In particular, a $C^{1+\beta}$ diffeomorphism is $C^{1,\alpha}$ for all $0 < \alpha < \beta$. We say that a stable holonomy $h : I \to J$ is $C^{1,\alpha}$ if in charts h has a $C^{1,\alpha}$ extension to a full ι-leaf segment containing I.

Let $\mathcal{M}$ be a Markov partition for f satisfying the disjointness property (see Appendix A). Suppose that M and N are Markov rectangles, and $x \in M$ and $y \in N$. We say that x and y are *ι-holonomically related* if (i) there is an ι'-leaf segment $\ell^{\iota'}(x, y)$ such that $\partial \ell^{\iota'}(x, y) = \{x, y\}$, and (ii) $\ell^{\iota'}(x, y) \subset \ell^{\iota'}(x, M) \cup \ell^{\iota'}(y, N)$. Let $P^\iota = P^\iota_{\mathcal{M}}$ be the set of all pairs (M, N) such that there are points $x \in M$ and $y \in N$ ι-holonomically related.

For every Markov rectangle $M \in \mathcal{M}$, choose an ι-spanning leaf segment ℓ^ι_M in M. Let $\mathcal{I}^\iota = \{\ell^\iota_M \colon M \in \mathcal{M}\}$. For every pair $(M, N) \in P^\iota$, there are maximal leaf segments $\ell^D_{(M,N)} \subset \ell^\iota_M$, $\ell^C_{(M,N)} \subset \ell^\iota_N$ such that there is a well-defined ι-holonomy $h^\iota_{(M,N)} : \ell^D_{(M,N)} \to \ell^C_{(M,N)}$ (see Appendix A.3). We call such holonomies $h^\iota_{(M,N)} : \ell^D_{(M,N)} \to \ell^C_{(M,N)}$ the ι-primitive holonomies associated to the Markov partition $\mathcal{M}$. The set $\mathcal{H}^\iota = \{h^\iota_{(M,N)} : \ell^D_{(M,N)} \to \ell^C_{(M,N)}; (M, N) \in P^\iota\}$ is a *complete set of ι-holonomies* (see Figs. 2 and 3).

For every leaf segment $\ell^\iota_M \in \mathcal{I}^\iota$, let $\hat{\ell}^\iota_M$ be the smallest full ι-leaf segment containing ℓ^ι_M (see definition in Appendix A). By the Stable Manifold Theorem, there are $C^{1+\alpha}$ diffeomorphisms $u^\iota_M : \hat{\ell}^\iota_M \to K^\iota_M \subset \mathbb{R}$.

Definition 1.4. A complete set of holonomies $\mathcal{H}^\iota$ is C^{1,HD^ι} if for every holonomy $h^\iota_{(M,N)} : \ell^D_{(M,N)} \to \ell^C_{(M,N)}$ in $\mathcal{H}^\iota$, the map $u^\iota_N \circ h^\iota_{(M,N)} \circ (u^\iota_M)^{-1}$ and its inverse have a C^{1,HD^ι} diffeomorphic extension to $\mathbb{R}$ such that the modulus of continuity does not depend upon $h^\iota_{(M,N)} \in \mathcal{H}^\iota$.

We prove, in Lemma 2, that for each proper codimension 1 attractor, for a diffeomorphism on a surface, there is only a finite number of holonomies in a complete set. In this case the uniformity hypothesis on the modulus of continuity of $h^\iota_{(M,N)} \in \mathcal{H}^\iota$ is redundant. However, for a Smale horseshoe this is not the case.

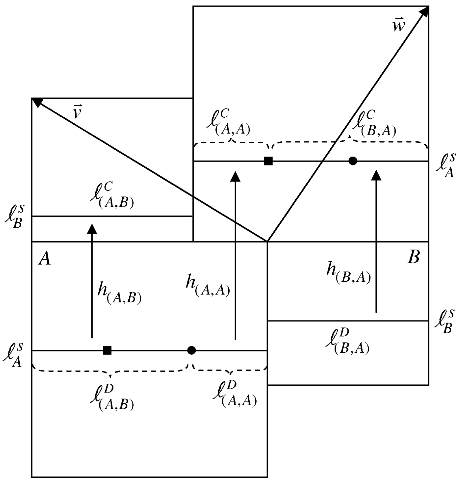

Fig. 2. The complete set of holonomies $\mathcal{H} = \{h_{(A,A)}, h_{(A,B)}, h_{(B,A)}, h^{-1}_{(A,A)}, h^{-1}_{(A,B)}, h^{-1}_{(B,A)}\}$ for the Anosov map $f : \mathbf{R}^2 \setminus (\mathbf{Z}\vec{v} \times \mathbf{Z}\vec{w}) \to \mathbf{R}^2 \setminus (\mathbf{Z}\vec{v} \times \mathbf{Z}\vec{w})$ defined by $f(x, y) = (x + y, y)$ and with Markov partition $\mathcal{M} = \{A, B\}$.

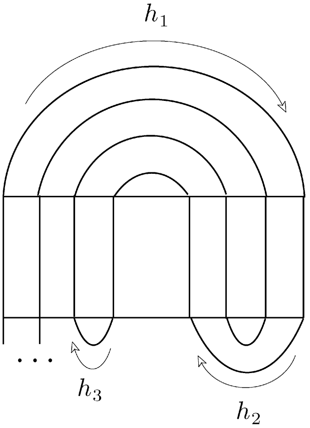

Fig. 3. The cardinality of the complete set of holonomies $\mathcal{H} = \{h_1, h_2, h_3, \dots\}$ is not finite.

Theorem 3. *Let Λ be a basic set for a $C^{1+\gamma}$ diffeomorphism f of a surface with $\gamma > HD^\iota$. If Λ contains a twinned pair of ι'-leaves, then the complete set of ι-holonomies $\mathcal{H}^\iota$ is not C^{1,HD^ι}.*

Putting together Theorems 1 and 3, we obtain the following corollary:

Corollary 2. *If Λ is a proper codimension* 1 *attractor for a $C^{1+\gamma}$ diffeomorphism on a surface with $\gamma > HD^s$, then the stable holonomies are never C^{1,HD^s}. In particular, they do not have $C^{1+\alpha}$ extensions for $\alpha > HD^s$.*

2. Twin leaves for codimension 1 attractors

We call an unstable leaf ℓ an *unstable free-leaf* if there is a full s-leaf segment I transversal to the leaf ℓ which is the union $I_1 \cup \{p\} \cup I_2$ of two disjoint (nonempty) full s-leaf seg-

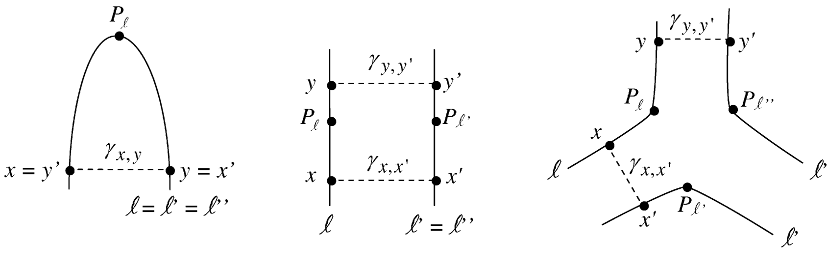

Fig. 4. Examples of sets $\mathcal{L}_m$ with cardinality 1, 2 and 3.

ments I_1 and I_2 such that I_1 and I_2 have a common endpoint $p \in \ell \cap \Lambda$ and I_2 does not intersect Λ.

By [11], the set $\mathcal{L}$ of all unstable free-leaves is nonempty and finite. Since the free-leaves are permuted by f, each one of these leaves ℓ contains a single periodic point P_ℓ. Furthermore, $\mathcal{L}$ is equal to the union of pairwise disjoint subsets $\mathcal{L}_1, \ldots, \mathcal{L}_j$ which are characterized by the following property: the leaves of each set $\mathcal{L}_m$ form the boundary of an open connected set $\mathcal{O}_m$ in M which does not intersect the basic set Λ.

Remark 2. We observe that by [8], $f|\Lambda$ is topologically conjugate to an Anosov or pseudo-Anosov map that has been unzipped along a finite set of leaves. It is these unzipped leaves which form $\mathcal{L}$. Each set $\mathcal{L}_m \subset \mathcal{L}$ corresponds to the unzipping a k-prong singularity where k is the number of leaves contained in $\mathcal{L}_m$ (see Fig. 4). The sets $\mathcal{L}_m$ of cardinality one and two correspond respectively to umbilic singularities and regular points.

Proof of Theorem 1. We claim that for each leaf $\ell \in \mathcal{L}_m$ there are two leaves $\ell', \ell'' \in \mathcal{L}_m$, two points $x \in \ell$ and $y \in \ell$ on different sides of the periodic point P_ℓ in ℓ and two points $x' \in \ell'$ and $y' \in \ell''$ such that x and x', and y and y' are the endpoints of two full s-leaf segments $\gamma_{x,x'}$ and $\gamma_{y,y'}$ whose interiors meet no unstable leaves of Λ. If the cardinality of $\mathcal{L}_m$ is greater or equal to three, then ℓ, ℓ' and ℓ'' are distinct leaves. If the cardinality of $\mathcal{L}_m$ is one, then $\ell = \ell' = \ell''$ and the claim just says that there are x and y in ℓ on either side of P_ℓ with x and y joined by a full s-leaf segment $\gamma_{x,y}$ whose interior meets no unstable leaves. If the cardinality of $\mathcal{L}_m$ is two, then $\ell' = \ell'' \neq \ell$ and $x', y' \in \ell'$ are on either side of the periodic point in ℓ'. This claim follows from the density of the unstable manifold in Λ and the local product structure as we pass to describe. If $x \in \ell$ then, for some $n > 0$, $f^n(x)$ lies inside of a small full s-leaf segment γ and, in γ, is contained between two points contained in Λ. We can then find a nontrivial full s-leaf segment γ' inside γ which also contains $f^n(x)$ so that to one side of $f^n(x)$ there is only a single point $w \neq f^n(x)$ in $\gamma' \cap \Lambda$. Let γ'' denote the part of γ between $f^n(x)$ and w. Then $f^{-n}(\gamma'')$ is a full s-leaf segment through x such that $x' = f^n(w)$ is the other endpoint of $f^{-n}(\gamma'')$. Since by construction $f^{-n}(\gamma'') \setminus \{x, x'\}$ meets no unstable leaves of Λ, $f^{-n}(\gamma'')$ is the required full s-leaf segment $\gamma_{x,x'}$, and ℓ' is the stable leaf passing through x'. One finds y' and ℓ'' by taking y on the other side of P_ℓ in ℓ and proceeding in a similar fashion which ends the proof of the claim. Let $\ell(x)$ be an unstable leaf segment containing x and having P_ℓ as one of its endpoints. Let $\ell'(x')$ be the unstable leaf containing x' such that there is a local holonomy $h : \ell(x) \to \ell'(x')$ with $h(x) = x'$ (and so $h(\ell(x)) = \ell'(x')$). Then, the pair $(\ell(x), \ell'(x'))$ form a twinned pair of leaves. $\square$

3. Non-existence of affine models

To prove Theorems 2 and 3, we are going to introduce the notion of transversely affine stable (respectively unstable) ratio functions for the attractor Λ. This consists of an affine structure on the stable (respectively unstable) leaves of Λ which is invariant under both the map f and the stable (respectively unstable) holonomies.

A *ratio function* $r(I:J)$ can be thought of as prescribing the ratio of the size of two leaf segments I and J in the same stable or unstable leaf. A ratio function $r(I:J)$ is positive and continuous in the endpoints of I and J. Moreover,

$$r(I:J) = r(J:I)^{-1} \quad \text{and} \quad r(I_1 \cup I_2 : K) = r(I_1 : K) + r(I_2 : K)$$

where I_1 and I_2 intersect at most in one of their endpoints.

Definition 3.1. We say that r is an *ι-ratio function* if (i) for all ι-leaf segments K, $r(I:J)$ $(I, J \subset K)$ defines a ratio function on K; (ii) r is invariant under f; and (iii) for every basic ι-holonomy $h: I \to J$ in a rectangle R

$$\left| \log \frac{r(hI_0 : hI_1)}{r(I_0 : I_1)} \right| \leqslant \mathcal{O}\big(\big(d_\Lambda(I,J)\big) r(K:I)\big)^\alpha \tag{2}$$

for all ι-leaf segments $I_0, I_1 \subset K$, where $0 < \alpha < 1$ depends upon r and the constant of proportionality also depends upon R.

Definition 3.2. Since r satisfies the condition (2) and defines an affine structure on each leaf that is f-invariant we say that it is a *transversely Hölder ι-ratio function*. If r is invariant under holonomies h (i.e. $r(I:J) = r(h(I):h(J))$) then we say that it is a *transversely affine ι-ratio function*.

Lemma 1. *If Λ contains a twinned pair of ι-leaves then there is not a transversely affine ι'-ratio function r.*

Proof. For simplicity of exposition we will consider the case $\iota = u$ and $\iota' = s$. The other case is similar by replacing f by f^{-1} and stable by unstable, and vice-versa. Let us suppose by contradiction that there is an affine model for f. For arguments sake assume that the twinned pair leaves are unstable. Let the full u-leaf segments I and J and the periodic points $p \in I \cap \Lambda$ and $q \in J \cap \Lambda$ be as in the definition of a twinned pair leaves. Let m be the common period of the periodic points. Fix $z \in I \cap \Lambda$ and $z' \in J \cap \Lambda$ such that z and z' are the endpoints of a full s-leaf segment which does not intersect Λ. Choose a full u-leaf segment K such that there is a holonomy $h : J \cap \Lambda \to K \cap \Lambda$. For every $n \neq 1$, let $y_n \in I \cap \Lambda$, $y'_n \in J \cap \Lambda$ and $y''_n \in K \cap \Lambda$ be such that $f^{mp}(y_n) = z$, $f^{mp}(y'_n) = z'$ and $h(y'_n) = y''_n$ (see Fig. 5). The ratio $r(y_n, y'_n, y''_n)$ between the length of the full u-leaf segment with endpoints y''_n and y'_n and the length of the full u-leaf segment with endpoints y'_n and y_n, when measured in a chart of the affine atlas, is well-defined and does not depend upon the chart considered.

Since the holonomy is affine, the value of the ratio $r(y_n, y'_n, y''_n)$ does not depend upon $n \neq 1$. Since f is also affine, $r(y_n, y'_n, y''_n)$ is equal to $r(z, z', f^{mp}(y''_n))$. Therefore, the value of the ratio $r(z, z', f^{mp}(y''_n))$ does not depend upon $n \neq 1$. But, by construction the sequence $f^{mp}(y''_n)$ converges to z' which implies that the ratio $r(z, z', f^{mp}(y''_n))$ converges to zero, which is absurd. □

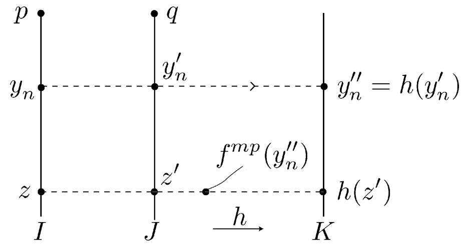

Fig. 5. The nonexistence of transversely affine ratio function.

Proof of Theorem 2. If there is an affine model for f on Λ then r is a transversely affine ι-ratio function, which contradicts Lemma 1. □

4. Nonexistence of uniformly C^{1,HD^ι} complete sets of holonomies

Lemma 2. *For a proper codimension* 1 *attractor the stable complete set of holonomies consists of a finite set of holonomies.*

However, there are cases where the complete set of holonomies is forced to be infinite. This is the case for systems like the Smale horseshoe (see Fig. 3).

Proof of Lemma 2. Since the u-leaf segments are manifolds, the number of holonomies in the complete sets of s-holonomies is two times the minimal number $\mathcal{N}$ of stable leaves which cover the s-boundaries of the rectangles contained in the Markov partition with the property that the interior of each one of these leaves is contained in at most two s-boundaries of Markov rectangles. □

Proof of Theorem 3. By Lemma 1 and Theorem 2 in [19], if f is $C^{1+\gamma}$, with $\gamma > HD^\iota$, and the complete set of ι-holonomies is C^{1,HD^ι}, then r is a transversely affine ι-ratio function. This contradicts Lemma 1. □

Acknowledgments

We thank IHES, CUNY, IMPA, Stony Brook and University of Warwick for their hospitality. We thank Calouste Gulbenkian Foundation, PRODYN-ESF, POCTI and POCI by FCT and Ministério da Ciência e da Tecnologia, and Centro de Matemática da Universidade do Porto for their financial support of A.A. Pinto and F. Ferreira, and the Wolfson Foundation and the UK Engineering and Physical Sciences Research Council for their financial support of D.A. Rand.

Appendix A

In this appendix, we present some basic facts on hyperbolic dynamics, that we include for clarity of the exposition.

A.1. Leaf segments

Let d be a metric on M, and define the map $f_\iota = f$ if $\iota = u$, or $f_\iota = f^{-1}$ if $\iota = s$. For $\iota \in \{s, u\}$, if $x \in \Lambda$ we denote the local ι-manifolds through x by

$$W^\iota(x, \varepsilon) = \bigl\{y \in M\colon\ d\bigl(f_\iota^{-n}(x), f_\iota^{-n}(y)\bigr) \leqslant \varepsilon,\ \text{for all } n \geqslant 0\bigr\}.$$

By the Stable Manifold Theorem (see [10] and [26]), these sets are respectively contained in the stable and unstable immersed manifolds

$$W^\iota(x) = \bigcup_{n \geqslant 0} f_\iota^n\bigl(W^\iota\bigl(f_\iota^{-n}(x), \varepsilon_0\bigr)\bigr)$$

which are the image of a $C^{1+\gamma}$ immersion $\kappa_{\iota,x} : \mathbb{R} \to M$. An *open* (respectively *closed*) *full ι-leaf segment* I is defined as a subset of $W^\iota(x)$ of the form $\kappa_{\iota,x}(I_1)$ where I_1 is an open (respectively closed) subinterval (nonempty) in $\mathbb{R}$. An *ι-open* (respectively *closed*) *leaf segment* is the intersection with Λ of a full open (respectively closed) ι-leaf segment such that the intersection contains at least two distinct points. If the intersection is exactly two points we call this ι-closed leaf segment an *ι-leaf gap*. An *ι-full leaf segment* is either an open or closed ι-full leaf segment. An *ι-leaf segment* is either an open or closed ι-leaf segment. The *endpoints* of a full ι-leaf segment are the points $\kappa_{\iota,x}(u)$ and $\kappa_{\iota,x}(v)$ where u and v are the endpoints of I_1. The *endpoints* of an ι-leaf segment I are the points of the minimal closed full ι-leaf segment containing I. The *interior* of a ι-leaf segment I is the complement of its boundary. In particular, an ι-leaf segment I has empty interior if, and only if, it is an ι-leaf gap. A map $c : I \to \mathbb{R}$ is an *ι-leaf chart* of an ι-leaf segment I if has an extension $c_E : I_E \to \mathbb{R}$ to a full ι-leaf segment I_E with the following properties: $I \subset I_E$ and c_E is a homeomorphism onto its image. An *ι-full leaf segment* is either an open or close full leaf segment.

A.2. Rectangles

Since Λ is a hyperbolic invariant set of a diffeomorphism $f : M \to M$, for $0 < \varepsilon < \varepsilon_0$ there is $\delta = \delta(\varepsilon) > 0$, such that for all points $w, z \in \Lambda$ with $d(w, z) < \delta$, $W^u(w, \varepsilon)$ and $W^s(z, \varepsilon)$ intersect in an unique point that we denote by $[w, z]$. Since we assume that the hyperbolic set has a *local product structure*, we have that $[w, z] \in \Lambda$. Furthermore, the following properties are satisfied: (i) $[w, z]$ varies continuously with $w, z \in \Lambda$; (ii) the bracket map is continuous on a δ-uniform neighbourhood of the diagonal in $\Lambda \times \Lambda$; and (iii) whenever both sides are defined $f([w, z]) = [f(w), f(z)]$. Note that the bracket map does not really depend on δ provided it is sufficiently small.

Let us underline that it is a standing hypothesis that all the hyperbolic sets considered here have such a local product structure.

A *rectangle* R is a subset of Λ which is (i) closed under the bracket, i.e. $w, z \in R \Rightarrow [w, z] \in R$, and (ii) proper, i.e. is the closure of its interior in Λ (see Fig. 6). This definition imposes that a rectangle has always to be proper which is more restrictive than the usual one which only insists on the closure condition.

If ℓ^s and ℓ^u are respectively stable and unstable leaf segments intersecting in a single point then we denote by $[\ell^s, \ell^u]$ the set consisting of all points of the form $[w, z]$ with $w \in \ell^s$ and $z \in \ell^u$. We note that if the stable and unstable leaf segments ℓ and ℓ' are closed

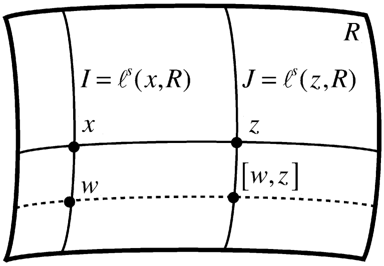

Fig. 6. Illustration of a basic holonomy.

then the set $[\ell, \ell']$ is a rectangle. Conversely in this 2-dimensional situations, any rectangle R has a product structure in the following sense: for each $x \in R$ there are closed stable and unstable leaf segments of Λ, $\ell^s(x, R) \subset W^s(x)$ and $\ell^u(x, R) \subset W^u(x)$ such that $R = [\ell^s(x, R), \ell^u(x, R)]$. The leaf segments $\ell^s(x, R)$ and $\ell^u(x, R)$ are called *stable and unstable spanning leaf segments* for R. For $\iota \in \{s, u\}$, we denote by $\partial \ell^\iota(x, R)$ the set consisting of the endpoints of $\ell^\iota(x, R)$, and we denote by $\operatorname{int} \ell^\iota(x, R)$ the set $\ell^\iota(x, R) \setminus \partial \ell^\iota(x, R)$. The *interior of* R is given by $\operatorname{int} R = [\operatorname{int} \ell^s(x, R), \operatorname{int} \ell^u(x, R)]$, and the *boundary of* R is given by $\partial R = [\partial \ell^s(x, R), \ell^u(x, R)] \cup [\ell^s(x, R), \partial \ell^u(x, R)]$.

A.3. Markov partitions

A *Markov partition of* f is a collection $\mathcal{R} = \{R_1, \ldots, R_k\}$ of such rectangles such that (i) $\Lambda \subset \bigcup_{i=1}^k R_i$; (ii) $R_i \cap R_j = \partial R_i \cap \partial R_j$ for all i and j; (iii) if $x \in \operatorname{int} R_i$ and $fx \in \operatorname{int} R_j$ then

(a) $f(\ell^s(x, R_i)) \subset \ell^s(fx, R_j)$ and $f^{-1}(\ell^u(fx, R_j)) \subset \ell^u(x, R_i)$,
(b) $f(\ell^u(x, R_i)) \cap R_j = \ell^u(fx, R_j)$ and $f^{-1}(\ell^s(fx, R_j)) \cap R_i = \ell^s(x, R_i)$.

The last condition means that $f(R_i)$ goes across R_j just once. In fact, it follows from condition (a) providing the rectangles R_j are chosen sufficiently small (see Mañé [13]). The rectangles which make up the Markov partition are called *Markov rectangles*.

We note that there is a Markov partition $\mathcal{R}$ of f with the following *disjointness property* (see Bowen [2], Newhouse and Palis [15] and Sinai [27]):

(i) if $0 < \delta_{f,s} < 1$ and $0 < \delta_{f,u} < 1$ then the stable and unstable leaf boundaries of any two Markov rectangles do not intersect;
(ii) if $0 < \delta_{f,\iota} < 1$ and $0 < \delta_{f,\iota'} = 1$ then the ι'-leaf boundaries of any two Markov rectangles do not intersect except, possibly, at their endpoints.

If $\delta_{f,s} = \delta_{f,u} = 1$, the disjointness property does not apply and so we consider that it is trivially satisfied for every Markov partition. For simplicity of our exposition, we consider Markov partitions that satisfy the disjointness property. This result is also used in [5–7,18,21–23,25].

A.4. Basic holonomies

Suppose that x and z are two points inside any rectangle R of Λ. Let I and J be two stable leaf segments respectively containing x and z and inside R. Then we define $h : I \to J$ by $h(w) =$

$[w, z]$ (see Fig. 6). Such maps are called the *basic stable holonomies*. They generate the pseudogroup of all stable holonomies. Similarly we define the basic unstable holonomies.

References

[1] C. Bonatti, R. Langevin, Difféomorphismes de Smale des surfaces, Astérisque 250 (1998).

[2] R. Bowen, Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms, Lecture Notes in Math., vol. 470, Springer-Verlag, New York, 1975.

[3] E. Cawley, The Teichmüller space of an Anosov diffeomorphism of T^2, Invent. Math. 112 (1993) 351–376.

[4] A. Denjoy, Sur les courbes définies par les équations différentielles á la surface du tore, J. Math. Pures Appl. 11 (9) (1932) 333–375.

[5] E. de Faria, W. de Melo, A.A. Pinto, Global hyperbolicity of renormalization for C^r unimodal mappings, Ann. of Math. 164 (2006) 731–824.

[6] F. Ferreira, A.A. Pinto, Explosion of smoothness from a point to everywhere for conjugacies between diffeomorphisms on surfaces, Ergodic Theory Dynam. Systems 23 (2003) 509–517.

[7] F. Ferreira, A.A. Pinto, Explosion of smoothness from a point to everywhere for conjugates between Markov families, Dyn. Syst. 16 (2) (2001) 193–212.

[8] A.A. Gaspar Ruas, Atratores hiperbólicos de codimensão um e classes de isotopia em superficies, Tese de Doutoramento, IMPA, Rio de Janeiro, 1982.

[9] J. Harrison, An introduction to fractals, in: Proc. Sympos. Appl. Math., vol. 39, Amer. Math. Soc., 1989, pp. 107–126.

[10] M. Hirsch, C. Pugh, Stable manifolds and hyperbolic sets, in: Proc. Sympos. Pure Math., vol. 14, Amer. Math. Soc., 1970, pp. 133–164.

[11] H. Kollmer, On the structure of axiom-A attractors of codimension one, in: XI Colóquio Brasileiro de Matemática, vol. II, 1997, pp. 609–619.

[12] B. Kra, J. Schmeling, Diophantine classes, dimension and Denjoy maps, Acta Arith. 105 (2002) 323–340.

[13] R. Mañé, Ergodic Theory and Differentiable Dynamics, Springer-Verlag, Berlin, 1987.

[14] H. Masur, Interval exchange transformations and measured foliations, Ann. of Math. (2) 115 (1) (1982) 169–200.

[15] S. Newhouse, J. Palis, Hyperbolic nonwandering sets on two-dimension manifolds, in: M. Peixoto (Ed.), Dynamical Systems, Academic Press, New York, 1973, pp. 293–301.

[16] A. Norton, Denjoy's Theorem with exponents, Proc. Amer. Math. Soc. 127 (10) (1999) 3111–3118.

[17] R.C. Penner, J.L. Harer, Combinatorics of Train Tracks, Princeton University Press, Princeton, New Jersey, 1992.

[18] A.A. Pinto, D.A. Rand, Solenoid functions for hyperbolic sets on surfaces, in: Boris Hasselblatt (Ed.), Dynamics, Ergodic Theory and Geometry: Dedicated to Anatole Katok, in: Math. Sci. Res. Inst. Publ., vol. 54, Cambridge University Press, 2007, pp. 145–178.

[19] A.A. Pinto, D.A. Rand, Rigidity of hyperbolic sets surfaces, J. London Math. Soc. 71 (2) (2004) 481–502.

[20] A.A. Pinto, D.A. Rand, Smoothness of holonomies for codimension 1 hyperbolic dynamics, Bull. London Math. Soc. 34 (2002) 341–352.

[21] A.A. Pinto, D.A. Rand, Teichmüller spaces and HR structures for hyperbolic surface dynamics, Ergodic Theory Dynam. Systems 22 (2002) 1905–1931.

[22] A.A. Pinto, D.A. Rand, Classifying C^{1+} structures on hyperbolical fractals: 1 The moduli space of solenoid functions for Markov maps on train-tracks, Ergodic Theory Dynam. Systems 15 (1995) 697–734.

[23] A.A. Pinto, D.A. Rand, Classifying C^{1+} structures on hyperbolical fractals: 2 Embedded trees, Ergodic Theory Dynam. Systems 15 (1995) 969–992.

[24] A.A. Pinto, D.A. Rand, F. Ferreira, F.A. Ferreira, Cantor exchange systems and renormalization, in press.

[25] A.A. Pinto, D. Sullivan, The circle and the solenoid. Dedicated to Anatole Katok on the occasion of his 60th birthday, DCDS-A 16 (2) (2006) 463–504.

[26] M. Schub, Global Stability of Dynamical Systems, Springer-Verlag, 1987.

[27] Ya. Sinai, Markov partitions and C-diffeomorphisms, Anal. Appl. 2 (1968) 70–80.

[28] W. Thurston, On the geometry and dynamics of diffeomorphisms of surfaces, Bull. Amer. Math. Soc. 19 (1988) 417–431.

[29] W. Thurston, The Geometry and Topology of Three-manifolds, Princeton University Press, Princeton, 1978.

[30] W. Veech, Gauss measures for transformations on the space of interval exchange maps, Ann. of Math. (2) 115 (2) (1982) 201–242.

[31] R.F. Williams, Expanding attractors, Publ. Inst. Hautes Etudes Sci. 43 (1974) 169–203.