# Fractional dynamics in the Rayleigh's piston

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A B S T R A C T

This paper studies the dynamics of the Rayleigh piston using the modeling tools of Fractional Calculus. Several numerical experiments examine the effect of distinct values of the parameters. The time responses are transformed into the Fourier domain and approximated by means of power law approximations. The description reveals characteristics usual in Fractional Brownian phenomena.

*Keywords:*
Thermodynamics
System modelling
Fractional calculus
Fractional Brownian motion
Power law

## 1. Introduction

During the last decades several papers addressed a conceptual example of statistical mechanics known as the "Rayleigh piston" [1,2]. This classical prototype system consists of a one-dimensional array of particles separated by means of an adiabatic piston. The particles in the two cylinders have non-zero random velocities and collide sporadically with the piston provoking its motion. While a very simple system, a kind of conceptual paradox occurs and considerable debate took place about the steady state operating conditions. Nevertheless, most of the technical literature addresses the relationship of the system final equilibrium conditions and the study of the complex dynamics has not attracted relevant attention.

This paper focus the dynamics of the Rayleigh piston in the perspective of Fractional Brownian motion (fBm) and Fractional Calculus (FC). The fBm was introduced by Kolmogorov [3]. Later Mandelbrot adopted the concept of fBm to model phenomena with self-similarity and long range effects [4]. The fBm is also called $1/f$ noise [5], where $f$ denotes frequency, because its spectrum is given by $1/f^{\alpha}$, $\alpha > 0$. The fBm is interpreted as a signature of complexity [6] and has been observed in many distinct areas [7], namely in economics and finance [8,9], geophysics [10–15], music and speech [16–18], biology [19–23] and others. During the last years the relation between fBm and FC was studied by some researchers [24–27]. FC emerged with the ideas of Leibniz and several important mathematicians contributed to its development [28–32]. However, only in the last decades [33,34] FC was recognized to be an important tool to study systems with long range memory phenomena [35–44]. FC generalizes the operations of integration and differentiation to non-integer orders and constitutes an efficient mathematical tool for describing natural phenomena with long-range memory effects and power law description. This paper addresses the Rayleigh piston and its characterization by means of fBm and FC concepts.

Having these ideas in mind, this paper focus on the fBm in the perspective of FC and is organized as follows. Section 2 introduces the "Rayleigh piston", develops the analysis in the Fourier domain, extracting several power-law parameters, and discusses the results in the perspective of dynamical systems. Finally, Section 3 outlines the main conclusions.

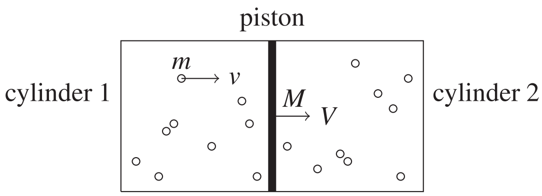


**Fig. 1.** The Rayleigh piston.

## 2. Preliminary concepts

The Rayleigh's piston is a system consisting of two cylinders, to be denoted as 1 and 2, containing some type of fluid, and separated by an adiabatic movable piston (Fig. 1). A brake maintains the piston at rest until time $t = 0$. The two fluids are in equilibrium with pressure, volume and temperature $\{p_i(0), V_i(0), T_i(0)\}$, $i = 1, 2$. The piston with mass $M$ undergoes random one-dimensional collisions with particles of mass $m$. Furthermore, there are $n_i$, $i = 1, 2$, particles per unit volume, with Maxwell distributed velocities at temperature $T_i$.

In steady state occurs a mechanical equilibrium and the pressures are identical, that is, $p_1(t \to \infty) = p_2(t \to \infty)$. However, nothing can be said about the final temperatures $T_1(t \to \infty)$ and $T_2(t \to \infty)$, since the laws of thermostatics are insufficient to predict them. The reader can follow the discussion about this *gedankenexperiment* in [45–54] and references therein.

In this paper we focus the dynamics of the motion of the piston for different operating conditions under the light of FC. At $t = 0$ the particles of cylinder $i$, $i = 1, 2$, are considered to have a one-dimensional probability distribution so that $v_i \sim e^{-\frac{v_i^2}{\sigma_i}}$. The collision phenomenon is modeled by means of the pair of initial and final velocities, $(V_i, v_i)$ and $(V_f, v_f)$, respectively. Elastic collisions satisfy the conservation of energy and momentum:

$$E_f + e_f = E_i + e_i, \tag{1a}$$

$$P_f + p_f = P_i + p_i, \tag{1b}$$

where subscripts $i$ and $f$ denote the initial and final states, $P = MV$ and $p = mv$ denote momenta and $E = \frac{1}{2}MV^2$ and $e = \frac{1}{2}mv^2$ the kinetic energies, of the piston and particles, respectively. Therefore, the velocities of the piston and the particle after collision, are given by:

$$V_f = V_i - \frac{2m}{m+M}(V_i - v_i), \tag{2a}$$

$$v_f = v_i - \frac{2M}{m+M}(v_i - V_i). \tag{2b}$$

During the following numerical simulations we adopt a time step of $h = 0.5 \cdot 10^{-3}$, an initial piston position $x(0) = 0$, and two identical cylinders with unit width.

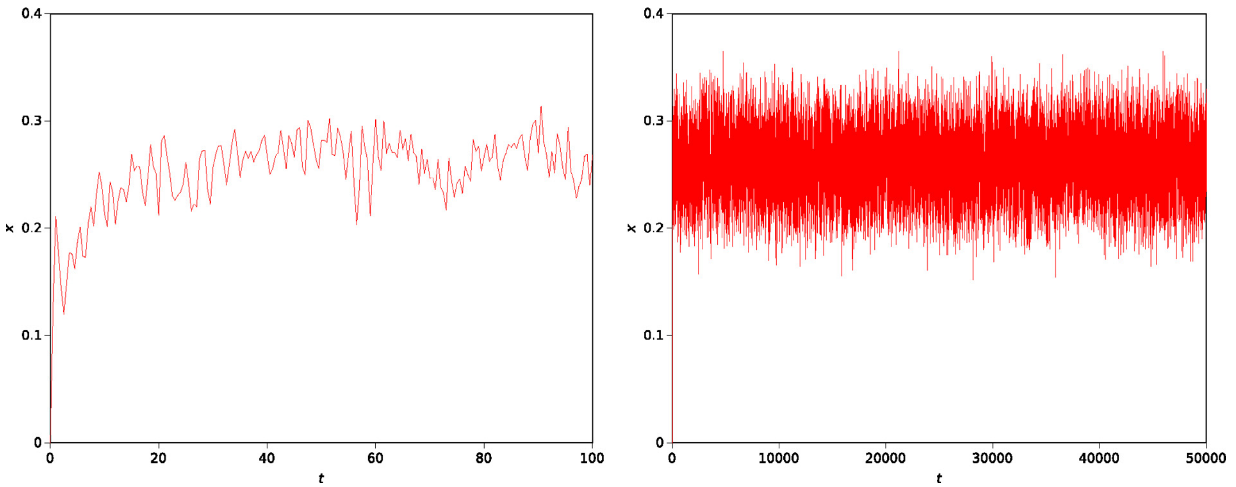


**Fig. 2.** Time response $x(t)$ of the piston position for $0 \le t \le 100$ (left) and $0 \le t \le 5 \cdot 10^4$ (right), with $M = 10$, $m = 1$, $n_1 = 1000$, $n_2 = 500$, $\sigma_1 = \sigma_2 = 1.0$, $h = 0.5 \cdot 10^{-3}$.

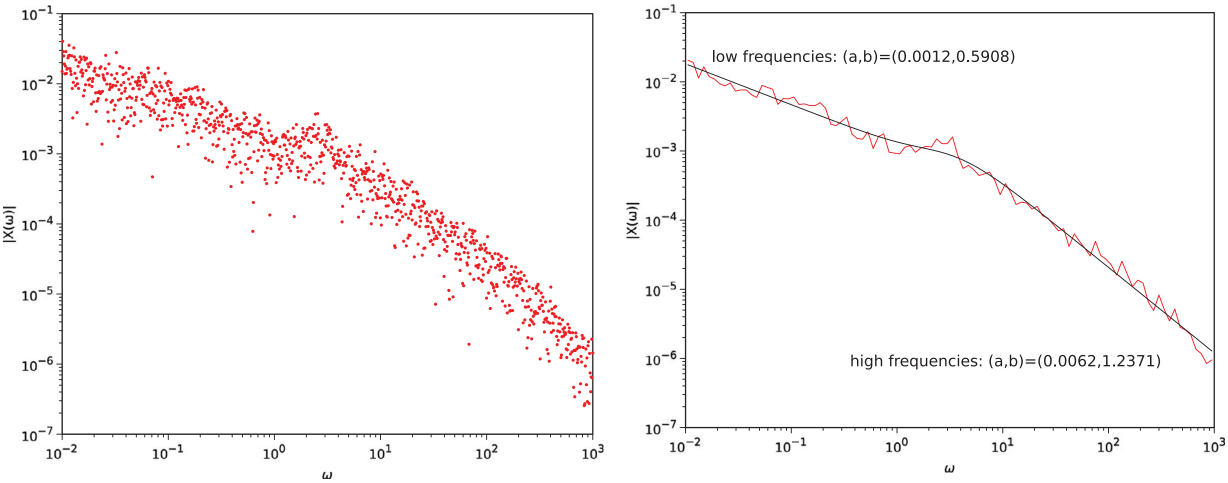


**Fig. 3.** Amplitude of the frequency response $|X(\omega)|$ versus $\omega$ of the piston position with $M = 10$, $m = 1$, $n_1 = 1000$, $n_2 = 500$, $\sigma_1 = \sigma_2 = 1.0$, $0 \leq t \leq 5 \cdot 10^4$, $h = 0.5 \cdot 10^{-3}$. Left: initial response, right: moving average and approximations (4) leading to the PL $(a_l, b_l) = (0.0012, 0.5908)$ and $(a_h, b_h) = (0.0062, 1.2371)$.

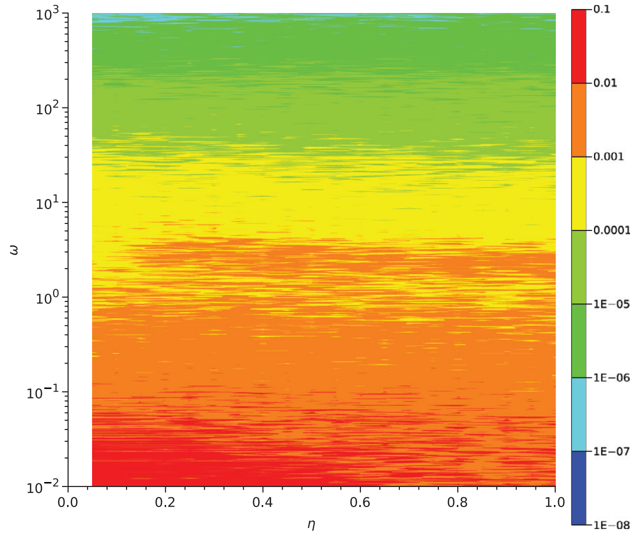


**Fig. 4.** Amplitude of the frequency response $|X(\omega)|$ versus $(\omega, \eta)$ of the piston position with $M = 10$, $m = 1$, $\sigma_1 = \sigma_2 = 1.0$, $0 \leq t \leq 5 \cdot 10^4$, $h = 0.5 \cdot 10^{-3}$.

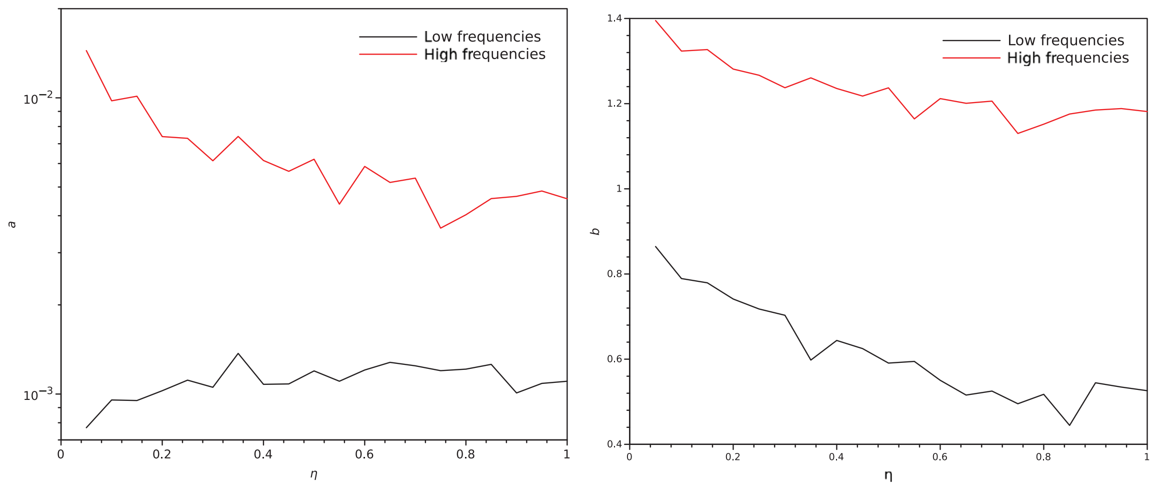


**Fig. 5.** Variation of parameters $a_l$ and $a_h$, and $b_l$ and $b_h$, versus $\eta = \frac{n_2}{n_1}$, for $0 \leq t \leq 5 \cdot 10^4$, with $M = 10$, $m = 1$, $\sigma_1 = \sigma_2 = 1.0$.

Fig. 2 shows the transient (left) and the long term behavior (right) of the piston position time response $x(t) = \int_0^t v(\tau)d\tau$, respectively, for $M = 10$, $m = 1$, $n_1 = 1000$, $n_2 = 500$, $\sigma_1 = \sigma_2 = 1.0$. In the transient we observe a fast initial evolution followed by a much slower response. In what concerns the steady-state response we verify that the long term behavior reveals fBm representative of the particles' dynamical effects.

Fig. 3 depicts the amplitude of the corresponding Fourier spectrum, $|\mathcal{F}\{x(t)\}| = |X(\omega)|$. At the left is represented the frequency response for 1000 points and at the right is shown a 10 point moving average and the approximation

$$|X(\omega)| = \left| K \frac{\left(1+\frac{s}{z}\right)^{\beta}}{s^{\alpha_1}\left(1+\frac{s}{p}\right)^{\alpha_2}} \right|_{s=j\omega}, \ K, z, p, \alpha_1, \alpha_2, \beta \in \mathbb{R}^+. \tag{3}$$

The experiments demonstrated that the values of $z$, $p$ and $\beta$ are of minor importance, being relevant the power law (PL) approximations at low and high frequencies of the type $|X(\omega)| \approx a\omega^{-b}$, $a, b \in \mathbb{R}^+$. Therefore, in the sequel we focus the attention in the low and high PL trendlines:

$$|X(\omega)| \approx a_l \omega^{-b_l} = K \cdot s^{-\alpha_1}, \ a_l, b_l \in \mathbb{R}^+, \tag{4a}$$

$$|X(\omega)| \approx a_h \omega^{-b_h} = K \frac{p^{\alpha_2}}{z^{\beta}} \cdot s^{-(\alpha_1+\alpha_2-\beta)}, \ a_h, b_h \in \mathbb{R}^+, \tag{4b}$$

at the low and high frequency ranges, respectively.

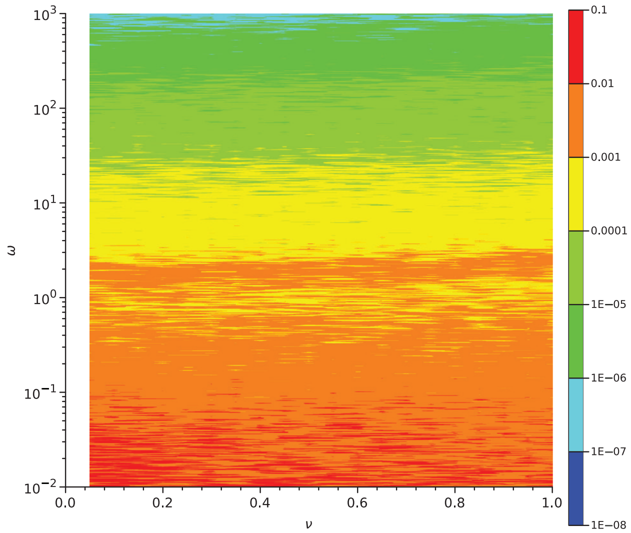


**Fig. 6.** Amplitude of the frequency response $|X(\omega)|$ versus $(\omega, \nu)$ of the piston position with $M = 10$, $m = 1$, $n_1 = n_2 = 1000$, $\sigma_1 = \sigma_2 = 1.0$, $0 \le t \le 5 \cdot 10^4$, $h = 0.5 \cdot 10^{-3}$.

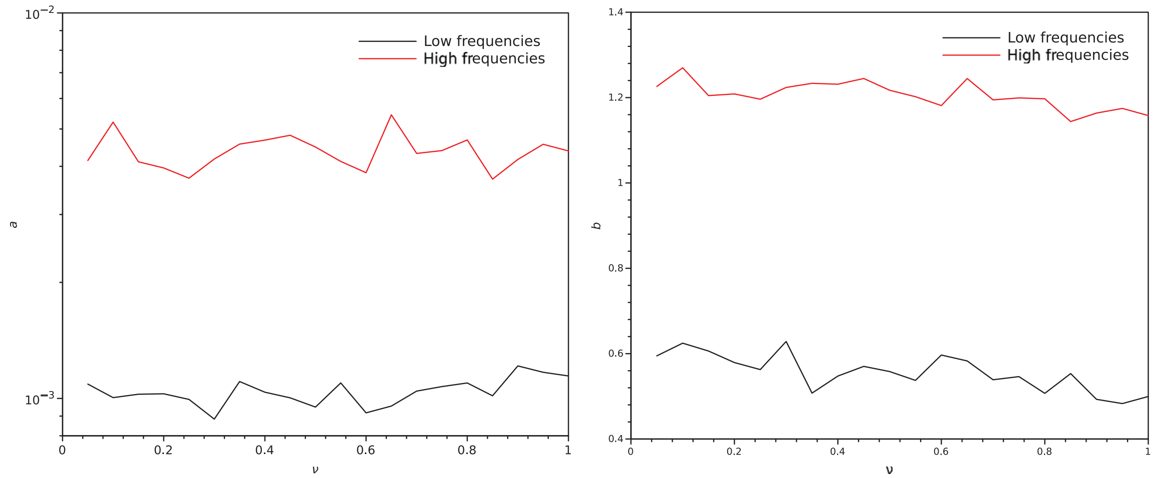


**Fig. 7.** Variation of parameters $a_l$ and $a_h$, and $b_l$ and $b_h$, versus $\nu = \frac{\sigma_2}{\sigma_1}$ of the piston position with $M = 10$, $m = 1$, $n_1 = n_2 = 1000$, $0 \le t \le 5 \cdot 10^4$, $h = 0.5 \cdot 10^{-3}$.

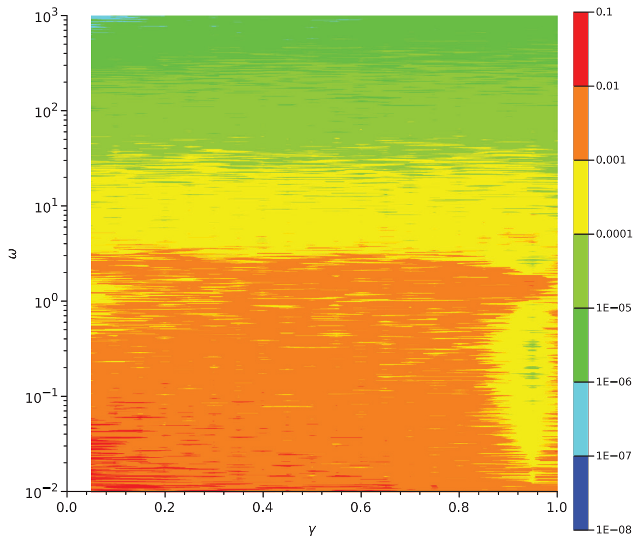


**Fig. 8.** Amplitude of the frequency response $|X(\omega)|$ versus $(\omega, \gamma)$ of the piston position with $n_1 = n_2 = 1000$, $\sigma_1 = \sigma_2 = 1.0$, $0 \le t \le 5 \cdot 10^4$, $h = 0.5 \cdot 10^{-3}$.

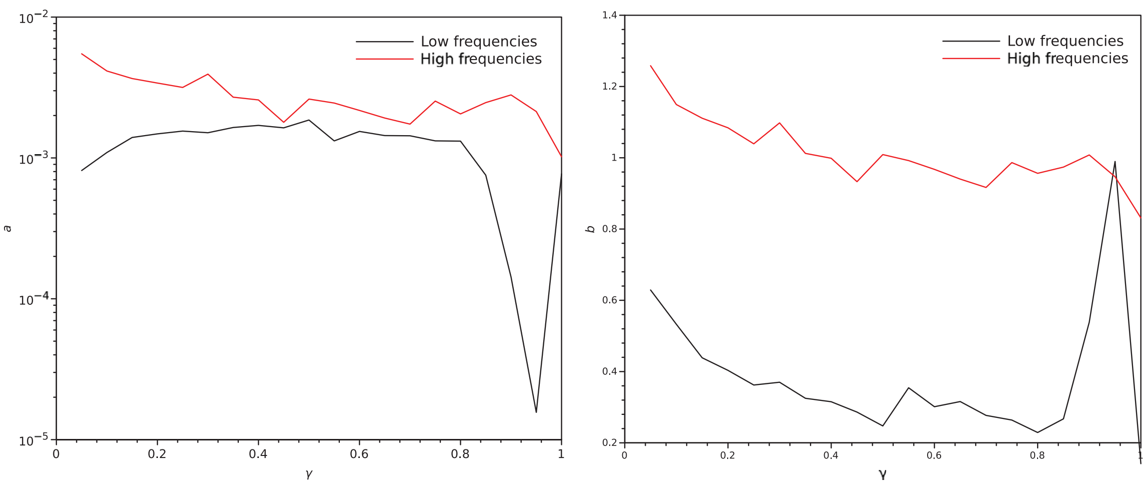


**Fig. 9.** Variation of parameters $a_l$ and $a_h$, and $b_l$ and $b_h$, versus $\gamma = \frac{m}{M}$, for $0 \le t \le 5 \cdot 10^4$, with $n_1 = n_2 = 5000$, $\sigma_1 = \sigma_2 = 1.0$.

We verify that $|\mathcal{F}\{x(t)\}|$ has distinct characteristics, but fractional in both cases.

Having these charts in mind, in a first set of experiments we vary $\eta = \frac{n_2}{n_1}$ while keeping constant $\nu = \frac{\sigma_2}{\sigma_1}$ and $\gamma = \frac{m}{M}$. Figs. 4 and 5 depict the locus $|X(\omega)|$ versus $(\omega, \nu)$ and the variation of parameters $a$ and $b$ (i.e., the pairs $a_l$, $a_h$, and $b_l$, $b_h$) versus $\eta$, respectively. Besides some "noise" in the charts, due to numerical approximations and the stochastic nature of the collision, we verify that at $b_l$ and $b_h$ are significantly distinct for low values of $\eta$, but tend to stabilize as $\eta \to 1$.

In a second set of experiments we keep constant $\eta$ and $\gamma$ while varying $\nu$. Figs. 6 and 7 depict the locus $|X(\omega)|$ versus $(\omega, \nu)$ and the variation of the pairs $a_l$, $a_h$, and $b_l$, $b_h$, versus $\nu$, respectively. The parameters exhibit a much smaller variation than in the previous case.

In a third set of experiments we keep constant $\eta$ and $\nu$ while varying $\gamma$. Figs. 8 and 9 depict the locus $|X(\omega)|$ versus $(\omega, \nu)$ and the variation of the pairs $a_l$, $a_h$, and $b_l$, $b_h$, versus $\gamma$, respectively. We note an interesting phenomenon for $0.8 < \gamma < 0.95$ with a kind of peak in the frequencies as shown in Fig. 10 (for $\gamma = 0.9$) with a considerable variation of the parameters $a_l$ and $b_l$.

In all cases was verified the existence of fBm, the presence of fractional order dynamics characterized by means of power law spectra, and the co-existence of distinct behaviors in the transient and steady-state time responses, that is to say, in the low and high frequencies.

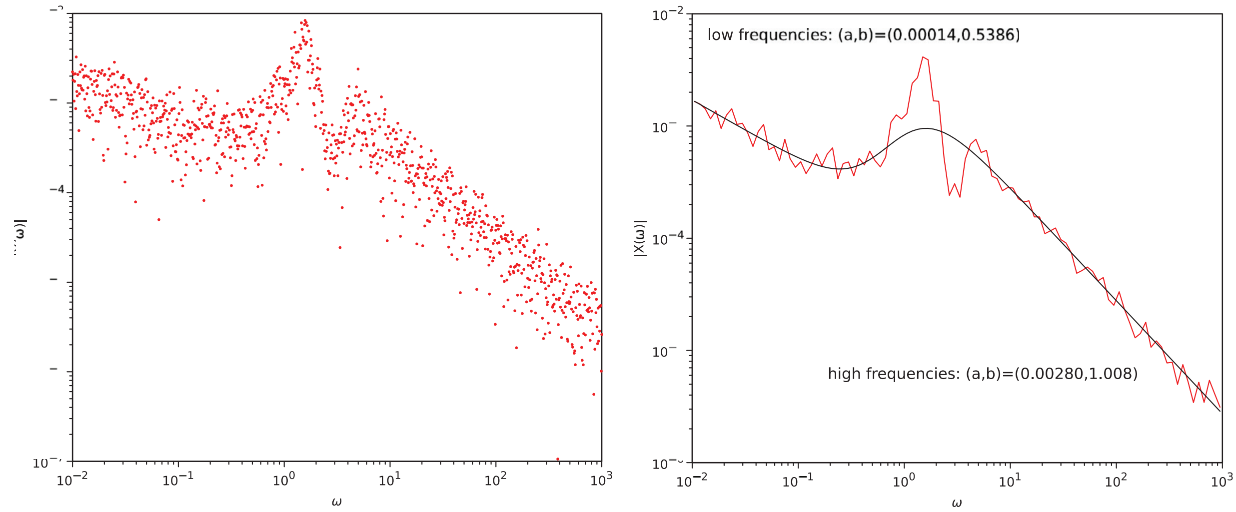


**Fig. 10.** Amplitude of the frequency response $|X(\omega)|$ versus $\omega$ of the piston position with $M = 10$, $m = 1$, $n_1 = 1000$, $n_2 = 500$, $\gamma = 0.9$, $0 \le t \le 5 \cdot 10^4$, $h = 0.5 \cdot 10^{-3}$. Left: initial response, right: moving average and PL approximations $(a_l, b_l) = (0.0012, 0.5908)$ and $(a_h, b_h) = (0.0062, 1.2371)$.

## 3. Conclusions

This paper studied the dynamical properties of Rayleigh piston. The novel contribution was in the viewpoint of fBm and FC. Several numerical experiments with distinct values for the system parameters, such as number of particles, their masses and their velocities, were conducted. The transient and steady-state behavior was characterized in the Fourier domain by means of power law approximations. The results demonstrated that fBm and FC are useful tools for investigating the complexity present in this classical system.

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