AN EFFICIENT NUMERICAL SCHEME FOR SOLVING MULTI-DIMENSIONAL FRACTIONAL OPTIMAL CONTROL PROBLEMS WITH A QUADRATIC PERFORMANCE INDEX

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ABSTRACT

The shifted Legendre orthogonal polynomials are used for the numerical solution of a new formulation for the multi-dimensional fractional optimal control problem (M-DFOCP) with a quadratic performance index. The fractional derivatives are described in the Caputo sense. The Lagrange multiplier method for the constrained extremum and the operational matrix of fractional integrals are used together with the help of the properties of the shifted Legendre orthonormal polynomials. The method reduces the M-DFOCP to a simpler problem that consists of solving a system of algebraic equations. For confirming the efficiency and accuracy of the proposed scheme, some test problems are implemented with their approximate solutions.

Key Words: Fractional optimal control problem, legendre polynomials, operational matrix, lagrange multiplier method, caputo derivatives, riemann–liouville integrals.

I. INTRODUCTION

In recent years, fractional calculus (theory of derivatives and integrals with any non-integer arbitrary order) gained considerable due to the considerable number of importance applications in different fields of physics and engineering such as solid mechanics [1], robotic bird [2], structure control [3], anomalous transport [4], continuum and statistical mechanics [5], fluid-dynamics [6], economics [7] and many other fields [8,9].

The operational matrix of fractional derivatives was derived for some types of orthogonal polynomials such as, the Legendre [10], Chebyshev [11] and Jacobi [12] polynomials and used to solve several types of fractional differential equations, (see [13–15]). On the other hand, the operational matrix of fractional integrals have been derived for many types of orthogonal polynomials such as, the Legendre [16], Chebyshev [17], Jacobi [18,19] and Laguerre [20] polynomials.

The optimal control problem refers to the minimization of a performance index subject to dynamic constraints on the state and control variables. If the fractional differential equations are used as the dynamic constraints, this leads to the fractional optimal control problems (FOCPs). Optimal control problems appear in engineering, science, economics, and many other fields. An extensive body of work exists in the area of optimal control of integer order dynamic systems, see [21–23], but limited work has been done in the area of FOCP. FOCPs have gained much attention for their many applications in engineering and physics. For example, it has been illustrated that materials with memory and hereditary effects, and dynamical processes, including gas diffusion and heat conduction in fractal porous media, can be more adequately modeled by fractional-order models than integer-order models [24]. Other applications of FOCPs are shown in [25–27]. For that reason, finding robust and accurate numerical methods for solving FOCPs has become an active research undertaking. In recent years, many researchers studied obtaining numerical solutions of FOCPs, (for instance see [28–33]). In most papers in this field, one-dimensional FOCPs were considered, where the problem contains one state variable, one control variable and one fractional differential equation as the dynamic constraint, (see [34,35]).

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Recently, Defterli [36] applied a numerical scheme to solve the two-dimensional FOCP with a quadratic performance index.

In this paper, we propose and develop a direct numerical technique for solving the multi-dimensional FOCP with a quadratic performance index

$$\text{Min.} \quad J = \frac{1}{2}\int_{t_0}^{t_1} \Big(b_1(t)x_1^2(t) + b_2(t)x_2^2(t) + \cdots + b_n(t)x_n^2(t) + a_0(t)u^2(t)\Big)dt, \tag{I.1}$$

subject to the dynamic constraints,

$$\begin{aligned} D^{\nu}x_1(t) &= b_{n+1}(t)x_1(t) + b_{n+2}(t)x_2(t) + \cdots + b_{2n}(t)x_n(t) + a_1(t)u(t),\\ D^{\nu}x_2(t) &= b_{2n+1}(t)x_1(t) + b_{2n+2}(t)x_2(t) + \cdots + b_{3n}(t)x_n(t) + a_2(t)u(t),\\ \vdots \quad &= \quad \vdots \qquad \vdots \qquad \vdots \qquad \vdots\\ D^{\nu}x_n(t) &= b_{n^2+1}(t)x_1(t) + b_{n^2+2}(t)x_2(t) + \cdots + b_{n^2+n}(t)x_n(t) + a_n(t)u(t),\\ x_1(0) &= x_1, \quad x_2(0) = x_2, \quad \cdots \quad , x_n(0) = x_n, \end{aligned} \tag{I.2}$$

where $t_0 \leq t \leq t_1$ and $0 \leq \nu \leq 1$.

The proposed numerical scheme consists of expanding the control variable $u(t)$ and the fractional derivatives of the state variables $D^{\nu}x_j(t)$, $j = 1, 2, \cdots, n$, by means of the Legendre orthonormal polynomials with unknown coefficients using the operational matrix of fractional integrals. Then, the system of equations, derived from the system of dynamic constraints (I.2), is adjoined to the performance index (I.1), using the Lagrange multiplier method. Finally, the M-DFOCP (I.1)-(I.2) is reduced to a system of algebraic equations that can be solved by an iterative method.

This article is organized as follows. In Section II, we introduce some definitions and notations of fractional calculus and we derive the operational matrix of fractional integrals for the shifted orthonormal Legendre polynomials. In Section III, the operational matrix of fractional integrals and the properties of the shifted Legendre orthonormal polynomials are adopted, together with the help of the Lagrange multiplier method in order to introduce an approximate solution for the M-DFOCP (I.1)-(I.2). In Section IV, three numerical examples are developed. The new results and those obtained by other methods are compared and discussed. Finally, Section V presents the main conclusion.

II. PRELIMINARIES AND NOTATIONS

2.1 Fractional calculus definitions

The Riemann–Liouville and Caputo fractional definitions are two often used definitions of fractional derivatives.

Definition 1.1. The integral of order $\gamma \geq 0$ (fractional) according to Riemann-Liouville is given by

$$\begin{aligned} I^{\gamma}f(t) &= \frac{1}{\Gamma(\gamma)}\int_0^t (t-y)^{\gamma-1}f(y)dy, \quad \gamma > 0, \quad t > 0,\\ I^0 f(t) &= f(t), \end{aligned} \tag{2.1}$$

where

$$\Gamma(\gamma) = \int_0^{\infty} t^{\gamma-1}e^{-t}dt$$

is gamma function.

The operator I^{ν} satisfies the following properties

$$\begin{aligned} I^{\gamma}I^{\delta}f(t) &= I^{\gamma+\delta}f(t),\\ I^{\gamma}I^{\delta}f(t) &= I^{\delta}I^{\gamma}f(t),\\ I^{\gamma}t^{\beta} &= \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\gamma)}\,t^{\beta+\gamma}. \end{aligned} \tag{2.2}$$

Definition 1.2. The Caputo fractional derivative of order γ is defined by

$$\begin{aligned} D^{\gamma}f(t) &= \frac{1}{\Gamma(m-\gamma)}\int_0^t (t-y)^{m-\gamma-1}\frac{d^m}{dy^m}f(y)dy,\\ &m-1 < \gamma \leq m, \ t > 0, \end{aligned} \tag{2.3}$$

where m is the ceiling function of γ.

The operator D^γ satisfies the following properties

$$
\begin{aligned}
D^\gamma C &= 0, \qquad (C \text{ is constant}),\\
I^\gamma D^\gamma f(t) &= f(t) - \sum_{i=0}^{m-1} f^{(i)}(0^+)\frac{t^i}{i!},\\
D^\gamma t^\beta &= \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\gamma)}\, t^{\beta-\gamma},\\
D^\gamma(\lambda f(t) + \mu g(t)) &= \lambda D^\gamma f(t) + \mu D^\gamma g(t).
\end{aligned} \tag{2.4}
$$

2.2 Shifted legendre polynomials

Assuming that the Legendre polynomial of degree k is denoted by $P_k(z)$ (defined on the interval $(-1, 1)$). Then $P_k(z)$ may be generated by the recurrence formulae

$$
\begin{aligned}
&P_{k+1}(z) = \frac{2k+1}{k+1} z P_k(z) - \frac{k}{k+1} P_{k-1}(z), \qquad 1 \le k,\\
&P_0(z) = 1, \qquad P_1(z) = z.
\end{aligned}
$$

Introducing $z = 2t - 1$, Legendre polynomials are defined on the interval $(0, 1)$ that may be called shifted Legendre polynomials $P_k^*(t)$ and generated using the following recurrence formulae

$$
\begin{aligned}
&P^*_{k+1}(t) = \frac{2k+1}{k+1}(2t-1)P^*_k(t) - \frac{k}{k+1} P^*_{k-1}(t), \qquad 1 \le k,\\
&P^*_0(t) = 1, \qquad P^*_1(t) = 2t - 1.
\end{aligned}
$$

The orthogonality relation is

$$
\int_0^1 P^*_j(t) P^*_k(t)dt = \begin{cases} \dfrac{1}{2k+1}, & \text{for } j = k,\\ 0, & \text{for } j \neq k. \end{cases} \tag{2.5}
$$

The explicit analytical form of shifted Legendre polynomial $P_k^*(t)$ of degree k may be written as

$$
P^*_k(t) = \sum_{i=0}^{k} (-1)^{k+i} \frac{(k+i)!}{(k-i)!\,(i!)^2}\, t^i. \tag{2.6}
$$

Introducing the shifted Legendre orthonormal polynomials $P^\star_k(t)$; $P^\star_k(t) \equiv \sqrt{2k+1}P^*_k(t)$, we have

$$
\int_0^1 P^\star_j(t) P^\star_k(t)dt = \begin{cases} 1, & \text{for } j = k,\\ 0, & \text{for } j \neq k, \end{cases} \tag{2.7}
$$

and

$$
P^\star_k(t) = \sqrt{2k+1} \sum_{i=0}^{k} (-1)^{k+i} \frac{(k+i)!}{(k-i)!\,(i!)^2}\, t^i. \tag{2.8}
$$

Any square integrable function $y(t)$ defined on the interval $(0, 1)$, may be expressed in terms of shifted Legendre polynomials $P^\star_k(t)$ as

$$
y(t) = \sum_{k=0}^{\infty} y_k P^\star_k(t),
$$

from which the coefficients y_k are given by

$$
y_k = \int_0^1 y(t) P^\star_k(t)dt, \qquad 0 \le k. \tag{2.9}
$$

If we approximate $y(t)$ by the first $(N+1)$-terms, then we can write

$$
y_N(t) = \sum_{k=0}^{N} y_k P^\star_k(t), \tag{2.10}
$$

which alternatively may be written in the matrix form:

$$
y_N(t) \simeq \mathbf{Y}^T \Delta_N(t), \tag{2.11}
$$

with

$$
\mathbf{Y} = \begin{pmatrix} y_0\\ y_1\\ \vdots\\ y_N \end{pmatrix}, \qquad \Delta_N(t) = \begin{pmatrix} P^\star_0(t)\\ P^\star_1(t)\\ \vdots\\ P^\star_N(t) \end{pmatrix}. \tag{2.12}
$$

2.3 Operational matrix for fractional integrals

Theorem 2.1. The fractional integral of order ν (in the sense of Riemann–Liouville) of the shifted Legendre polynomial vector $\Delta_N(t)$ is given by

$$
I^\nu \Delta_N(t) = \mathbf{I}^{(\nu)} \Delta_N(t), \tag{2.13}
$$

where $\mathbf{I}^{(\nu)}$ is the $(N+1)\times(N+1)$ operational matrix of fractional integral of order ν and is defined by

$$\mathbf{I}^{(\nu)} = \begin{pmatrix} \Theta_\nu(0,0) & \Theta_\nu(0,1) & \cdots & \Theta_\nu(0,j) & \cdots & \Theta_\nu(0,N) \\ \Theta_\nu(1,0) & \Theta_\nu(1,1) & \cdots & \Theta_\nu(1,j) & \cdots & \Theta_\nu(1,N) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \Theta_\nu(i,0) & \Theta_\nu(i,1) & \cdots & \Theta_\nu(i,j) & \cdots & \Theta_\nu(i,N) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \Theta_\nu(N,0) & \Theta_\nu(N,1) & \cdots & \Theta_\nu(N,j) & \cdots & \Theta_\nu(N,N) \end{pmatrix},$$

where

$$\Theta_\nu(i,j) = \sum_{k=0}^{i} \delta_\nu(i,j,k) \tag{2.14}$$

and

$$\delta_\nu(i.j,k) = \sqrt{(2j+1)(2i+1)} \sum_{l=0}^{j} \frac{(-1)^{i+j+k+l}(i+k)!(l+j)!}{(i-k)!k!\Gamma(k+\nu+1)(j-l)!(l!)^2(k+l+\nu+1)}.$$

Proof. Using (2.4) and (2.8), the fractional integral of order ν for the shifted Legendre polynomials $P_i^\star(t)$ is given by

$$\begin{aligned} I^\nu P_i^\star(t) &= \sqrt{2i+1} \sum_{k=0}^{i} (-1)^{i+k} \frac{(i+k)!}{(i-k)!\,(k!)^2}\, I^\nu t^k, \\ &= \sqrt{2i+1} \sum_{k=0}^{i} (-1)^{i+k} \frac{(i+k)!}{(i-k)!\,k!\,\Gamma(k+\nu+1)}\, t^{k+\nu}. \end{aligned} \tag{2.15}$$

Now we can approximate $t^{k+\nu}$ by $N+1$ terms of shifted Legendre polynomials $P_j^\star(t)$ as:

$$t^{k+\nu} = \sum_{j=0}^{N} \mu_{kj} P_j^\star(t), \tag{2.16}$$

where μ_{kj} is given as in (2.9) with $y(t) = t^{k+\nu}$, then

$$\begin{aligned} \mu_{kj} &= \int_0^1 t^{k+\nu} P_j^\star(t)dt \\ &= \sqrt{2j+1} \sum_{l=0}^{j} (-1)^{j+l} \frac{(j+l)!}{(j-l)!\,(l!)^2} \int_0^1 t^{l+k+\nu} dt \\ &= \sqrt{2j+1} \sum_{l=0}^{j} (-1)^{j+l} \frac{(j+l)!}{(j-l)!\,(l!)^2\,(k+\nu+l+1)}. \end{aligned} \tag{2.17}$$

Employing (2.15)–(2.17), we have

$$\begin{aligned} I^\nu P_i^\star(t) &= \sqrt{2i+1} \sum_{k=0}^{i} \sum_{j=0}^{N} (-1)^{i+k} \frac{(i+k)!}{(i-k)!\,k!\,\Gamma(k+\nu+1)} \\ &\quad \times \mu_{kj} P_j^\star(t) \\ &= \sum_{j=0}^{N} \Theta_\nu(i,j) P_j^\star(t), \end{aligned} \tag{2.18}$$

where $\Theta(i,j)$ is given in (2.14).

Finally, we can rewrite (2.18) in a vector form as

$$\begin{aligned} I^\nu P_i^\star(t) \simeq \big[&\Theta_\nu(i,0),\ \Theta_\nu(i,1),\ \cdots, \\ &\Theta_\nu(i,j),\ \cdots,\ \Theta_\nu(i,N)\big] \Delta_N(t). \end{aligned} \tag{2.19}$$

Eq. 2.19 completes the proof.

III. THE NUMERICAL SCHEME

In this section, we discuss our numerical scheme to approximate the solution of the M-DFOCP in the following form:

$$\text{Min.} \quad J = \frac{1}{2} \int_{t_0}^{t_1} \Big(\sum_{i=1}^{n} \left[b_i(t) x_i^2(t) \right] + a_0(t) u^2(t) \Big) dt, \tag{3.1}$$

subjected to the dynamic constraints,

$$\begin{aligned} D^\nu x_j(t) &= \sum_{i=1}^{n} \left[b_{jn+i}(t) x_i(t) \right] + a_j(t) u(t), \qquad j = 1, 2, \cdots, n. \\ x_j(0) &= x_j, \end{aligned} \tag{3.2}$$

First, we can approximate $D^\nu x_j(t)$, $(j = 1, 2, \cdots, n)$ and $u(t)$ by the shifted Legendre orthonormal polynomials $P_k^\star(t)$ as

$$\begin{aligned} D^\nu x_j(t) &\simeq \mathbf{C}_j^T \Delta_N(t), \qquad j = 1, 2, \cdots, n, \\ u(t) &\simeq \mathbf{U}^T \Delta_N(t), \end{aligned} \tag{3.3}$$

where $\mathbf{C}_j$, $(j = 1, 2, \cdots, n)$ and $\mathbf{U}$ are unknown coefficients matrices that can be written as

$$\mathbf{U} = \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_N \end{pmatrix}, \qquad \mathbf{C}_j = \begin{pmatrix} c_{j,0} \\ c_{j,1} \\ \vdots \\ c_{j,N} \end{pmatrix}, \qquad j = 1, 2, \cdots, n. \tag{3.4}$$

Using (2.4), we have

$$I^\nu D^\nu x_j(t) = x_j(t) - x_j(0), \qquad j = 1, 2, \cdots, n, \tag{3.5}$$

also adopting (2.13) together with (3.3), we get

$$I^\nu D^\nu x_j(t) \simeq \mathbf{C}_j^T \mathbf{I}^{(\nu)} \Delta_N(t), \qquad j = 1, 2, \cdots, n. \tag{3.6}$$

Using (3.5) and (3.6), we can write

$$x_j(t) \simeq \mathbf{C}_j^T \mathbf{I}^{(\nu)} \Delta_N(t) + x_j(0), \qquad j = 1, 2, \cdots, n. \tag{3.7}$$

By approximating $x_j(0)$ by the shifted Legendre orthonormal polynomials $P_k^\star(t)$ as

$$x_j(0) \simeq \mathbf{F}_j^T \Delta_N(t), \qquad j = 1, 2, \cdots, n, \tag{3.8}$$

where

$$\mathbf{F}_j = \begin{pmatrix} x_j \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad j = 1, 2, \cdots, n, \tag{3.9}$$

we can approximate $x_j(t)$ as

$$x_j(t) \simeq \left(\mathbf{C}_j^T \mathbf{I}^{(\nu)} + \mathbf{F}_j^T\right) \Delta_N(t), \qquad j = 1, 2, \cdots, n. \tag{3.10}$$

Also, we approximate $b_i(t)$ $(i = 1, 2, \cdots, n(n+1))$ and $a_j(t)$ $(j = 0, 1, \cdots, n)$ by the the shifted Legendre orthonormal polynomials $P_k^\star(t)$ as

$$\begin{aligned} b_i(t) &\simeq \mathbf{B}_i^T \Delta_N(t), \qquad i = 1, 2, \cdots, n(n+1), \\ a_j(t) &\simeq \mathbf{A}_j^T \Delta_N(t), \qquad j = 0, 1, \cdots, n, \end{aligned} \tag{3.11}$$

where

$$\mathbf{B}_i = \begin{pmatrix} b_{i,0} \\ b_{i,1} \\ \vdots \\ b_{i,N} \end{pmatrix}, \qquad \mathbf{A}_j = \begin{pmatrix} a_{j,0} \\ a_{j,1} \\ \vdots \\ a_{j,N} \end{pmatrix}, \qquad i = 1, 2, \cdots, n(n+1), \quad j = 0, 1, \cdots, n. \tag{3.12}$$

and

$$\begin{aligned} b_{i,k} &= \int_0^1 b_i(t) P_k^\star(t) dt, \\ &\quad k = 0, 1, \cdots, N, \ i = 1, 2, \cdots, n(n+1), \\ a_{j,k} &= \int_0^1 a_j(t) P_k^\star(t) dt, \\ &\quad k = 0, 1, \cdots, N, \ j = 0, 1, \cdots, n. \end{aligned} \tag{3.13}$$

For general functions $b_i(t)$ $(i = 1, 2, \cdots, n(n+1))$ and $a_j(t)$ $(j = 0, 1, \cdots, n)$, it is more difficult to compute the previous integrals exactly. Using the Legendre-Gauss quadrature formula, we can approximate the coefficients $b_{i,k}$ and $a_{j,k}$ as

$$\begin{aligned} b_{i,k} &= \sum_{\epsilon=0}^{N} b_i(t_{N,\epsilon}) P_k^\star(t_{N,\epsilon}) \varpi_{N,\epsilon}, \\ &\quad k = 0, 1, \cdots, N, \ i = 1, 2, \cdots, n(n+1), \\ a_{j,k} &= \sum_{\epsilon=0}^{N} a_j(t_{N,\epsilon}) P_k^\star(t_{N,\epsilon}) \varpi_{N,\epsilon}, \\ &\quad k = 0, 1, \cdots, N, \ j = 0, 1, \cdots, n, \end{aligned}$$

where $t_{N,\epsilon}$, $0 \le \epsilon \le N$ are the zeros of Legendre–Gauss quadrature in the interval $(0, 1)$, with $\varpi_{N,\epsilon}$, $0 \le \epsilon \le N$ are corresponding Christoffel numbers.

Using (3.3), (3.10) and (3.11), we can approximate $J \equiv J[\mathbf{C}_1, \mathbf{C}_2, \cdots, \mathbf{C}_n, \mathbf{U}]$ as

$$J_N \simeq \frac{1}{2}\int_{t_0}^{t_1}\Big(\sum_{i=1}^{n}\big[\big(\mathbf{B}_i^T\Delta_N(t)\big)\big(\mathbf{C}_i^T\mathbf{I}^{(\nu)} + d_i^T\big)\Delta_N(t)\Delta_N^T(t) \big(\mathbf{C}_i^T\mathbf{I}^{(\nu)} + d_i^T\big)^T\big] + \big(\mathbf{A}_0^T\Delta_N(t)\big) \times \big(\mathbf{U}^T\Delta_N(t)\Delta_N^T(t)\mathbf{U}\big)\Big)dt. \tag{3.14}$$

Employing (3.3), (3.10) and (3.11), the dynamic constraints (3.2) can be approximated as

$$\mathbf{C}_j^T\Delta_N(t) - \sum_{i=1}^{n}\big[\mathbf{B}_{jn+i}^T\Delta_N(t)\Delta_N^T(t)\big(\mathbf{C}_i^T\mathbf{I}^{(\nu)} + d_i^T\big)^T\big] - \mathbf{A}_j^T\Delta_N(t)\Delta_N^T(t)\mathbf{U} = 0, \quad j = 1, 2, \cdots, n. \tag{3.15}$$

Let

$$\mathbf{B}_{jn+i}^T\Delta_N(t)\Delta_N^T(t) \simeq \Delta_N^T(t)\mathbf{G}_{jn+i}^T, \qquad i,j = 1, 2, \cdots, n, \tag{3.16}$$

$$\mathbf{A}_j^T\Delta_N(t)\Delta_N^T(t) \simeq \Delta_N^T(t)\mathbf{H}_j^T, \qquad j = 1, 2, \cdots, n, \tag{3.17}$$

where $\mathbf{G}_{jn+i}$ and $\mathbf{H}_j$ are $N \times N$ matrices.

In order to compute $\mathbf{G}_{jn+i}$ $(i,j = 1, 2, \cdots, n)$ and $\mathbf{H}_j$ $(j = 1, 2, \cdots, n)$, we may write (3.16) and (3.17) as

$$\sum_{k=0}^{N} b_{jn+i,k}P_k^\star(t)P_l^\star(t) = \sum_{k=0}^{N}\mathbf{G}_{jn+i(lk)}P_k^\star(t), \qquad i, j = 1, 2, \cdots, n, \ l = 0, 1, \cdots, N, \tag{3.18}$$

$$\sum_{k=0}^{N} a_{j,k}P_k^\star(t)P_l^\star(t) = \sum_{k=0}^{N}\mathbf{H}_{j(lk)}P_k^\star(t), \qquad j = 1, 2, \cdots, n, \ l = 0, 1, \cdots, N. \tag{3.19}$$

Multiplying both sides of (3.18) and (3.19) by $P_m^\star(t)$, $m = 0, 1, \cdots, N$ and integrating from 0 to 1, we have

$$\sum_{k=0}^{N} b_{jn+i,k}\int_0^1 P_k^\star(t)P_l^\star(t)P_m^\star(t)dt = \sum_{k=0}^{N}\mathbf{G}_{jn+i(lm)} \times \int_0^1 P_k^\star(t)P_m^\star(t)dt, \tag{3.20}$$

$$\sum_{k=0}^{N} a_{j,k}\int_0^1 P_k^\star(t)P_l^\star(t)P_m^\star(t)dt \sum_{k=0}^{N}\mathbf{H}_{j(lm)}\int_0^1 P_k^\star(t)P_m^\star(t)dt, \tag{3.21}$$

$$i,j = 1, 2, \cdots, n, \qquad l,m = 0, 1, \cdots, N.$$

By means of the orthogonality relation (2.7), we get

$$\mathbf{G}_{jn+i(lm)} = \sum_{k=0}^{N} b_{jn+i,k}\int_0^1 P_k^\star(t)P_l^\star(t)P_m^\star(t)dt, \quad \mathbf{H}_{j(lm)} = \sum_{k=0}^{N} a_{j,k}\int_0^1 P_k^\star(t)P_l^\star(t)P_m^\star(t)dt, \quad i,j = 1, 2, \cdots, n, \quad l,m = 0, 1, \cdots, N. \tag{3.22}$$

Using (3.16) and (3.17), Eq. 3.15 may be written as

$$\mathbf{C}_j^T\Delta_N(t) - \sum_{i=1}^{n}\big[\Delta_N^T(t)\mathbf{G}_{jn+i}^T\big(\mathbf{C}_i^T\mathbf{I}^{(\nu)} + d_i^T\big)^T\big] - \Delta_N^T(t)\mathbf{H}_j^T\mathbf{U} = 0, \quad j = 1, 2, \cdots, n, \tag{3.23}$$

or

$$\Big(\mathbf{C}_j^T - \sum_{i=1}^{n}\big[\mathbf{G}_{jn+i}^T\big(\mathbf{C}_i^T\mathbf{I}^{(\nu)} + d_i^T\big)^T\big] - \mathbf{H}_j^T\mathbf{U}\Big)\Delta_N(t) = 0, \quad j = 1, 2, \cdots, n. \tag{3.24}$$

Thus, the dynamic constraints (3.2) are converted into the following linear system of algebraic equations:

$$\mathbf{C}_j^T - \sum_{i=1}^{n} [\mathbf{G}_{jn+i}^T (\mathbf{C}_i^T \mathbf{I}^{(\nu)} + d_i^T)^T] - \mathbf{H}_j^T \mathbf{U} = 0, \quad j = 1, 2, \cdots, n. \tag{3.25}$$

Let

$$J^\star[\mathbf{C}_1, \mathbf{C}_2, \cdots, \mathbf{C}_n, \mathbf{U}, \lambda] = J[\mathbf{C}_1, \mathbf{C}_2, \cdots, \mathbf{C}_n, \mathbf{U}] + \sum_{j=1}^{n} \Big(\mathbf{C}_j^T - \sum_{i=1}^{n} [\mathbf{G}_{jn+i}^T (\mathbf{C}_i^T \mathbf{I}^{(\nu)} + d_i^T)^T] - \mathbf{H}_j^T \mathbf{U} \Big) \lambda_j, \tag{3.26}$$

where

$$\lambda_j = \begin{pmatrix} \lambda_{j,0} \\ \lambda_{j,1} \\ \vdots \\ \lambda_{j,N} \end{pmatrix}, \qquad j = 1, 2, \cdots, n, \tag{3.27}$$

is the unknown Lagrange multiplier.

The necessary conditions for the optimality of the performance index (3.1) subjected to the dynamic constraints (3.2) are

$$\begin{aligned} \frac{\partial J^\star}{\partial c_{j,k}} &= 0, \qquad j = 1, 2, \cdots, n, \\ \frac{\partial J^\star}{\partial u_k} &= 0, \qquad k = 0, 1, \cdots, N. \\ \frac{\partial J^\star}{\partial \lambda_{j,k}} &= 0, \qquad j = 1, 2, \cdots, n, \end{aligned} \tag{3.28}$$

The system of algebraic equations introduced above can be solved by using any standard iteration method for the unknown coefficients $c_{j,k}$, u_k and $\lambda_{j,k}$, $j = 1, 2, \cdots, n$, $k = 0, 1, \cdots, N$. Consequently, $\mathbf{C}_j$, $\mathbf{U}$ and λ_j given in (3.4) and (3.27) can be obtained.

IV. NUMERICAL EXPERIMENTS

In order to demonstrate the validity and accuracy of the proposed numerical scheme, we solve three problems and we compare the results obtained using the novel algorithm and those obtained using other methods.

4.1 One-dimensional FOCP

As the first example, we consider the following one-dimensional FOCP studied in [34,37]

$$\text{Min.} \quad J = \frac{1}{2} \int_0^1 \big(x^2(t) + u^2(t) \big) dt, \tag{4.1}$$

subjected to the dynamic constraints,

$$\begin{aligned} D^\nu x(t) &= -x(t) + u(t), \\ x(0) &= 1. \end{aligned} \tag{4.2}$$

The exact solution of this problem for $\nu = 1$ is

$$\begin{aligned} x(t) &= \cosh(\sqrt{2}t) + \beta \sinh(\sqrt{2}t), \\ u(t) &= (1 + \sqrt{2}\beta) \cosh(\sqrt{2}t) + (\sqrt{2} + \beta) \sinh(\sqrt{2}t), \end{aligned} \tag{4.3}$$

where

$$\beta = -\frac{\cosh(\sqrt{2}t) + \sqrt{2} \sinh(\sqrt{2}t)}{\sqrt{2} \cosh(\sqrt{2}t) + \sinh(\sqrt{2}t)}.$$

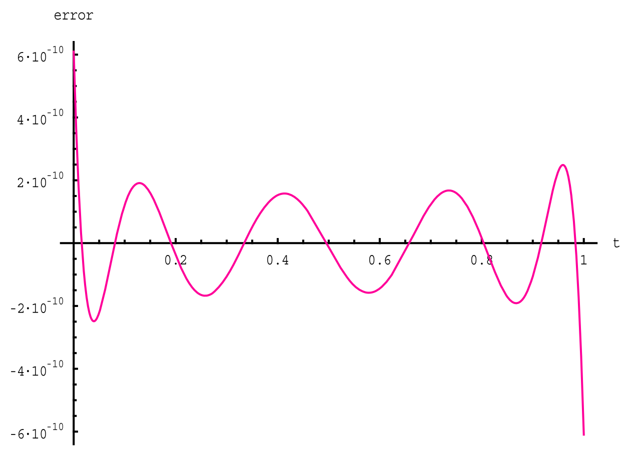

Fig. 1. Absolute error of $x(t)$ at $N = 8$ with $\nu = 1$ for problem (4.1).

In [34], the Lagrange multiplier method and the calculus of variations were used together with the formula for fractional integration by parts to obtain approximate solutions of the control variable $u(t)$ and the state variable $x(t)$. The authors used $N = 10, 20, 40, 80, 160, 230$, but achieved reasonable results for the approximate values of $u(t)$ and $x(t)$ only when adopting a large f N (see Figs 1–4 in [34]). Also, Jafari and Tajadodi [37] used the operational matrices of Bernstein polynomials to approximate the solution of this problem.

Figs 1 and 2 depict the absolute errors of the state variable $x(t)$ and the control variable $u(t)$ at $N = 8$ and $\nu = 1$. Figs 3 and 4 present the approximate values of $x(t)$ and $u(t)$ as functions of time when $N = 6$ and various values of ν namely, $\nu = 0.80, 0.90, 0.99$, and 1. In Table I,

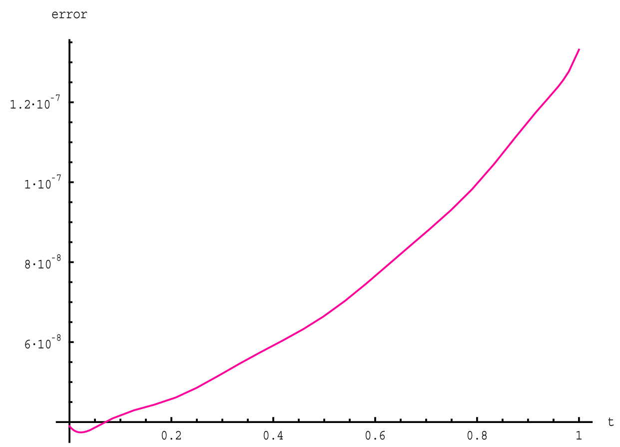

Fig. 2. Absolute error of $u(t)$ at $N = 8$ with $\nu = 1$ for problem (4.1).

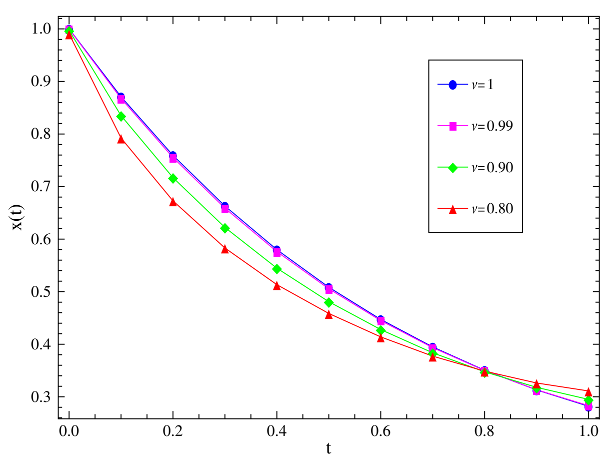

Fig. 3. Approximate solutions of $x(t)$ at $N = 6$ and $\nu = 1,\ 0.99,\ 0.90$ and 0.80 for problem (4.1).

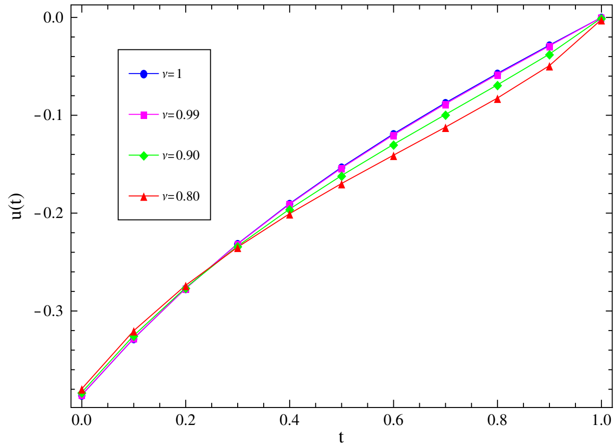

Fig. 4. Approximate solutions of $u(t)$ at $N = 6$ and $\nu = 1,\ 0.99,\ 0.90$ and 0.80 for problem (4.1).

we compare the absolute errors of $x(t)$ using our method with those achieved using this in [37]. Tables II and III list the absolute errors of $x(t)$ and $u(t)$ for $\nu = 1$ and various values of N.

From Figs 1 and 2 and Tables I–III, it is clear that adding few terms of shifted Legendre orthonormal polynomials, leads to good approximations of the exact state and control variables. Figs 3 and 4 reveal that as ν approaches to 1, the solution for the integer order system is recovered.

4.2 Two-dimensional FOCP

As a two-dimensional FOCP, we consider the following problem [33,36]

$$\text{Min.} \quad J = \frac{1}{2}\int_0^1 \left(x_1^2(t) + x_2^2(t) + u^2(t)\right)dt, \tag{4.4}$$

subjected to the dynamic constraints,

$$\begin{aligned} D^\nu x_1(t) &= -x_1(t) + x_2(t) + u(t),\\ D^\nu x_2(t) &= -2x_2(t),\\ x_1(0) &= x_2(0) = 1. \end{aligned} \tag{4.5}$$

The exact solution of this problem for $\nu = 1$ is

$$\begin{aligned} x_1(t) &= -\frac{3}{2}e^{-2t} + 2.48164e^{-\sqrt{2}t} + 0.018352e^{\sqrt{2}t},\\ x_2(t) &= e^{-2t},\\ u(t) &= \frac{1}{2}e^{-2t} - 1.02793e^{-\sqrt{2}t} + 0.0443056e^{\sqrt{2}t}. \end{aligned}$$

Table I. Comparing of the new method with the one proposed in [37] for $x(t)$ at $\nu = 1$ for problem (4.1).

t	Method in [37]			Our method		
	$N = 3$	$N = 4$	$N = 5$	$N = 3$	$N = 4$	$N = 5$
0.1	$3.41.10^{-4}$	$4.77.10^{-5}$	$1.34.10^{-5}$	$3.30.10^{-4}$	$3.66.10^{-5}$	$2.39.10^{-6}$
0.2	$5.08.10^{-4}$	$3.25.10^{-5}$	$2.12.10^{-5}$	$4.86.10^{-4}$	$1.00.10^{-5}$	$1.21.10^{-6}$
0.3	$1.12.10^{-4}$	$7.74.10^{-6}$	$3.24.10^{-5}$	$7.78.10^{-5}$	$2.64.10^{-5}$	$1.72.10^{-6}$
0.4	$2.87.10^{-4}$	$2.13.10^{-5}$	$4.73.10^{-5}$	$3.34.10^{-4}$	$2.53.10^{-5}$	$6.82.10^{-7}$
0.5	$3.97.10^{-4}$	$6.43.10^{-5}$	$6.20.10^{-5}$	$4.57.10^{-4}$	$4.23.10^{-6}$	$1.93.10^{-6}$
0.6	$1.50.10^{-4}$	$1.03.10^{-4}$	$7.49.10^{-5}$	$2.30.10^{-4}$	$2.91.10^{-5}$	$3.10.10^{-7}$
0.7	$2.93.10^{-4}$	$1.12.10^{-4}$	$8.88.10^{-5}$	$2.02.10^{-4}$	$2.14.10^{-5}$	$1.90.10^{-6}$
0.8	$6.29.10^{-4}$	$9.14.10^{-5}$	$1.07.10^{-5}$	$5.21.10^{-3}$	$1.72.10^{-5}$	$9.16.10^{-7}$
0.9	$3.71.10^{-4}$	$9.41.10^{-5}$	$1.31.10^{-4}$	$2.42.10^{-4}$	$3.46.10^{-5}$	$2.49.10^{-6}$

Table II. Absolute errors of $x(t)$ at $\nu = 1$ and various values of N for problem (4.1).

t	$N = 6$	$N = 8$	$N = 10$
0.1	$6.86398.10^{-8}$	$1.21608.10^{-10}$	$5.77379.10^{-12}$
0.2	$1.00670.10^{-7}$	$3.87690.10^{-11}$	$8.27005.10^{-13}$
0.3	$1.33050.10^{-8}$	$1.05482.10^{-10}$	$9.84246.10^{-12}$
0.4	$9.00998.10^{-8}$	$1.52765.10^{-10}$	$8.83471.10^{-12}$
0.5	$9.27207.10^{-9}$	$1.27758.10^{-11}$	$3.52051.10^{-12}$
0.6	$9.13792.10^{-8}$	$1.44500.10^{-10}$	$1.11767.10^{-11}$
0.7	$4.44009.10^{-9}$	$1.23409.10^{-10}$	$4.95581.10^{-12}$
0.8	$9.93486.10^{-8}$	$1.63731.10^{-11}$	$5.44142.10^{-12}$
0.9	$8.05744.10^{-8}$	$1.06073.10^{-10}$	$1.02076.10^{-11}$

Defterli [36] used the Grünwald–Letnikov definition to approximate the Riemann–Liouville fractional derivatives for approximating its solution. Defterli considered $N = 8, 16, 32, 64, 128$ and achieved reasonable results for the approximate values of the control variable, $u(t)$, and the two state variables, $x_1(t)$ and $x_2(t)$, only for a large number of N. Also, Yousefi *et al.* [33] introduced this problem and applied the Legendre multiwavelet collocation method (LMWCM) for solving it numerically.

In Figs 5–7, we plot the approximate values of the state variables, $x_1(t)$ and $x_2(t)$, and the control variable $u(t)$ for $N = 8$ and various choices of ν, $\nu = 0.80, 0.90, 0.99$ and 1. In Figs 8–10, we present the absolute errors of $x_1(t)$, $x_2(t)$ and $u(t)$ at $N = 10$ and $\nu = 1$. Table IV shows the maximum absolute errors (MAEs) of $x_1(t)$, $x_2(t)$ and $u(t)$ using our scheme at $\nu = 1$ and various choices of N. Table V lists the absolute errors of $x_2(t)$ at $\nu = 1$ and

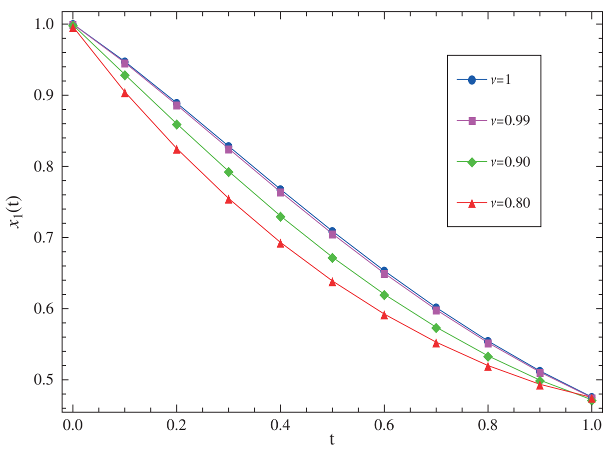

Fig. 5. Approximate solutions of $x_1(t)$ at $N = 8$ and $\nu = 1,\ 0.99,\ 0.90$ and 0.80 for problem (4.4).

Table III. Absolute errors of $u(t)$ at $\nu = 1$ and various values of N for problem (4.1).

t	$N = 3$	$N = 5$	$N = 7$	$N = 9$
0.1	$1.09654.10^{-4}$	$7.08189.10^{-7}$	$5.79289.10^{-11}$	$5.44732.10^{-12}$
0.2	$1.42091.10^{-4}$	$3.99869.10^{-7}$	$8.95548.10^{-10}$	$7.88896.10^{-13}$
0.3	$8.61549.10^{-6}$	$4.98380.10^{-7}$	$1.32975.10^{-9}$	$9.50503.10^{-12}$
0.4	$1.13021.10^{-4}$	$2.49280.10^{-7}$	$1.39678.10^{-11}$	$9.25248.10^{-12}$
0.5	$1.37870.10^{-4}$	$5.81011.10^{-7}$	$1.30231.10^{-9}$	$2.24265.10^{-12}$
0.6	$5.73271.10^{-5}$	$4.96686.10^{-8}$	$3.64046.10^{-10}$	$1.12853.10^{-11}$
0.7	$7.57957.10^{-5}$	$5.92683.10^{-7}$	$1.21586.10^{-9}$	$7.22766.10^{-12}$
0.8	$1.60967.10^{-4}$	$2.40909.10^{-7}$	$1.13464.10^{-9}$	$3.48032.10^{-12}$
0.9	$6.26788.10^{-5}$	$7.61099.10^{-7}$	$2.20743.10^{-10}$	$9.91018.10^{-12}$

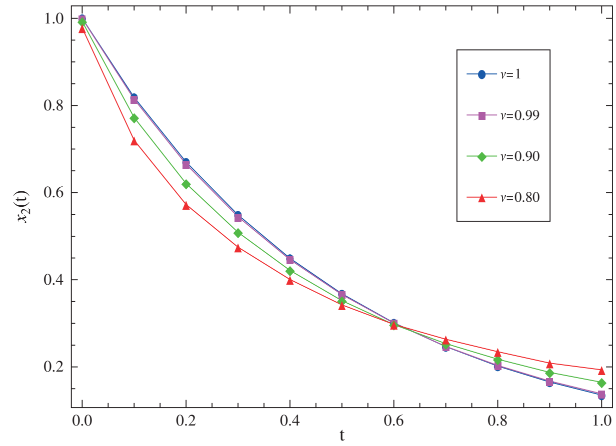

Fig. 6. Approximate solutions of $x_2(t)$ at $N = 8$ and $\nu = 1,\ 0.99,\ 0.90$ and 0.80 for problem (4.4).

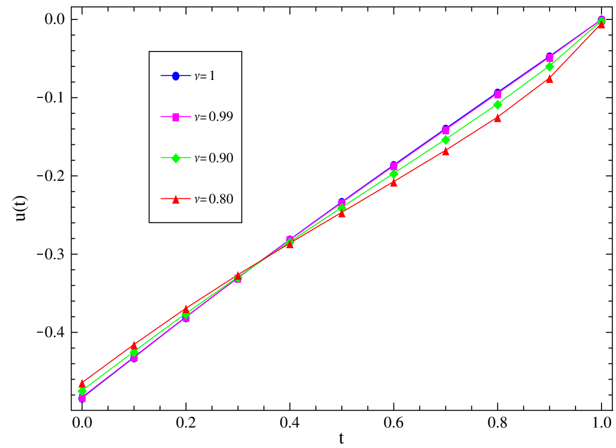

Fig. 7. Approximate solutions of $u(t)$ at $N = 8$ and $\nu = 1,\ 0.99,\ 0.90$ and 0.80 for problem (4.4).

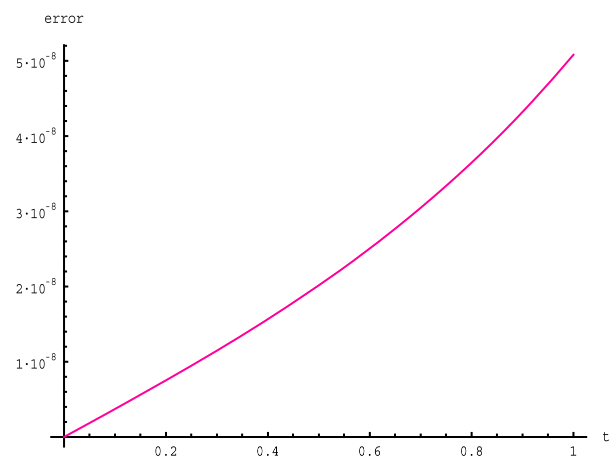

Fig. 8. Absolute error of $x_1(t)$ at $N = 10$ with $\nu = 1$ for problem (4.4).

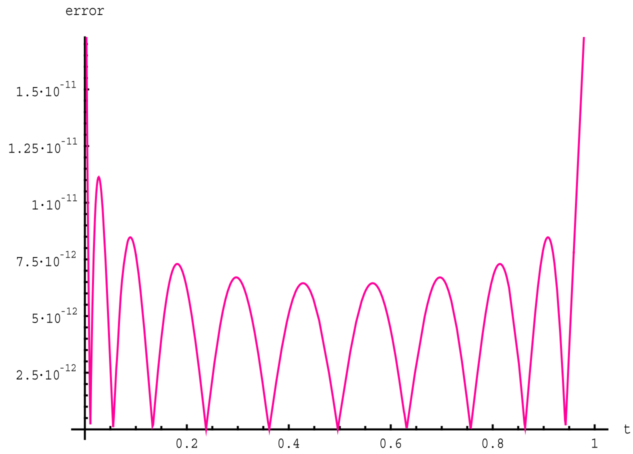

Fig. 9. Absolute error of $x_2(t)$ at $N = 10$ with $\nu = 1$ for problem (4.4).

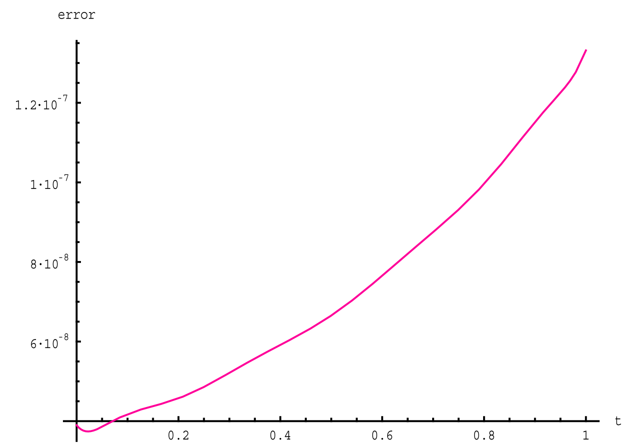

Fig. 10. Absolute error of $u(t)$ at $N = 10$ with $\nu = 1$ for problem (4.4).

Table IV. MAEs of $x_1(t),\ x_2(t)$ and $u(t)$ at $\nu = 1$ and various values of N for problem (4.4).

N	$x_1(t)$	$x_2(t)$	$u(t)$
3	$2.528709.10^{-3}$	$3.759398.10^{-3}$	$8.536543.10^{-4}$
4	$3.888715.10^{-4}$	$4.113533.10^{-4}$	$9.095703.10^{-5}$
5	$4.003185.10^{-5}$	$3.702743.10^{-5}$	$1.351619.10^{-5}$
6	$3.502786.10^{-6}$	$2.828678.10^{-6}$	$1.053620.10^{-6}$
7	$2.419059.10^{-7}$	$1.876229.10^{-7}$	$2.124256.10^{-7}$
8	$6.575392.10^{-8}$	$1.099375.10^{-8}$	$1.286924.10^{-7}$
9	$5.000459.10^{-8}$	$5.768396.10^{-10}$	$1.319097.10^{-7}$

various values of N. In order to demonstrate that our scheme is more accurate than the LMWCM [33], we compare in Tables VI–VIII the absolute errors of $x_1(t),\ x_2(t)$

Table V. Absolute errors of $x_2(t)$ at $\nu = 1$ and various values of N for problem (4.4).

t	$N = 5$	$N = 7$	$N = 9$	$N = 11$
0.1	$1.395.10^{-5}$	$1.482.10^{-9}$	$1.755.10^{-10}$	$1.833.10^{-13}$
0.2	$7.572.10^{-6}$	$3.552.10^{-8}$	$1.449.10^{-10}$	$3.969.10^{-14}$
0.3	$9.842.10^{-6}$	$5.139.10^{-8}$	$6.718.10^{-11}$	$9.892.10^{-14}$
0.4	$5.458.10^{-6}$	$1.486.10^{-9}$	$6.240.10^{-11}$	$2.063.10^{-13}$
0.5	$1.129.10^{-5}$	$5.058.10^{-8}$	$1.406.10^{-10}$	$2.664.10^{-13}$
0.6	$1.229.10^{-6}$	$1.322.10^{-8}$	$8.527.10^{-11}$	$2.313.10^{-13}$
0.7	$1.141.10^{-5}$	$4.756.10^{-8}$	$4.350.10^{-11}$	$1.362.10^{-13}$
0.8	$4.895.10^{-6}$	$4.356.10^{-8}$	$1.344.10^{-10}$	$7.757.10^{-14}$
0.9	$1.483.10^{-5}$	$7.911.10^{-9}$	$1.699.10^{-10}$	$2.117.10^{-13}$

and $u(t)$, at $\nu = 1$ and $N = 6$, with those obtained using the LMWCM [33].

4.3 Three-dimensional FOCP

In order to obtain that the proposed method can be applied for high dimensions problems, we consider the following problem as a three-dimensional FOCP

$$\text{Min.} \quad J = \frac{1}{2}\int_0^1 \left(x_1^2(t)+x_2^2(t)-x_3^2(t)+u^2(t)\right)dt, \quad (4.7)$$

subjected to the dynamic constraints,

$$\begin{aligned} D^\nu x_1(t) &= -2x_1(t) - tx_2(t) + u(t), \\ D^\nu x_2(t) &= 3x_1(t) + x_3(t) - u(t), \\ D^\nu x_3(t) &= tx_1(t) + x_2(t), \\ x_1(0) &= x_2(0) = x_3(0) = 1. \end{aligned} \quad (4.8)$$

This problem is solved by the numerical method introduced above. In Figs 11–14, we plot the approximate values of the state variables $x_1(t)$, $x_2(t)$, $x_3(t)$ and the control variable $u(t)$ at $N = 3$ with various choices of ν, $\nu = 0.60, 0.80, 0.90, 0.99$ and 1. As shown in Figs 11–14, by using the presented method we achieve satisfactory result with at most three elements of the shifted Legendre orthonormal basis, which demonstrates the efficiency of the presented method for high dimensions problems.

Table VI. Comparing the new method with the LMWCM [33] for $x_1(t)$ at $\nu = 1$ for problem (4.4).

t	LMWCM [33]			Our method		
	$N = 3$	$N = 4$	$N = 6$	$N = 3$	$N = 4$	$N = 6$
0.1	$4.496.10^{-4}$	$6.283.10^{-5}$	$6.947.10^{-6}$	$7.444.10^{-4}$	$1.590.10^{-4}$	$7.373.10^{-7}$
0.2	$1.152.10^{-3}$	$3.861.10^{-4}$	$5.662.10^{-6}$	$9.097.10^{-4}$	$3.592.10^{-5}$	$1.101.10^{-6}$
0.3	$2.906.10^{-3}$	$4.751.10^{-4}$	$1.420.10^{-6}$	$1.418.10^{-5}$	$1.169.10^{-4}$	$1.928.10^{-7}$
0.4	$3.764.10^{-3}$	$2.562.10^{-4}$	$5.483.10^{-7}$	$7.544.10^{-4}$	$1.027.10^{-4}$	$9.998.10^{-7}$
0.5	$3.342.10^{-3}$	$1.135.10^{-4}$	$3.537.10^{-6}$	$8.713.10^{-4}$	$2.658.10^{-5}$	$1.193.10^{-7}$
0.6	$1.763.10^{-3}$	$4.026.10^{-4}$	$5.847.10^{-6}$	$3.189.10^{-4}$	$1.266.10^{-4}$	$9.780.10^{-7}$
0.7	$4.660.10^{-4}$	$4.290.10^{-4}$	$3.908.10^{-6}$	$5.379.10^{-4}$	$8.487.10^{-5}$	$1.166.10^{-7}$
0.8	$2.556.10^{-3}$	$1.591.10^{-4}$	$1.154.10^{-7}$	$1.050.10^{-3}$	$8.129.10^{-5}$	$1.124.10^{-6}$
0.9	$3.512.10^{-3}$	$2.141.10^{-4}$	$1.843.10^{-6}$	$3.622.10^{-4}$	$1.456.10^{-4}$	$8.710.10^{-7}$

Table VII. Comparing the new method with the LMWCM [33] for $x_2(t)$ at $\nu = 1$ for problem (4.4).

t	LMWCM [33]			Our method		
	$N = 3$	$N = 4$	$N = 6$	$N = 3$	$N = 4$	$N = 6$
0.1	$7.175.10^{-4}$	$5.965.10^{-5}$	$1.802.10^{-6}$	$1.043.10^{-3}$	$1.680.10^{-4}$	$6.053.10^{-7}$
0.2	$1.630.10^{-3}$	$4.057.10^{-4}$	$1.281.10^{-6}$	$1.409.10^{-3}$	$4.112.10^{-5}$	$9.096.10^{-7}$
0.3	$4.397.10^{-3}$	$5.105.10^{-4}$	$1.754.10^{-6}$	$1.357.10^{-4}$	$1.226.10^{-4}$	$1.430.10^{-7}$
0.4	$6.018.10^{-3}$	$2.878.10^{-4}$	$2.109.10^{-6}$	$1.058.10^{-3}$	$1.113.10^{-4}$	$8.084.10^{-7}$
0.5	$5.810.10^{-3}$	$1.039.10^{-4}$	$6.633.10^{-7}$	$1.338.10^{-3}$	$2.471.10^{-5}$	$1.083.10^{-7}$
0.6	$3.814.10^{-3}$	$4.207.10^{-4}$	$2.867.10^{-6}$	$5.927.10^{-4}$	$1.336.10^{-4}$	$8.232.10^{-7}$
0.7	$6.659.10^{-4}$	$4.649.10^{-4}$	$1.555.10^{-6}$	$6.962.10^{-4}$	$9.287.10^{-5}$	$6.466.10^{-8}$
0.8	$2.517.10^{-3}$	$1.924.10^{-4}$	$1.528.10^{-6}$	$1.562.10^{-3}$	$8.327.10^{-5}$	$8.936.10^{-7}$
0.9	$4.218.10^{-3}$	$1.995.10^{-4}$	$1.285.10^{-6}$	$6.409.10^{-4}$	$1.557.10^{-4}$	$7.454.10^{-7}$

Table VIII. Comparing the new method with the LMWCM [33] for $u(t)$ at $\nu = 1$ for problem (4.4).

t	LMWCM [33]			Our method		
	$N = 3$	$N = 4$	$N = 6$	$N = 3$	$N = 4$	$N = 6$
0.1	$1.883.10^{-4}$	$1.046.10^{-5}$	$2.597.10^{-6}$	$2.287.10^{-4}$	$3.568.10^{-5}$	$1.711.10^{-7}$
0.2	$3.498.10^{-4}$	$8.057.10^{-5}$	$2.121.10^{-6}$	$3.118.10^{-4}$	$4.319.10^{-6}$	$3.728.10^{-7}$
0.3	$9.956.10^{-4}$	$8.937.10^{-5}$	$8.640.10^{-7}$	$3.045.10^{-5}$	$3.022.10^{-5}$	$6.595.10^{-9}$
0.4	$1.399.10^{-4}$	$2.583.10^{-5}$	$5.089.10^{-7}$	$2.329.10^{-4}$	$2.387.10^{-5}$	$2.301.10^{-7}$
0.5	$1.411.10^{-3}$	$6.734.10^{-5}$	$1.346.10^{-6}$	$2.953.10^{-4}$	$7.706.10^{-6}$	$1.135.10^{-7}$
0.6	$1.039.10^{-3}$	$1.341.10^{-5}$	$1.960.10^{-6}$	$1.323.10^{-4}$	$2.948.10^{-5}$	$3.717.10^{-7}$
0.7	$4.216.10^{-4}$	$1.350.10^{-4}$	$1.319.10^{-6}$	$1.514.10^{-4}$	$1.704.10^{-5}$	$5.660.10^{-8}$
0.8	$1.969.10^{-4}$	$6.878.10^{-5}$	$9.161.10^{-6}$	$3.437.10^{-4}$	$2.207.10^{-5}$	$2.197.10^{-7}$
0.9	$4.782.10^{-4}$	$1.375.10^{-5}$	$1.003.10^{-7}$	$1.393.10^{-4}$	$3.393.10^{-5}$	$3.873.10^{-7}$

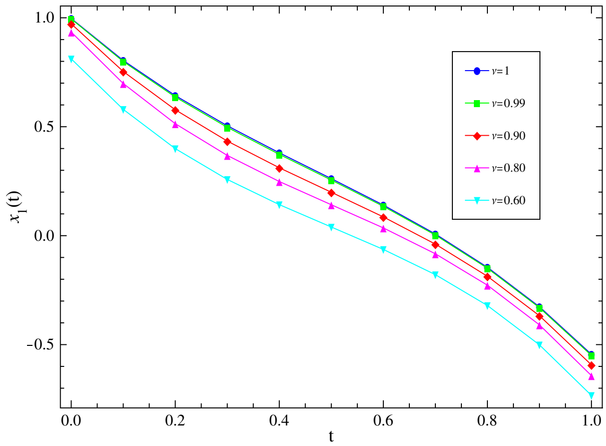

Fig. 11. Approximate solutions of $x_1(t)$ at $N = 3$ and $\nu = 1,\ 0.99,\ 0.90,\ 0.80$ and 0.60 for problem (4.7).

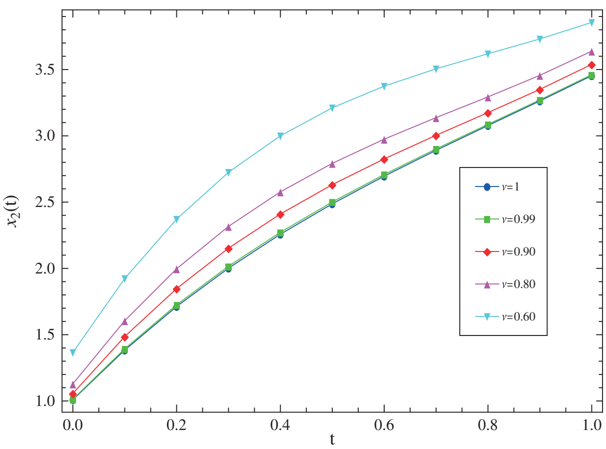

Fig. 12. Approximate solutions of $x_2(t)$ at $N = 3$ and $\nu = 1,\ 0.99,\ 0.90,\ 0.80$ and 0.60 for problem (4.7).

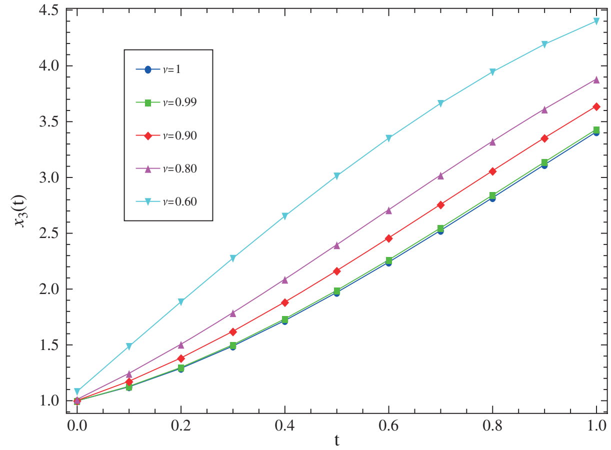

Fig. 13. Approximate solutions of $x_3(t)$ at $N = 3$ and $\nu = 1,\ 0.99,\ 0.90,\ 0.80$ and 0.60 for problem (4.7).

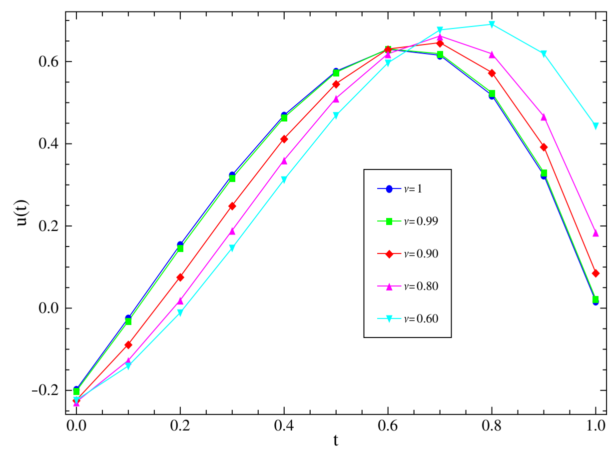

Fig. 14. Approximate solutions of $u(t)$ at $N = 3$ and $\nu = 1,\ 0.99,\ 0.90,\ 0.80$ and 0.60 for problem (4.7).

V. CONCLUSION

In this paper, a new formulation of the M-DFOCP was considered. The Lagrange multiplier method for the constrained extremum and the operational matrix of fractional integrals are used, together with the help of the properties of the shifted Legendre orthonormal polynomials, to solve the M-DFOCP numerically, by reducing it to a solution of a system of algebraic equations. The fractional derivatives are described in the Caputo sense, while the fractional integrals are described in the Riemann–Liouville sense. In order to clarify the validity and accuracy of the proposed scheme, three numerical examples were presented with their exact and approximate solutions.

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