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On Local Fractional Continuous Wavelet Transform

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Abstract

In this manuscript a fractal modeling for wavelet analysis is investigated by using the local fractional transform in fractal space based on the local fractional calculus. An illustrative example of local fractional wavelet is shown.

Keywords

Wavelet transform
Local fractional Fourier transforms
Local fractional wavelet
Local fractional calculus

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1 Introduction

Wavelet transform has applied successfully to signals analysis, data compression and sound analysis (see for more details Refs. [1-6] and the references therein). Although there is scaled and shifted versions of a mother wavelet, daughter wavelets is structured in [3- 5] as

$$\varphi_{a,b}(t) = \frac{1}{a^{\frac{1}{2}}} \varphi\left(\frac{t-b}{a}\right) \quad (1)$$

where a is the dyadic dilation, b is the dyadic position, and $a^{-\frac{1}{2}}$ is the normalization factor. The expression of a one-dimensional wavelet transform for a given continuous signal $f(t)$ is given by [3-5]

$$W_{\varphi}f(a,b) = \int_{-\infty}^{\infty} f(t) \overline{\varphi_{a,b}(t)} dt \quad (2)$$

and the reconstruction formula becomes [3-5]

$$f(x) = C_{\varphi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{a^2} W_{\varphi}f(a,b) \varphi_{a,b}(t) da db \quad (3)$$

where

$$C_{\varphi} = \int_{-\infty}^{\infty} \frac{|f(x)|^2}{|x|^{\alpha}} dx. \quad (4)$$

Recently, fractional wavelet transform, as a generalization of the classical wavelet transform was proposed in [7]. A one-dimensional fractal wavelet transform of a continuous signal $f(t)$ has the form [7]

$$W_{\varphi}f(a,b) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} B(x,t) f(t) \overline{\varphi_{a,b}(x)} dt dx \quad (5)$$

where $B(x,t)$ denotes a bulk optics kernel.

The reconstructing formula of the input is defined as given below [7]

$$f(x) = \frac{1}{C_{\varphi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{a^3} W_{\varphi}f(a,b) B_{-}(x,t) \varphi_{a,b}(t) da db dt \quad (6)$$

We notice that the fractional wavelet transform was applied to the image encryption [8], to the simultaneous spectral analysis [9] and to the composite signals [10-11]. For other definition of fractional wavelet transform (see reference [12] and the references therein).

Having in mind to study the fractal signals (local fractional continuous signals), a new local fractional wavelet transform was developed in [13] based on the local fractional Fourier transform [14] via local fractional calculus [15-18].

In this manuscript, we investigate the local fractional Fourier transform method to deal with the local fractional wavelet transforms by implementing the local fractional calculus.

The organization of the paper is as follows. Section 2 presents the concept of local fractional Fourier transform and wavelet. In Section 3, the derivation of the local fractional continuous wavelet

transform is considered. The wave space is mentioned in Section 4. An illustrative example is shown in Section 5. Finally, the conclusions are presented in Section 6.

2 Local fractional Fourier transform and wavelet

Let $f(x)$ be local fractional continuous, then we denote [18]

$$f(x) \in C_\alpha(-\infty, \infty) \quad (7)$$

The space of local fractional continuous functions $C_{p,\alpha}[a,b]$ under P -norm is given by [13]

$$\|f\|_{p,\alpha} = \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b |f(x)|^p (dx)^\alpha \right)^{\frac{1}{p}}, \text{ for } 1 \leq p < \infty, \quad (8)$$

where the operator is local fractional operator.

The space $L_{p,\alpha}[\square]$ norm on $C_{p,\alpha}[\square]$ is defined by [13]

$$\|f\|_{p,\alpha} = \left(\frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} |f(x)|^p (dx)^\alpha \right)^{\frac{1}{p}} < \infty \quad (9)$$

for $1 \leq p < \infty$. This is infinite for a and b .

The local fractional Fourier transforms in fractal space is defined as [13, 14]

$$F_\alpha \{f(x)\} = f_\omega^{F,\alpha}(\omega) := \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} E_\alpha(-i^\alpha \omega^\alpha x^\alpha) f(x) (dx)^\alpha \quad (10)$$

Its inverse formula is suggested as [13, 14]

$$f(x) = F_\alpha^{-1} (f_\omega^{F,\alpha}(\omega)) := \frac{1}{(2\pi)^\alpha} \int_{-\infty}^{\infty} E_\alpha(i^\alpha \omega^\alpha x^\alpha) f_\omega^{F,\alpha}(\omega) (d\omega)^\alpha, \quad x > 0. \quad (11)$$

Let $\varphi(x) \in L_{2,\alpha}[\square]$ and

$$\varphi_\omega^{F,\alpha}(\omega) = \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} E_\alpha(-i^\alpha \omega^\alpha x^\alpha) \varphi(x) (dx)^\alpha, \quad 0 < \alpha \leq 1. \quad (12)$$

When

$$\varphi_\omega^{F,\alpha}(\omega) = \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} \varphi(x) (dx)^\alpha = 0, \quad 0 < \alpha \leq 1, \quad (13)$$

$\varphi(x)$ is called a local fractional wavelet [13].

Let $\varphi(x) \in L_{2,\alpha}[\square]$. Then we have

$$\|\varphi_{a,b,\alpha}(t)\|_{1,\alpha}^2 = \frac{1}{\alpha^\alpha \Gamma(1+\alpha)} \int_{-\infty}^{\infty} |\varphi_{a,b,\alpha}(t)|^2 (dx)^\alpha = \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} |\varphi(t)|^2 (dx)^\alpha = \|\varphi\|_{1,\alpha}^2, \quad (14)$$

where

$$\varphi_{a,b,\alpha}(t) = \frac{1}{a^{\frac{\alpha}{2}}} \varphi\left(\frac{t-b}{a}\right), \quad (15)$$

$a, b \in \mathbb{R}$ and $a \neq 0$.

3 Local fractional continuous wavelet transform

Let $\varphi \in L_{2,\alpha}[\mathbb{R}]$. Then we arrive at the following relation

$$\begin{aligned} & \|\varphi_{a,b,\alpha}(t)\|_{2,\alpha}^4 \\ &= \frac{1}{a^\alpha \Gamma(1+\alpha)} \int_{-\infty}^{\infty} |\varphi_{a,b,\alpha}(t)|^2 (dt)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} |\varphi(t)|^2 (dt)^\alpha \\ &= \|\varphi\|_{2,\alpha}^4 \end{aligned} \quad (16)$$

where $\varphi_{a,b,\alpha}(t) = \frac{1}{a^{\frac{\alpha}{2}}} \varphi\left(\frac{t-b}{a}\right)$, $a, b \in \mathbb{R}$ and $a \neq 0$.

Likewise, we get

$$\|\varphi_{b,\alpha}(t)\|_{2,\alpha}^4 = \|\varphi\|_{2,\alpha}^4. \quad (17)$$

Taking $\varphi_{b,\alpha}(t)$ in place of $\varphi_{a,b,\alpha}(t) E_\alpha(-i^\alpha \omega^\alpha t^\alpha)$, we obtain

$$\Theta_{\varphi_{a,b,\alpha}} f(a,b) = \langle f(t), \varphi_{a,b,\alpha}(t) \rangle = \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} f(t) \overline{\varphi_{a,b,\alpha}(t)} (dt)^\alpha. \quad (18)$$

In special case of $f(t) = 1$, we have the following relation

$$\Theta_{\varphi_{a,b,\alpha}}(a,b) = \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} \overline{\varphi_{a,b,\alpha}(t)} (dt)^\alpha \quad (19)$$

such that

$$\frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} \Theta_{\varphi_{a,b,\alpha}}(a,b) \varphi_{a,b,\alpha}(t) (db)^\alpha = |a|^\alpha \left[\frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} \varphi(t) (dt)^\alpha \right]^2. \quad (20)$$

Hence, there exists the following relation

$$\frac{1}{\Gamma^2(1+\alpha)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a^{-2\alpha} \Theta_{\varphi_{a,b,\alpha}}(a,b) \varphi_{a,b,\alpha}(t) (da)^\alpha (db)^\alpha = \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} |x|^{-\alpha} (dx)^\alpha. \quad (21)$$

In general, we also deduce the following identities:

$$f(x) = \frac{\int_{-\infty}^{\infty} \frac{|f(x)|^2}{|x|^\alpha} (dx)^\alpha}{\Gamma^3(1+\alpha)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a^{-2\alpha} \Theta_{\varphi_{a,b,\alpha}} f(a,b) \varphi_{a,b,\alpha}(t) (db)^\alpha (da)^\alpha \quad (22)$$

and

$$\Theta_{\varphi_{a,b,\alpha}} f(a,b) = \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} f(t) \overline{\varphi_{a,b,\alpha}(t)} (dt)^\alpha. \quad (23)$$

Now we establish the following relations

$$\Theta_{\varphi_{a,b,\alpha}} f(a,b) = \frac{a^{-\frac{\alpha}{2}}}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} f(t) \overline{\varphi_{a,b,\alpha}(t)} (dt)^\alpha, 0 < \alpha \leq 1, \quad (24)$$

and

$$f(x) = \frac{\int_{-\infty}^{\infty} \frac{|f(x)|^2}{|x|^\alpha} (dx)^\alpha}{\Gamma^3(1+\alpha)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a^{-2\alpha} \Theta_{\varphi_{a,b,\alpha}} f(a,b) \varphi_{a,b,\alpha}(t) (da)^\alpha (db)^\alpha. \quad (25)$$

Hence, the local fractional continuous wavelet transform takes the form [13]

$$W_{\varphi,\alpha} f(a,b) = \frac{a^{-\frac{\alpha}{2}}}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} f(t) \overline{\varphi_{a,b,\alpha}(t)} (dt)^\alpha, 0 < \alpha \leq 1, \quad (26)$$

and the inversion formula of local fractional continuous wavelet transform is derived as [14]

$$f(x) = \frac{C_{\varphi,\alpha}}{\Gamma^2(1+\alpha)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a^{-2\alpha} W_{\varphi,\alpha} f(a,b) \varphi_{a,b,\alpha}(t) (da)^\alpha (db)^\alpha, 0 < \alpha \leq 1, \quad (27)$$

where

$$C_{\varphi,\alpha} = \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} \frac{|f(x)|^2}{|x|^\alpha} (dx)^\alpha, 0 < \alpha \leq 1. \quad (28)$$

4 The wavelet space

In order to differ from classical wavelet and fractional wavelet, here we structure a wavelet space as follows.

A wavelet space is defined by

$$W_{\varphi,\alpha}[\square] = \left\{ (\varphi, \alpha) : W_{\varphi,\alpha} f(a,b) = \frac{a^{-\frac{\alpha}{2}}}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} f(t) \overline{\varphi_{a,b,\alpha}(t)} (dt)^\alpha, 0 < \alpha \leq 1 \right\}. \quad (29)$$

When fractal dimension α is equal to 1, from (29) we deduce [3-5]

$$W_{\varphi,1}[\square] = \left\{ (\varphi, 1) : W_{\varphi,1}f(a, b) = a^{-\frac{1}{2}} \int_{-\infty}^{\infty} f(t) \overline{\varphi_{a,b,1}(t)} dt, \alpha = 1 \right\}, \quad (30)$$

where $f(t)$ is continuous and $W_{\varphi,1}f(a, b) \in W_{\varphi,1}[\square]$.

Taking the fractal dimension $0 < \alpha < 1$, we derive a formula

$$W_{\varphi,\alpha}f(a, b) = \frac{a^{-\frac{\alpha}{2}}}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} f(t) \overline{\varphi_{a,b,\alpha}(t)} (dt)^{\alpha} \quad (31)$$

with $W_{\varphi,\alpha}f(a, b) \in W_{\varphi,\alpha}[\square]$, where $f(t)$ is a local fractional continuous function.

5 Illustrative example

In order to construct the local fractional continuous wavelet, we suppose that $\phi(t)$ is $m\alpha$ times local fractional differentiable function.

We define the local fractional wavelet $\varphi(t)$ by the following expression

$$\varphi(t) = \frac{d^{m\alpha} y}{dx^{m\alpha}}, \quad (32)$$

where the differential operator is local fraction operator proposed by Yang et al. [18] (other definition, see [19] and therein).

Then we get

$$\frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} \varphi(t) \frac{t^{m\alpha}}{\Gamma(1+m\alpha)} (dt)^{\alpha} = 0 \quad (m = 0, 1, 2, \dots, m). \quad (33)$$

Let us consider the following non-differentiable signal, namely,

$$\phi_{H(\alpha)}(t) = \begin{cases} \frac{t^\alpha}{\Gamma(1+\alpha)}, & 0 \leq t < \frac{1}{2}, \\ \frac{(1-t)^\alpha}{\Gamma(1+\alpha)}, & \frac{1}{2} \leq t < 1, \\ 0, & \text{else.} \end{cases} \quad (34)$$

For $0 \leq t < \frac{1}{2}$, we obtain

$$\begin{aligned} & \frac{d^\alpha \phi_{H(\alpha)}(t)}{dt^\alpha} \\ &= \frac{d^\alpha t^\alpha}{dt^\alpha \Gamma(1+\alpha)} \\ &= 1. \end{aligned} \quad (35)$$

For $\frac{1}{2} \leq t < 1$, we obtain

$$\begin{aligned} & \frac{d^\alpha \phi_{H(\alpha)}(t)}{dt^\alpha} \\ &= \frac{d^\alpha (1-t)^\alpha}{dt^\alpha \Gamma(1+\alpha)}. \\ &= -1 \end{aligned} \quad (36)$$

In view of Eqs.(35)-(36), we get a local fractional wavelet given by

$$\varphi_{H(\alpha)}(t) = \begin{cases} 1, & 0 \leq t < \frac{1}{2}, \\ -1, & \frac{1}{2} \leq t < 1, \\ 0, & \text{else.} \end{cases} \quad (37)$$

Following Eq. (37), we obtain

$$\frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} \varphi_{H(\alpha)}(t) (dt)^\alpha = 0 \quad (38)$$

and

$$\frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} \varphi_{H(\alpha)}^2(t) (dt)^\alpha = 1. \quad (39)$$

In view of Eq. (15), taking $a = 2^{-j}$ and $b = k2^{-j}$, we have

$$\begin{aligned} & \varphi_{a,b,\alpha}(t) \\ &= \frac{1}{a^{\frac{\alpha}{2}}} \varphi\left(\frac{t-b}{a}\right) \\ &= \varphi_{j,k,\alpha}(t) \\ &= \varphi_{2^{-j},k2^{-j},\alpha}(t) \\ &= 2^{\frac{j\alpha}{2}} \varphi(2^j t - k) \end{aligned} \quad (40)$$

for integers $j, k \in \mathbb{Z}$.

Hence, we get the following equation:

$$\varphi_{H(\alpha)}^{j,k}(t) = 2^{\frac{j\alpha}{2}} \varphi_{H(\alpha)}(2^j t - k). \quad (41)$$

Thus, we conclude that

$$\begin{aligned} & \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} \left[\varphi_{H(\alpha)}^{j,k}(t) \right]^2 (dt)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} \left[2^{\frac{j\alpha}{2}} \varphi_{H(\alpha)}(2^j t - k) \right]^2 (dt)^\alpha \\ &= 2^{j\alpha} \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} \left[\varphi_{H(\alpha)}(2^j t - k) \right]^2 (dt)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} \left[\varphi_{H(\alpha)}(2^j t - k) \right]^2 (d(2^j t - k))^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} \left[\varphi_{H(\alpha)}(t) \right]^2 (dt)^\alpha \\ &= 1 \end{aligned} \quad (42)$$

and

$$\begin{aligned} & \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} \varphi_{H(\alpha)}^{j,k}(t) (dt)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)} \int_0^{\frac{1}{2}} 2^{\frac{j\alpha}{2}} \varphi_{H(\alpha)}(2^j t - k) (dt)^\alpha - \frac{1}{\Gamma(1+\alpha)} \int_{\frac{1}{2}}^1 2^{\frac{j\alpha}{2}} \varphi_{H(\alpha)}(2^j t - k) (dt)^\alpha \\ &= 0. \end{aligned} \quad (43)$$

6 Conclusions

A novel local fractional transformation was investigated by using Fourier transform based on local fractional calculus. This transform takes advantage of dealing with the functions in fractal space. The wave space is considered and an illustrative example is shown in present work.

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