30th Benelux Meeting
on
Systems and Control

March 15 – 17, 2011

Lommel, Belgium

Book of Abstracts

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Book of Abstracts 30th Benelux Meeting on Systems and Control

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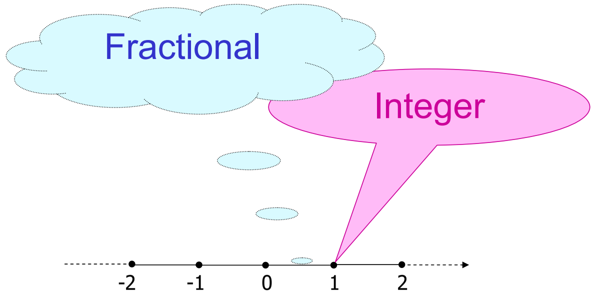
Application of Fractional Calculus in Engineering Sciences

J. Tenreiro Machado

March 15-17, 2011

Integer *vs* fractional numbers



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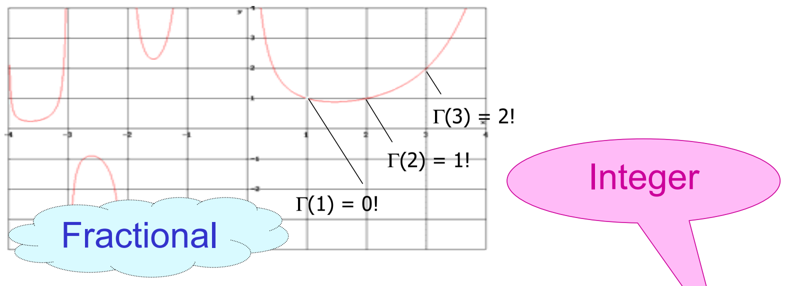
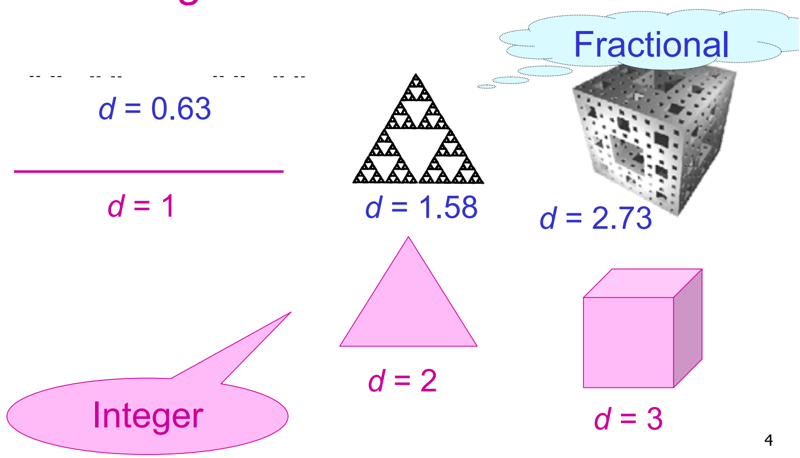
Factorial *vs* Gamma function



$$\Gamma(n)=\int_0^{\infty} t^{n-1}e^{-t}dt \Rightarrow \Gamma(n+1)=n(n-1)\cdots 1=n!$$

3

Integer *vs* Fractal dimension



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Integer *vs* Fractional derivative

- $D^1(e^{ax}) = a\, e^{ax}$
- $D^2(e^{ax}) = a^2\, e^{ax}$
- $D^3(e^{ax}) = a^3\, e^{ax}$
- ....
- $D^n(e^{ax}) = a^n\, e^{ax}$
- $D^\alpha(e^{ax}) = a^\alpha\, e^{ax}$

n -integer

α - complex

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Harry Potter and the Philosopher's Stone Platform 9-3/4, King's Cross Station, London

Guillaume de l'Hôpital

(1661–1704)

What is the meaning of $D^{½}y$?

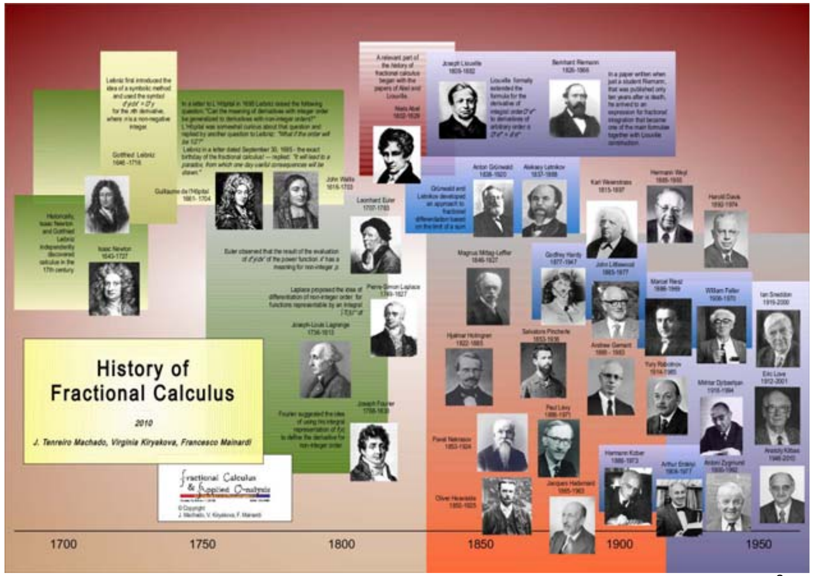
1695

intimate connection between derivatives and infinite series

Gottfried Wilhelm Leibniz

(1646–1716)

This is an apparent paradox from which, one day, useful consequences will be drawn…



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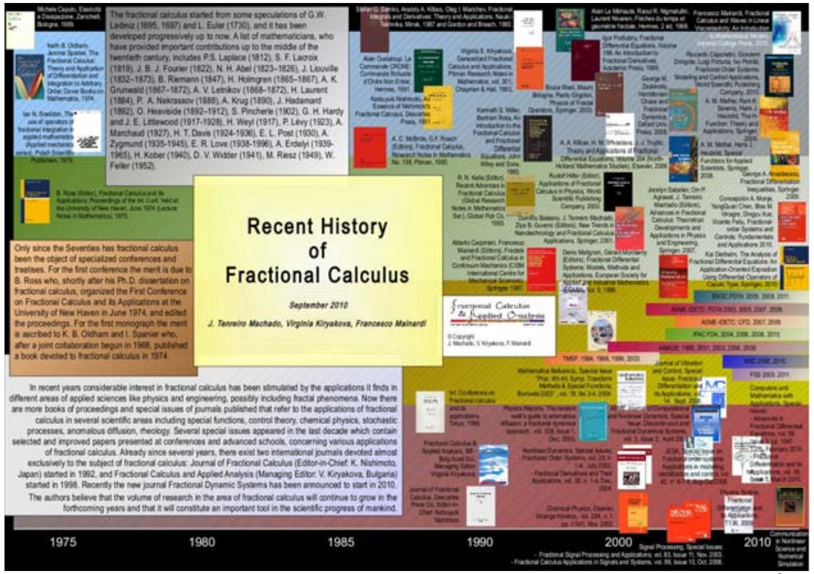

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Fractional Calculus

- The name "fractional calculus" is actually a misnomer
- The designation "integration and differentiation of arbitrary order" is more appropriate

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Motivation: sin(*ax*) function

- $D^1[\sin(ax)] = a^1 \sin(ax + 1\,\pi/2)$
- $D^2[\sin(ax)] = a^2 \sin(ax + 2\,\pi/2)$
- $D^3[\sin(ax)] = a^3 \sin(ax + 3\,\pi/2)$
- $D^4[\sin(ax)] = a^4 \sin(ax + 4\,\pi/2)$
- ...
- $D^\alpha[\sin(ax)] = a^\alpha \sin(ax + \alpha\,\pi/2)$

Weyl derivative ${}_{-\infty}D_t^\alpha$

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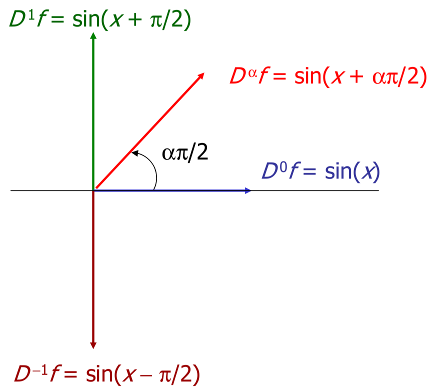
Vector interpretation of D^α for the function $f = \sin(x)$



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Power-Law Function

- A *power law* is a relationship that exhibits the property of *scale invariance*

$$f(x) = ax^b,\ a, b \in \Re$$

- For a constant *c*

$$f(cx) = a(cx)^b = c^b f(x) \sim f(x)$$

which implies the *scale invariance*

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Fractional Integral (1)

Cauchy's integral and power-law function

$${}_aD^{-1} f(t) = \int_a^t f(\tau)d\tau$$

$${}_aD^{-2} f(t) = \int_a^t (t-\tau) f(\tau)d\tau$$

$${}_aD^{-n} f(t) = \frac{1}{(n-1)!}\int_a^t (t-\tau)^{n-1} f(\tau)d\tau$$

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Fractional Integral (2)

$${}_aD^{-n} f(t) = \frac{1}{\Gamma(n)}\int_a^t (t-\tau)^{n-1} f(\tau)d\tau$$

Replacing *n* by *q* > 0:

$${}_aD^{-q} f(t) = \frac{1}{\Gamma(q)}\int_a^t \frac{f(\tau)}{(t-\tau)^{1-q}}d\tau$$

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Fractional Integral (3)

Laplace transform and Convolution.
Without loss of generality let *a* = 0:

$${}_0D^{-q} f(t) = \frac{1}{\Gamma(q)}\int_0^t \frac{f(\tau)}{(t-\tau)^{1-q}}d\tau = \frac{t^{q-1}}{\Gamma(q)} * f(t)$$

The symbol * denotes the convolution in the perspective of the Laplace transform:

$$F(s) = L\{f(t)\}_0 = \int_0^{+\infty} f(t)e^{-st}dt$$

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Fractional Integral (4)

Denoting $h(t)$ as the unit step function and knowing the property:

$$L\{f(t) * g(t)\} = F(s)G(s)$$

results:

$$L\left\{\frac{t^{q-1}}{\Gamma(q)} h(t)\right\} = \frac{1}{s^q},\ q > 0$$

The Laplace transform of the fractional integral results:

$$L\left\{\frac{t^{q-1}}{\Gamma(q)} h(t)\right\} = \frac{1}{s^q},\ q > 0$$

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Definitions of fractional derivatives-1

Bernhard Riemann (1826–1866) Joseph Liouville (1809–1882)

Riemann-Liouville definition

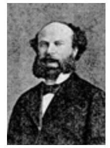
$${}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t \frac{f(\tau)d\tau}{(t-\tau)^{\alpha-n+1}}$$

$$n-1 < \alpha < n$$

Grünwald-Letnikov definition

$${}_aD_t^\alpha f(t) = \lim \frac{1}{h^\alpha} \sum_{k=0}^{\left[\frac{t-a}{h}\right]} (-1)^k \binom{\alpha}{k} f(t-kh)$$

$[x]$ – integer part of x



Aleksey Letnikov (1837-1888) Anton Grünwald (1838-1920)

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Definitions of fractional derivatives -2

Michele Caputo

Caputo definition

$${}_a^CD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)d\tau}{(t-\tau)^{\alpha-n+1}}$$

$$n-1 \le \alpha < n$$

Pierre-Simon Laplace (1749-1827)

Laplace definition

$$D^\alpha x(t) = L^{-1}\left\{s^\alpha X(s) - \sum_{k=0}^{n-1} s^k D^{\alpha-k-1} x(t)\Big|_{t=0}\right\}$$

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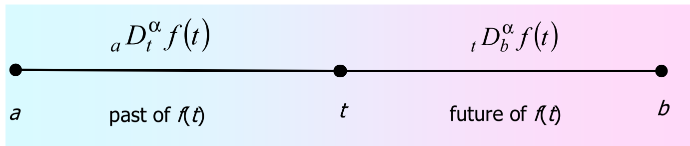
Left and Right fractional derivatives

- Left-sided $${}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t \frac{f(\tau)d\tau}{(t-\tau)^{\alpha-n+1}}$$
- Right-sided $${}_tD_b^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dt}\right)^n \int_t^b \frac{f(\tau)d\tau}{(t-\tau)^{\alpha-n+1}}$$



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Properties (1)

The fractional derivative of the function $f(t) = t^{\mu}$, $\mu>0$, $\alpha>0$:

$$D^{\alpha} t^{\mu} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\mu+1)} t^{\alpha-\mu}$$

The fractional derivative of the constant c :

$$D^{\alpha} c = \frac{c}{\Gamma(1-\alpha)} t^{-\alpha}$$

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Properties (2)

It is always true for $\alpha>0$, $\beta>0$:

$$D^{-\alpha} D^{-\beta} = D^{-\alpha-\beta}$$

It is not always true that:

$$D^{\alpha} D^{\beta} = D^{\alpha+\beta}$$

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Grünwald-Letnikov definition

$$D^1[f(x)] = \lim_{h\to 0} \frac{f(x) - f(x-h)}{h}$$

$$D^2[f(x)] = \lim_{h\to 0} \frac{f(x) - 2f(x-h) + f(x-2h)}{h^2}$$

$$D^3[f(x)] = \lim_{h\to 0} \frac{f(x) - 3f(x-h) + 3f(x-2h) - f(x-3h)}{h^3}$$

$$\binom{3}{0} \quad -\binom{3}{1} \quad \binom{3}{2} \quad -\binom{3}{3}$$

$$(-1)^k \binom{\alpha}{k} = (-1)^k \frac{\Gamma(\alpha+1)}{k!\,\Gamma(\alpha-k+1)}$$

$$D^{1/2}[f(x)] = \lim_{h\to 0} \frac{f(x) - 1/2\, f(x-h) - 1/8\, f(x-2h) - 1/16\, f(x-3h) - \cdots}{h^{1/2}}$$

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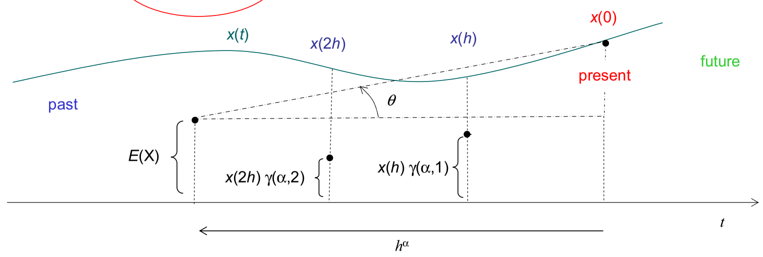
A probabilistic perspective…

$$D^{1/2}[f(x)] = \lim_{h\to 0} \frac{f(x) - 1/2\, f(x-h) - 1/8\, f(x-2h) - 1/16\, f(x-3h) - \cdots}{h^{1/2}}$$

$$\frac{1}{2} + \frac{1}{8} + \frac{1}{16} + \cdots = 1$$

non uniform time variation



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Fractional-Order Integrals of Several Functions

$\varphi(x),\ x \in \Re$	$\left(I_+^{\alpha}\varphi\right)(x),\ x \in \Re,\ \alpha \in C$
$(x-a)^{\beta-1}$	$\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(x-a)^{\alpha+\beta-1},\ \mathrm{Re}(\beta)>0$
$e^{\lambda x}$	$\lambda^{-\alpha}e^{\lambda x},\ \mathrm{Re}(\lambda)>0$
$\begin{cases}\sin(\lambda x)\\ \cos(\lambda x)\end{cases}$	$\lambda^{-\alpha}\begin{cases}\sin(\lambda x-\alpha\pi/2)\\ \cos(\lambda x-\alpha\pi/2)\end{cases},\ \lambda>0, \mathrm{Re}(\alpha)>1$
$e^{\lambda x}\begin{cases}\sin(\gamma x)\\ \cos(\gamma x)\end{cases}$	$\frac{e^{\lambda x}}{\left(\lambda^2+\gamma^2\right)^{\alpha/2}}\begin{cases}\sin(\gamma x-\alpha\phi)\\ \cos(\gamma x-\alpha\phi)\end{cases},\ \phi=\arctan(\gamma/\lambda),\ \gamma>0, \mathrm{Re}(\lambda)>1$

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Approximations of fractional derivatives

- Two methods:
  - Frequency-based approach
  - Discrete-time approach

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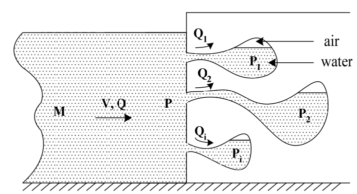
Natural phenomenon



Differential equation of fractional-order 1<α<2:

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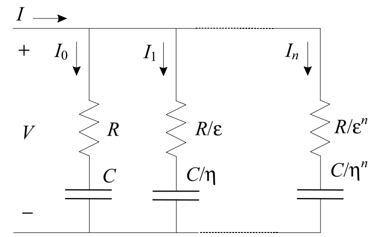
Frequency-based approach



- Recursive circuit:

$$I=\sum_{i=0}^{n} I_i \qquad R_{i+1}=\frac{R_i}{\varepsilon} \qquad C_{i+1}=\frac{C_i}{\eta}$$

- Admittance:

$$Y(j\omega)=\frac{I(j\omega)}{V(j\omega)}=\sum_{i=0}^{n}\frac{j\omega C\varepsilon^i}{j\omega CR+(\eta\varepsilon)^i}$$

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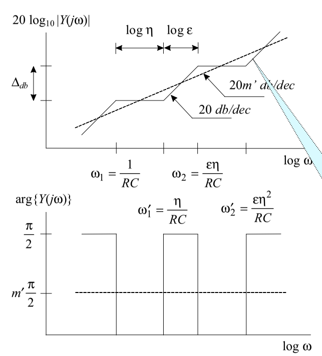
Frequency-based FD approximation



- Recursive relationships of pole/zero frequencies:

$$\frac{\omega'_{i+1}}{\omega'_i} = \frac{\omega_{i+1}}{\omega_i} = \varepsilon\eta \qquad \frac{\omega_i}{\omega'_i} = \varepsilon \qquad \frac{\omega'_{i+1}}{\omega_i} = \eta$$

- Average slope:

$$m' = \frac{\log\varepsilon}{\log\varepsilon + \log\eta}$$

- Approach to D^α $(0 < \alpha < 1)$:

$$m' = \alpha$$

limited bandwidth

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Discrete-time FD approximation (1)

Grünwald-Letnikov definition:

$$D^\alpha x(t) = \lim_{h\to 0}\left[\frac{1}{h^\alpha}\sum_{k=0}^{\infty}(-1)^k \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)} x(t-kh)\right]$$

$h \approx T$, T - sampling period:

$$\frac{Z\{D^\alpha x(t)\}}{X(z)} \approx \frac{1}{T^\alpha}\sum_{k=0}^{\infty}\frac{(-1)^k\,\Gamma(\alpha+1)}{k!\,\Gamma(\alpha-k+1)} z^{-k} = \left(\frac{1-z^{-1}}{T}\right)^\alpha$$

Series approximation

Fraction approximation

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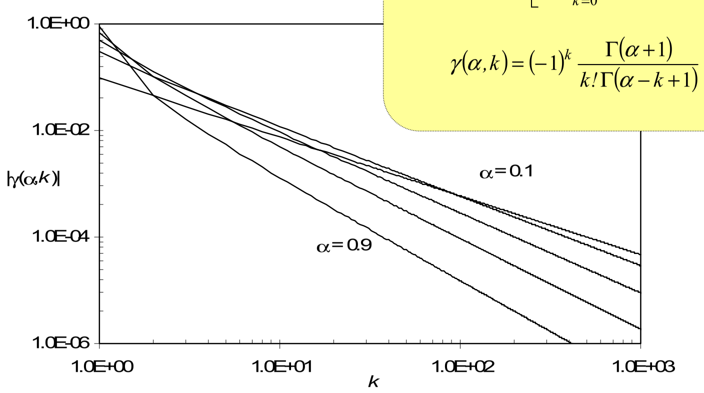
Discrete-time FD approximation (2)

$$D^\alpha[x(t)] = \lim_{h\to 0}\left[\frac{1}{h^\alpha}\sum_{k=0}^{\infty}\gamma(\alpha,k)x(t-kh)\right]$$

$$\gamma(\alpha,k) = (-1)^k \frac{\Gamma(\alpha+1)}{k!\,\Gamma(\alpha-k+1)}$$



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Discrete-time FD approximation (3)

i. Grünwald-Letnikov definition:

$$D^\alpha x(t) = \lim_{h\to 0}\left[\frac{1}{h^\alpha}\sum_{k=0}^{\infty}(-1)^k \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)} x(t-kh)\right]$$

ii. $h \approx T$ where T is the sampling period:

$$\frac{Z\{D^\alpha x(t)\}}{X(z)} \approx \frac{1}{T^\alpha}\sum_{k=0}^{\infty}\frac{(-1)^k\,\Gamma(\alpha+1)}{k!\,\Gamma(\alpha-k+1)} z^{-k} = \left(\frac{1-z^{-1}}{T}\right)^\alpha$$

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Discrete-time FD approximation (4)

Truncated series

$$\left(\frac{1-z^{-1}}{T}\right)^{1/2}=\frac{1}{T^{1/2}}\left(1-\frac{1}{2}z^{-1}-\frac{1}{8}z^{-2}-\frac{1}{16}z^{-3}-\frac{5}{128}z^{-4}-\cdots\right)$$

$$Z\left\{D^{1/2}x(t)\right\}=U(z)=\frac{1}{T^{1/2}}\left[1-\frac{1}{2}z^{-1}-\frac{1}{8}z^{-2}-\frac{1}{16}z^{-3}-\frac{5}{128}z^{-4}-\cdots\right]X(z)$$

$$D^{1/2}x(t)\Big|_{t=kT}\Rightarrow u(kT)=\frac{1}{T^{1/2}}\left\{x(kT)-\frac{1}{2}x[(k-1)T]-\frac{1}{8}x[(k-2)T]-\cdots\right\}$$

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Discrete-time FD approximation (5)

Fraction approximation

$$\left(\frac{1-z^{-1}}{T}\right)^{1/2}\approx\frac{1}{T^{1/2}}\frac{-\frac{7}{64}z^{-3}+\frac{7}{8}z^{-2}-\frac{7}{4}z^{-1}+1}{-\frac{1}{64}z^{-3}+\frac{3}{8}z^{-2}-\frac{5}{4}z^{-1}+1}$$

$$\frac{U(z)}{X(z)}=\frac{1}{T^{1/2}}\frac{-\frac{7}{64}z^{-3}+\frac{7}{8}z^{-2}-\frac{7}{4}z^{-1}+1}{-\frac{1}{64}z^{-3}+\frac{3}{8}z^{-2}-\frac{5}{4}z^{-1}+1}$$

$$D^{1/2}x(t)\Big|_{t=kT}\Rightarrow \begin{aligned}u(kT)=&\frac{1}{T^{1/2}}\left\{x(kT)-\frac{7}{4}x[(k-1)T]+\frac{7}{8}x[(k-2)T]-\frac{7}{64}x[(k-3)T]\right\}+\\&+\left\{\frac{5}{4}u[(k-1)T]+\frac{3}{8}u[(k-2)T]+\frac{1}{64}u[(k-3)T]\right\}\end{aligned}$$

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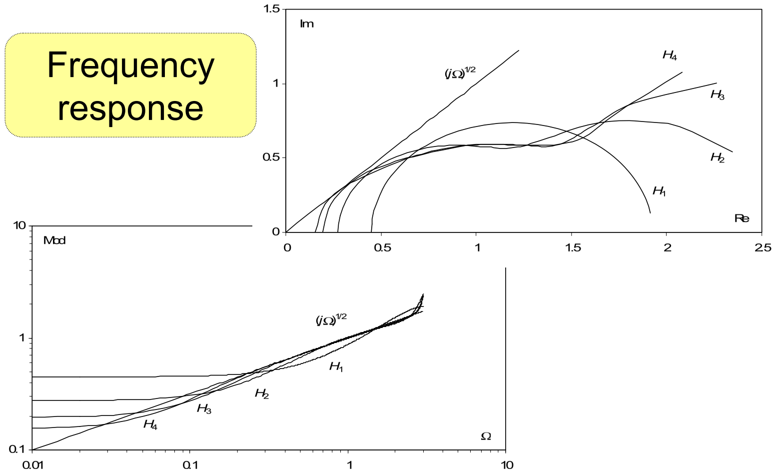
Discrete-time FD approximation (6)

Frequency response



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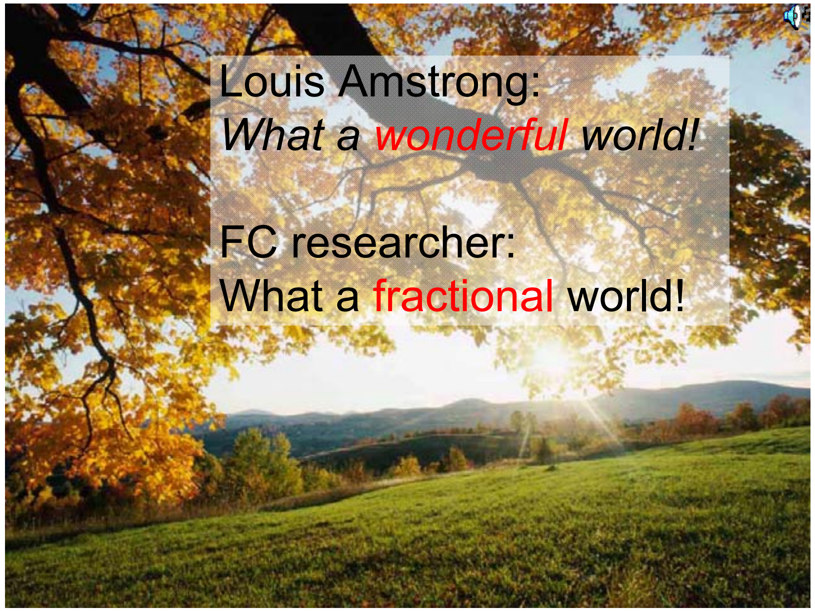
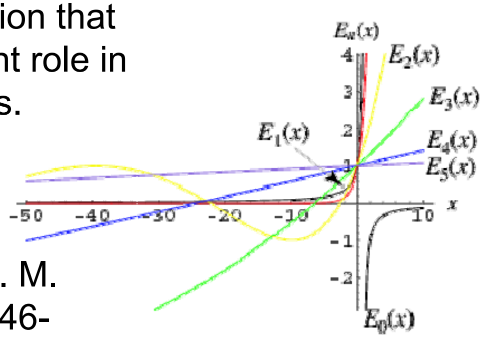


Mittag-Leffler Function (1)

The Mittag-Leffler function is a generalization of the exponential function that plays an important role in fractional calculus.
The function was developed by the Scandinavian mathematician G. M. Mittag-Leffler (1846-1927).



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Mittag-Leffler Function (2)

The function $E_\alpha(z)$ is defined by:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}$$

In particular, when $\alpha = 1$ and $\alpha = 2$, we have

$$E_1(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = \exp(z)$$

$$E_2(z) = \cosh\left(\sqrt{z}\right)$$

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Mittag-Leffler Function (3)

The function $E_{\alpha,\beta}(z)$ is defined by:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}$$

When $\beta = 1$, $E_{\alpha,\beta}(z)$ coincides with the Mittag-Leffler function:

$$E_{\alpha,1}(z) = E_\alpha(z)$$

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Mittag-Leffler Function (4)

In particular, we have:

$$E_{1/2,1}(z) = exp\left(z^2\right) erfc(-z)$$

$$E_{2,1}\left(-z^2\right) = cos(z)$$

$$E_{0,1}(z) = \frac{1}{1-z}$$

$$E_{1,2}(z) = \frac{e^z - 1}{z}$$

$$E_{2,2}(z) = \frac{\sinh(z)}{z}$$

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Mittag-Leffler Function (5)

An important characteristic of the Mittag-Leffler function is its asymptotic behavior. In the case where the argument $z \leq 0$, the Mittag-Leffler function decreases monotonically. In particular for large values of z we can write:

$$E_{\alpha,\alpha}(z) \approx \frac{\alpha}{\Gamma(1-\alpha)} \frac{1}{z^2}, \alpha \neq 1$$

$$E_{\alpha,\upsilon}(-z) \approx \frac{\alpha}{\Gamma(\upsilon-\alpha)} \frac{1}{z}, \alpha \neq \upsilon$$

$$E_{\alpha,1}(-z) \approx \frac{\alpha}{\Gamma(1-\alpha)} \frac{1}{z}, \alpha \neq 1$$

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Mittag-Leffler Function (6)

Three very important related improper integrals define the Laplace transformation of the one- and two-parameter Mittag-Leffler function:

$$\int_0^\infty t^{-\frac{1}{2}} E_{\frac{1}{2},\frac{1}{2}}\left(\pm a\sqrt{t}\right) e^{-st} dt = \frac{1}{\sqrt{s} \mp a}$$

$$\int_0^\infty t^{\beta-1} E_{\alpha,\beta}\left(\pm a t^\alpha\right) e^{-st} dt = \frac{s^{\alpha-\beta}}{s^\alpha \mp a}$$

$$\int_0^\infty E_\alpha\left(\pm a t^\alpha\right) e^{-st} dt = \frac{s^\alpha}{s\left(s^\alpha \mp a\right)}$$

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Integer *vs* Fractional Mechanics…

- Spring
  - Hooke law — $F = kx$
- Viscous friction
  - Newton fluid — $F = k\dot{x}$
- Mass
  - Newton 2nd law — $F = k\ddot{x}$

$F = kx^{(\alpha)}$

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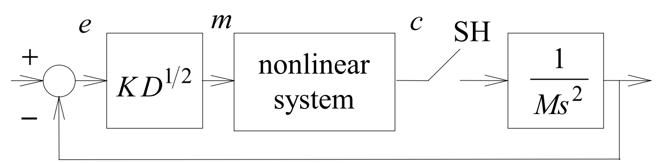
Applications: Control Systems



$$Z\{D^\alpha[x(t)]\} \approx \left[\frac{1}{T^\alpha} \sum_{k=0}^{n} \frac{(-1)^k \Gamma(\alpha+1)}{k!\,\Gamma(\alpha-k+1)} z^{-k}\right] X(z) = Trunc_n\left\{\left(\frac{1-z^{-1}}{T}\right)^\alpha\right\} X(z)$$

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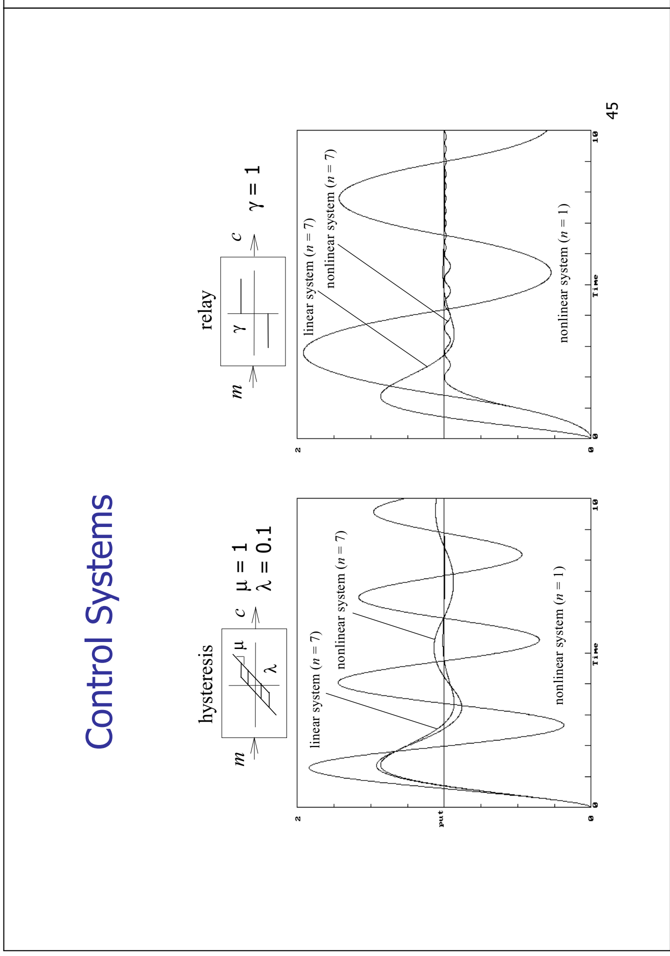
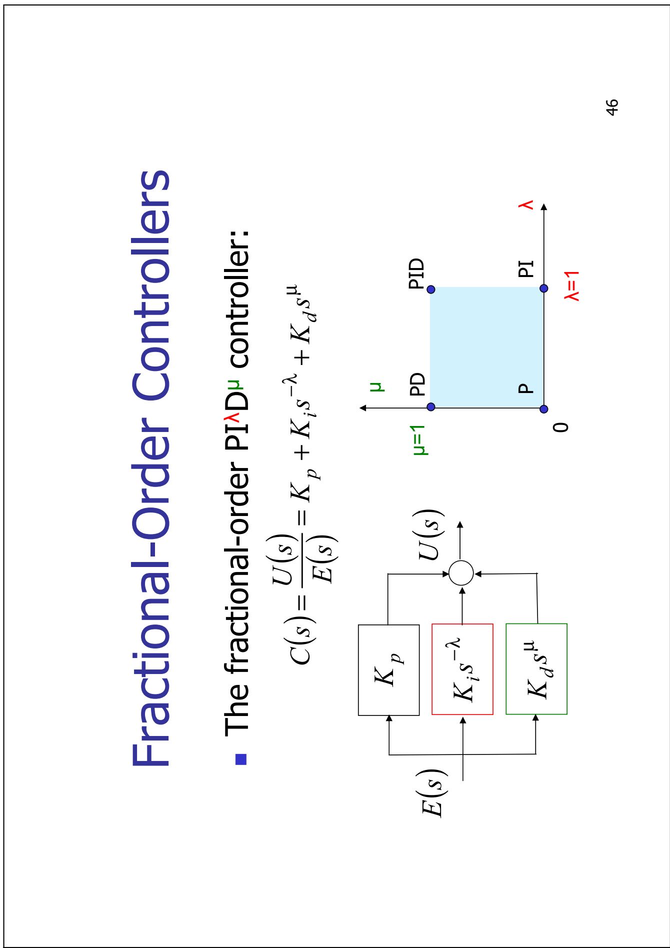
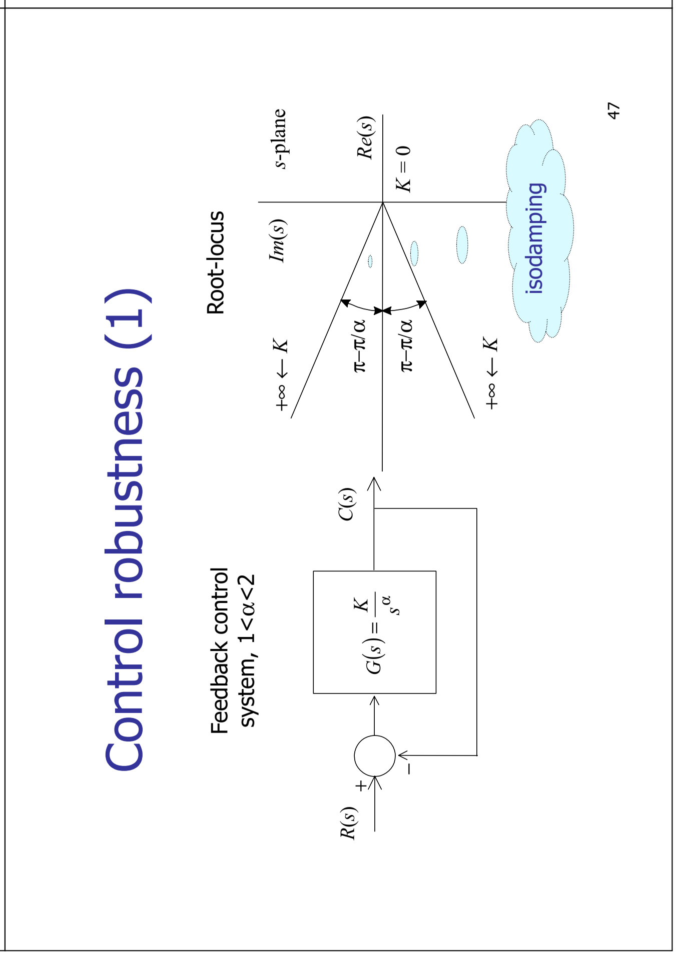
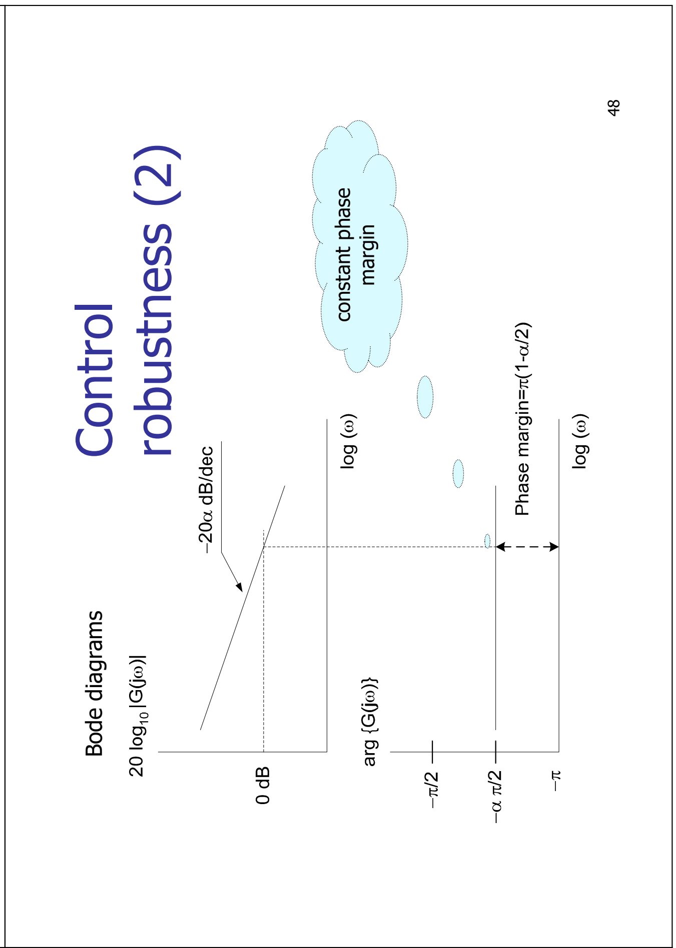

Control Systems
hysteresis
m
c
μ = 1
λ = 0.1
relay
γ = 1
linear system (n = 7)
nonlinear system (n = 7)
nonlinear system (n = 1)
45


Fractional-Order Controllers
The fractional-order PI^λD^μ controller:
$C(s)=\frac{U(s)}{E(s)}=K_p+K_i s^{-\lambda}+K_d s^{\mu}$
E(s)
U(s)
K_p
$K_i s^{-\lambda}$
$K_d s^{\mu}$
μ
μ=1
PD
PID
P
PI
0
λ=1
λ
46


Control robustness (1)
Feedback control system, 1<α<2
R(s)
C(s)
$G(s)=\frac{K}{s^{\alpha}}$
Root-locus
s-plane
Im(s)
Re(s)
K = 0
+∞ ← K
π−π/α
isodamping
47


Control robustness (2)
Bode diagrams
20 log10 |G(jω)|
0 dB
−20α dB/dec
log (ω)
arg {G(jω)}
−π/2
−α π/2
−π
Phase margin=π(1-α/2)
constant phase margin
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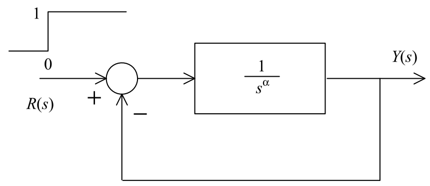
Mittag-Leffler Function: Example



$$\frac{Y(s)}{R(s)}=\frac{1}{s^{\alpha}+1} \quad R(s)=\frac{1}{s}\ \text{(step input)} \ \Rightarrow \ Y(s)=\frac{1}{s(s^{\alpha}+1)} \ \overset{L^{-1}}{\Rightarrow} \ \boxed{y(t)=1-E_{\alpha}(-t^{\alpha})}$$

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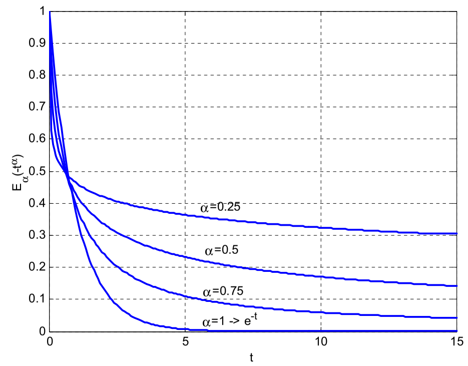
Mittag-Leffler Function $E_{\alpha}(-t^{\alpha})$, $0 < \alpha \le 1$



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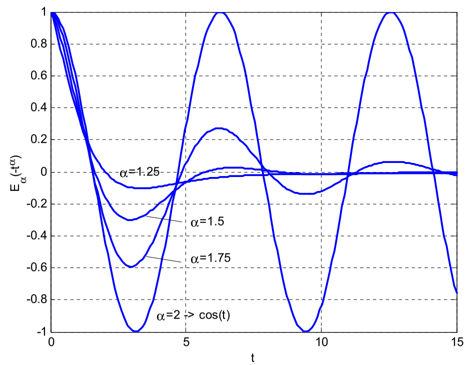
Mittag-Leffler Function $E_{\alpha}(-t^{\alpha})$, $1 < \alpha \le 2$



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Stability of fractional order systems (1)

A general fractional-order linear system can be described by a fractional differential equation of the form:

$$a_n D^{\alpha_n} y(t) + a_{n-1} D^{\alpha_{n-1}} y(t) + \cdots + a_0 D^{\alpha_0} y(t) =$$
$$b_m D^{\beta_m} u(t) + b_{m-1} D^{\beta_{m-1}} u(t) + \cdots + b_0 D^{\beta_0} u(t)$$

where $D^q = {}_0D^q_t$ denotes the Riemann-Liouville or Caputo's fractional derivative.

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Stability of fractional order systems (2)

- The corresponding transfer function of *incommensurate* real orders is of the following form:

$$G(s)=\frac{b_m s^{\beta_m}+b_{m-1}s^{\beta_{m-1}}+\cdots+b_0 s^{\beta_0}}{a_n s^{\alpha_n}+a_{n-1}s^{\alpha_{n-1}}+\cdots+a_0 s^{\alpha_0}}=\frac{Q\left(s^{\beta_k}\right)}{P\left(s^{\alpha_k}\right)}$$

- The incommensurate order system can also be expressed in *commensurate* form by:

$$G(s)=\frac{b_m s^{m/v}+\cdots+b_1 s^{1/v}+b_0 s^{\beta_0}}{a_n s^{n/v}+\cdots+a_1 s^{1/v}+a_0 s^{\alpha_0}}$$

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Stability of fractional order systems (3)

- In the particular case of *commensurate* order systems, it holds that $\alpha_k = qk$; $\beta_k = qk$; ($0<q<1$, k - Z), and the transfer function has the form:

$$G(s)=K_0\frac{\sum_{k=0}^{M} b_k\left(s^q\right)^k}{\sum_{k=0}^{N} a_k\left(s^q\right)^k}=K_0\frac{Q\left(s^q\right)}{P\left(s^q\right)}$$

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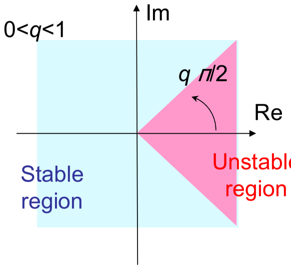
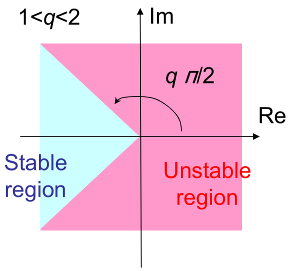
Stability of fractional order systems (4)

- A commensurate order system described by a rational transfer function $G(s)$ is stable if only if $|\arg(\lambda_i)|>q\,\pi/2$, for all i, λ_i = *i-th root of* $P(s^q)$.





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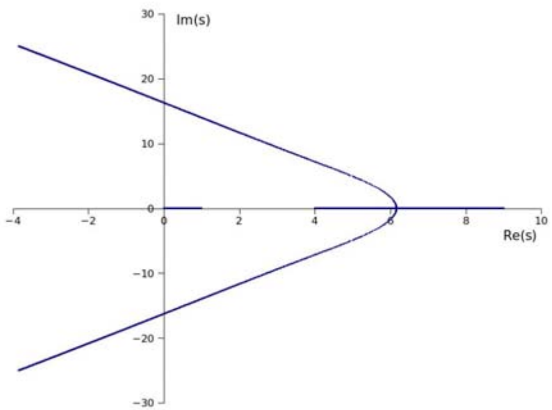
A gallery of FC root locus (1)

$$Q(s)=s^2-3s^{1.5}-2s+2s^{0.5}+12+k\left(s^{0.5}-1\right)$$



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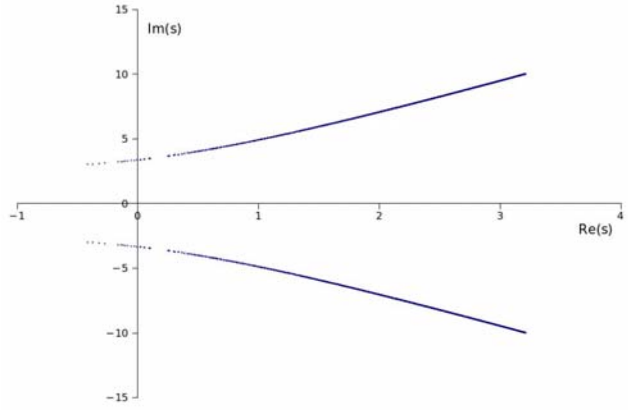
A gallery of FC root locus (2)

$$Q(s) = 0.7943s^{2.5708} + 5.2385s^{0.8372} + 1.5560 + k$$



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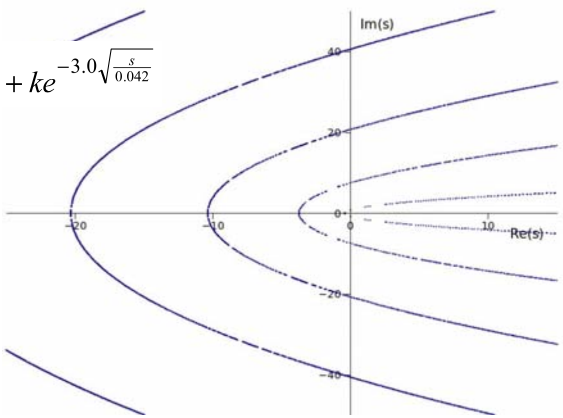
A gallery of FC root locus (3)

$$Q(s) = 1 + ke^{-3.0\sqrt{\frac{s}{0.042}}}$$



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Electromagnetism: Skin effect (1)

Maxwell equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \boldsymbol{\delta} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \cdot \mathbf{D} = \rho$$

For a sinusoidal field we have:

$$\frac{\partial^2 E}{\partial r^2} + \frac{1}{r}\frac{\partial E}{\partial r} = \gamma\mu \frac{\partial E}{\partial t}$$

For a conductor of length l_0 results:

$$\tilde{Z} = \frac{q l_0}{2\pi\, r_0 \sigma} \frac{J_0(q r_0)}{J_1(q r_0)}$$

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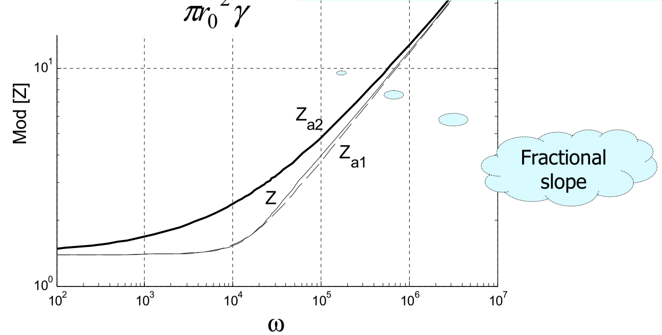
Electromagnetism: Skin effect (2)

$$\omega \to 0 \Rightarrow \tilde{Z} \approx \frac{l_0}{\pi r_0^{\,2} \gamma}$$

$$\omega \to \infty \Rightarrow \tilde{Z} \approx \frac{l_0}{2\pi r_0} \sqrt{\frac{\omega\mu}{2\gamma}} (1 + j)$$



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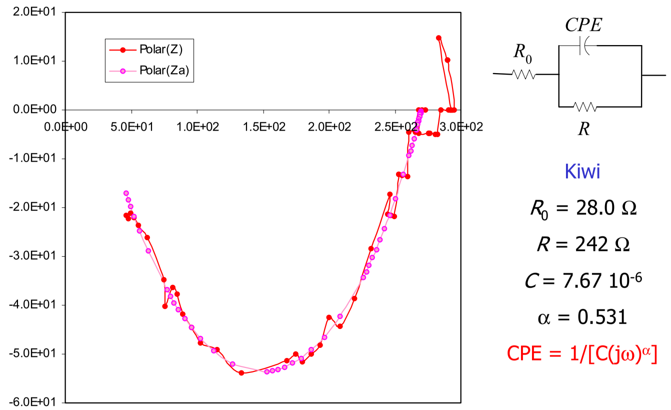
Electrical impedance of fruits and vegetables



Kiwi

R_0 = 28.0 Ω

R = 242 Ω

C = 7.67 10^{-6}

α = 0.531

CPE = 1/[C(jω)^α]

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Electrical impedance of fruits and vegetables

Vegetable / fruits	R_0 [Ω]	R_1 [Ω]	C $[m^{-2/\alpha}\ kg^{-1/\alpha}\ s^{(\alpha+3)/\alpha}\ A^{2/\alpha}]$	α
Garlic	1	9.7 10^3	1.81 10^{-7}	0.609
Potato	57	3.15 10^3	2.40 10^{-7}	0.677
Tomato	35.04	240.30	5.00 10^{-6}	0.565
Kiwi	28.04	242.00	7.67 10^{-6}	0.531
Pear	44.04	409.00	1.14 10^{-6}	0.619

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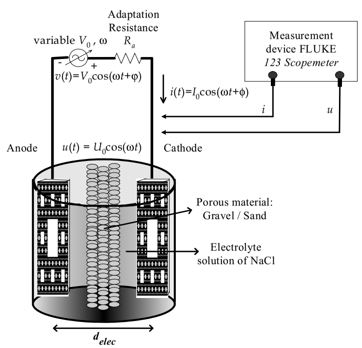
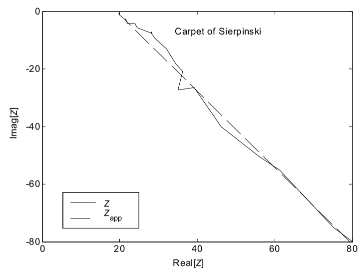
Electrical impedance of electrolyte apparatus



$$Z_{\text{app}}(j\omega) = R + \frac{1}{(j\omega C_F)^{\alpha}}$$



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Electrical impedance of electrolyte apparatus

Numerical values of the parameters for the series circuit with R and CPE (electrodes with the carpet of Sierpinski)

Surface	d_{elec}	R [Ω]	C_F $[m^{-2/\alpha}kg^{-1/\alpha}s^{(\alpha+3)/\alpha}A^{2/\alpha}]$	α
S	AS5	19.10	1.02×10^{-4}	0.589
S	AS5G	58.00	1.40×10^{-5}	0.500
S	AS5S	90.90	3.90×10^{-5}	0.540
$S/3$	AS5	27.10	5.30×10^{-17}	0.175
S	AS10	13.00	2.66×10^{-4}	0.690

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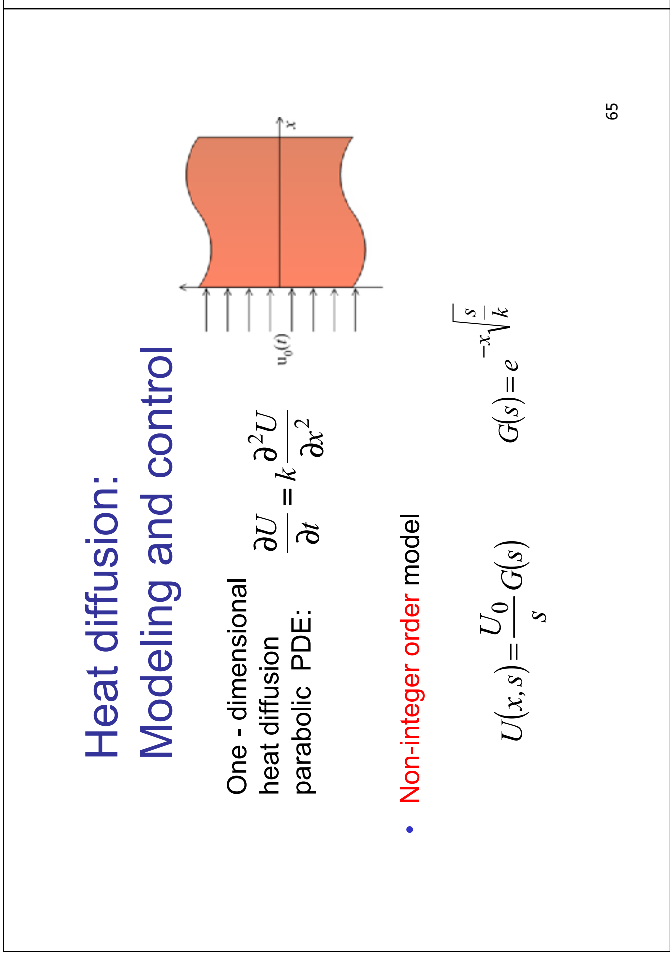
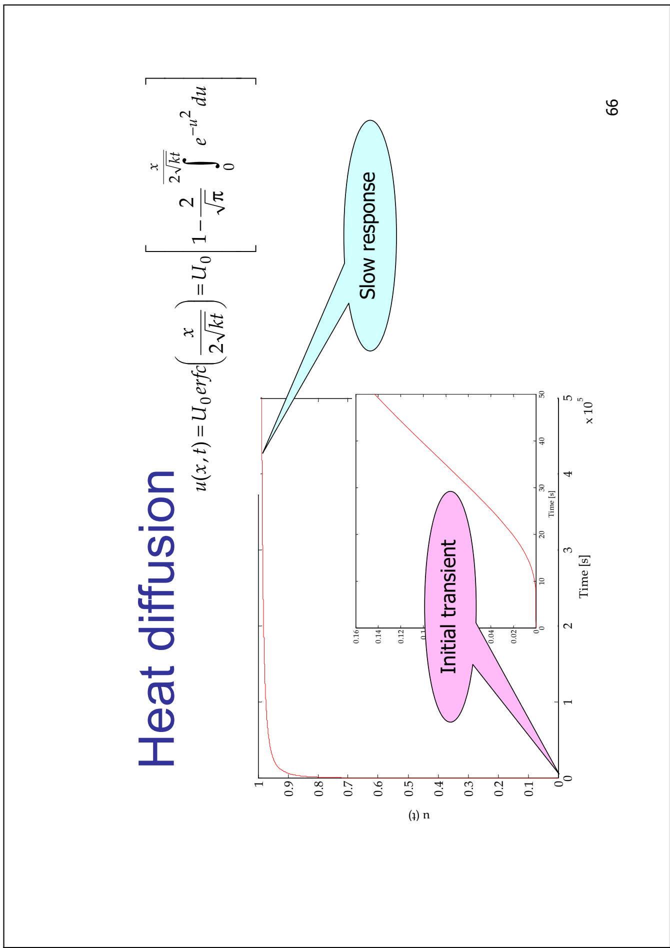
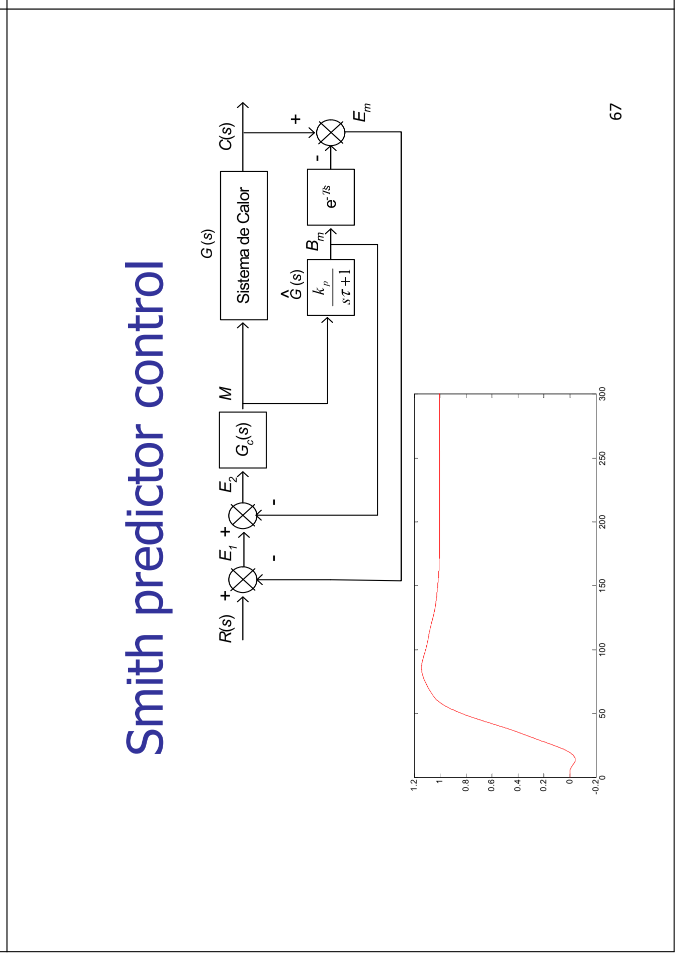
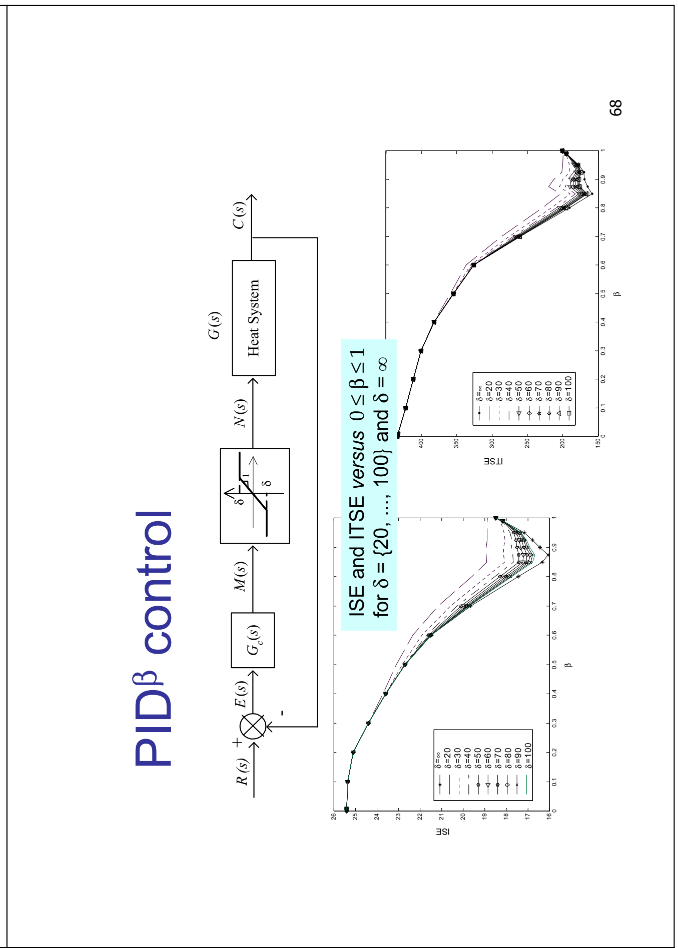

Heat diffusion:
Modeling and control
One - dimensional heat diffusion parabolic PDE:
$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}$
$u_0(t)$
x
• Non-integer order model
$U(x,s) = \frac{U_0}{s}G(s)$
$G(s) = e^{-x\sqrt{\frac{s}{k}}}$
65


Heat diffusion
$u(x,t) = U_0 erfc\left(\frac{x}{2\sqrt{kt}}\right) = U_0\left[1 - \frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2\sqrt{kt}}} e^{-u^2}du\right]$
Slow response
Initial transient
u (t)
Time [s]
x 10^5
66


Smith predictor control
R(s)
E_1
E_2
$G_c(s)$
M
G (s)
Sistema de Calor
C(s)
$\hat{G}(s)$
$\frac{k_p}{s\tau+1}$
B_m
e^{-Ts}
E_m
67


PIDβ control
R (s)
E (s)
$G_c(s)$
M(s)
N(s)
G (s)
Heat System
C(s)
ISE and ITSE versus 0 ≤ β ≤ 1
for δ = {20, …, 100} and δ = ∞
ISE
ITSE
β
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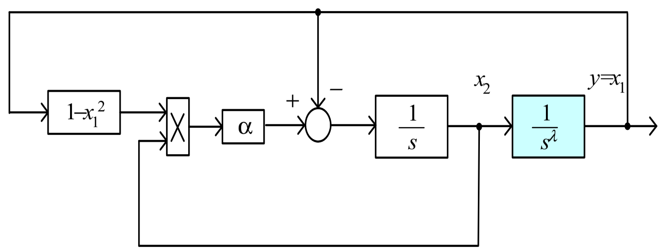
Van der Pol oscillator of fractional order

$$x^{(1+\lambda)} + \alpha\left(x^2 - 1\right)x^{(\lambda)} + x = 0, \quad 0 < \lambda < 1$$



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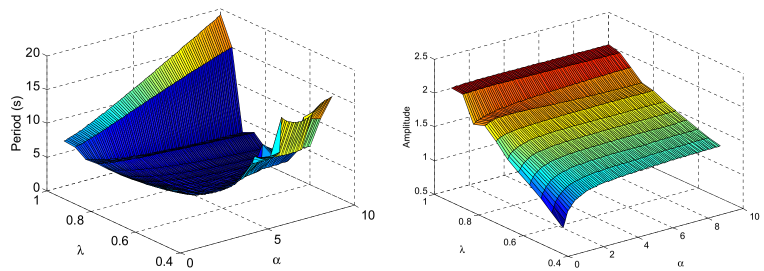
Limit cycle



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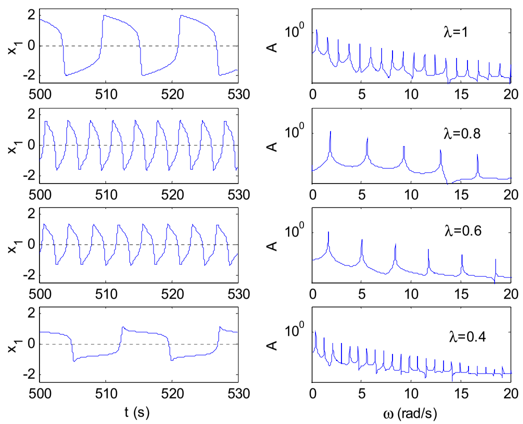
Time/Frequency responses



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Van der Pol oscillator of fractional order

- The fractional Van der Pol Oscillator depicts a richer behavior than the classical system
- The fractional order l acts as a modulation parameter that can be used to display a broader range of output regimes

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Electromagnetism: Fractional potential

Quadrupole $\varphi = \frac{ql^2(3\cos^2\theta - 1)}{4\pi\varepsilon_0}\frac{1}{r^3} + C, r >> l$

Dipole $\varphi = \frac{ql\cos\theta}{4\pi\varepsilon_0}\frac{1}{r^2} + C, r >> l$

Singular charge $\varphi = \frac{q}{4\pi\varepsilon_0}\frac{1}{r} + C$

Linear infinite conductor $\varphi = -\frac{\lambda}{2\pi\varepsilon_0}\ln r + C, C \in \Re$

Infinite Plane $\varphi = -\frac{\sigma}{2\varepsilon_0} r + C, C \in \Re$

Differentiation

Integration

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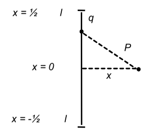
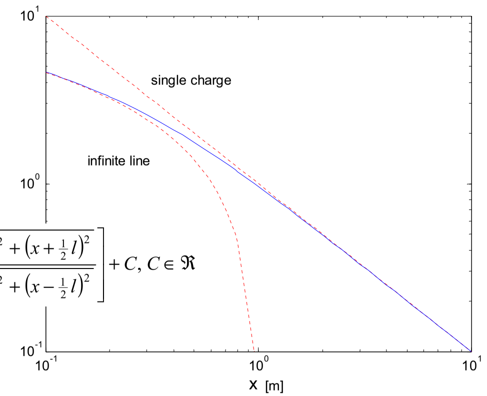
Electromagnetism: fractional potential



$$\varphi = \frac{1}{4\pi\varepsilon_0}\frac{q}{l}\ln\left[\frac{x + \frac{1}{2}l + \sqrt{y^2 + (x + \frac{1}{2}l)^2}}{x - \frac{1}{2}l + \sqrt{y^2 + (x - \frac{1}{2}l)^2}}\right] + C, C \in \Re$$



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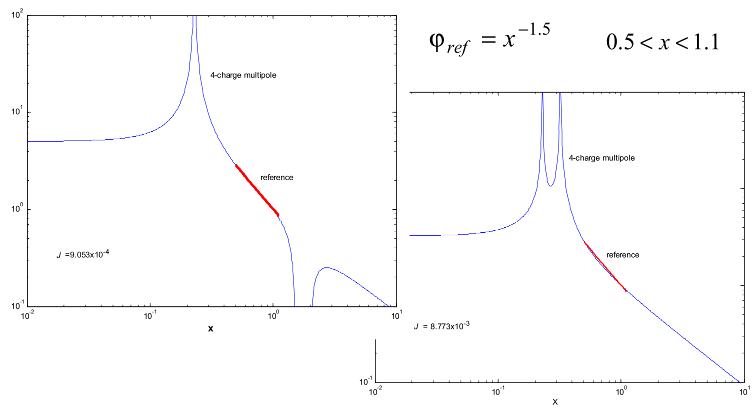
Electromagnetism: fractional potential



$\varphi_{ref} = x^{-1.5}$ $0.5 < x < 1.1$

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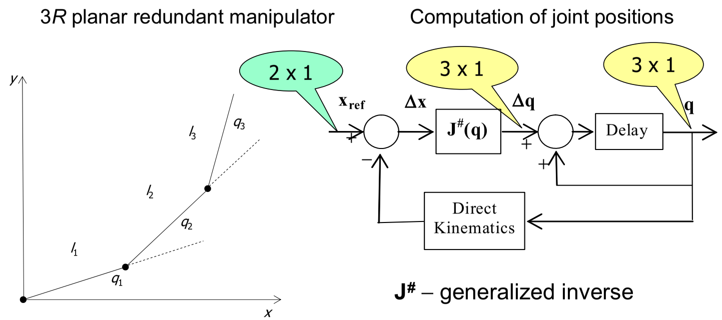
Redundant manipulators: Trajectory planning



J# – generalized inverse

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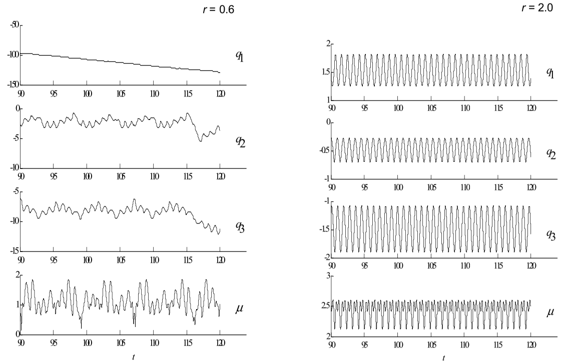
Joint positions and manipulability *vs* time for $\rho = 0.5$, $r = \{0.6, 2.0\}$



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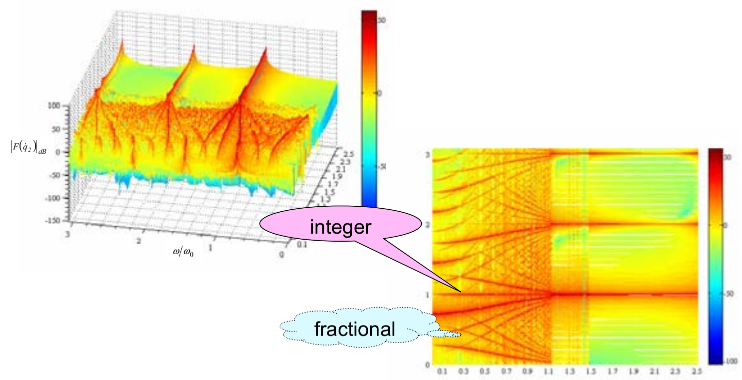
F{dq_2/dt} *vs* (r ,ω/ω_0) for ρ=0.5



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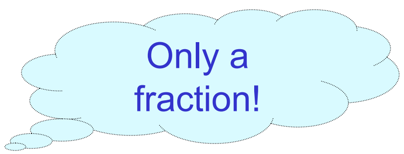
Conclusions



- Fractional models capture phenomena and properties that classical integer-order neglect
- Recent studies encourage the dynamical analysis and control of systems based on FC

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  - 4th FDTA, Washington, USA, 2011.

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  - CFD, Las Vegas, USA, 2007.
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- Journal Advances in Difference Equations, 2011.
- Computers & Mathematics with Applications, 2011.
- International Journal of Bifurcation and Chaos, 2011.
- Signal Image and Video Processing, 2012.

More information in: J. Tenreiro Machado, Virginia Kiryakova, Francesco Mainardi, Recent History of Fractional Calculus, Communications in Nonlinear Science and Numerical Simulations, Elsevier, vol. 16, Issue 3, pp. 1140-1153, March 2011.

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