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ELECTRIC FRACTIONAL ORDER POTENCIAL

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Abstract *– In this study we apply the concept of fractional calculus to electromagnetism and we develop a new fractional order approximation method to the electrical potential.*

1. Introduction

The Maxwell equations play a fundamental role in the well established formulation of the electromagnetic theory [1]. These equations lead to the derivation of precise mathematical models useful in many applications in physics and engineering. The Maxwell equations involve only the integer-order calculus and, therefore, it is natural that the resulting classical models adopted in electrical engineering reflect this perspective. Recently, a closer look of some phenomena present in electrical systems, such as motors, transformers and lines [2, 3], and the motivation towards the development of comprehensive models, seem to point out the requirement for a fractional calculus (FC) approach [4]. In an alternative perspective several authors [5] have verified that well-known expressions for the electrical potential are related through integer-order integral and derivatives and have proposed its generalization based on the concept of fractional-order poles. Nevertheless, the mathematical generalization towards FC lacks a comprehensive method for its practical implementation.

Bearing these ideas in mind, we address the analysis and the synthesis of fractional-order multipoles. In section 2 we start by recalling the method for approximating fractional-order transfer functions based on integer order expressions. In section 3 we review classical expressions for the static electric potential and we study them in the perspective of FC. Based on this re-evaluation in section 4 we develop a numerical method for implementing fractional-order electrical potential approximations. Finally, in section 5 we draw the main conclusions.

2. Approximating Fractional-Order Transfer Functions

Fractional calculus is a natural extension of the classical mathematics. In fact, since the foundation of the differential calculus the generalization of the concept of derivative and integral to a non-integer order has been the subject of distinct approaches. Due to this reason there are several definitions [6-8] which are proved to be equivalent.

The Laplace definition for a derivative of order $\alpha \in C$ is a direct generalization of the classical integer-order scheme with the multiplication of the signal transform by the s operator yielding:

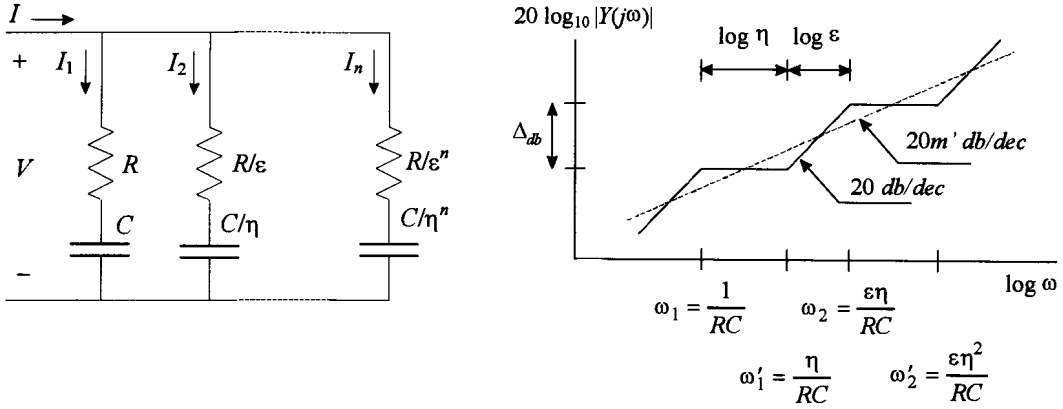
$$L\{D_{0+}^{\alpha}\varphi\} = s^{\alpha}L\{\varphi\},\ \mathrm{Re}(\alpha) \geq 0 \quad (1)$$

This means that frequency-based analysis methods have a straightforward adaptation to FC. The practical implementation of (1) requires an infinite number of poles and zeros obeying a recursive relationship [9]. Nevertheless, in a real approximation the finite number of poles and zeros yields a ripple in the frequency response and a limited bandwidth.

In order to analyze the frequency-based approach to (1) let us consider the recursive circuit represented on Fig. 1a) such that:

$$I = \sum_{i=1}^{n} I_i\ ,\ R_{i+1} = \frac{R_i}{\varepsilon}\ ,\ C_{i+1} = \frac{C_i}{\eta} \quad (2)$$

where η and ε are scale factors, I is the current due to an applied voltage V and R_i and C_i are the resistance and capacitance elements of the i^{th} branch of the circuit.



a) b)

Fig. 1. a) Electrical circuit with a recursive association of resistance and capacitance elements, b) Bode diagram of amplitude of $Y(j\omega)$.

The admittance $Y(j\omega)$ is given by:

$$Y(j\omega) = \frac{I(j\omega)}{V(j\omega)} = \sum_{i=0}^{n} \frac{j\omega C\varepsilon^{i}}{j\omega CR + (\eta\varepsilon)^{i}} \quad (3)$$

Figure 1b) shows the asymptotic Bode diagrams of amplitude of $Y(j\omega)$.

The pole and zero frequencies (ω_i and ω_i') obey the recursive relationships:

$$\frac{\omega_{i+1}'}{\omega_i'} = \frac{\omega_{i+1}}{\omega_i} = \varepsilon\eta\ ,\ \frac{\omega_i}{\omega_i'} = \varepsilon\ ,\ \frac{\omega_{i+1}'}{\omega_i} = \eta \quad (4)$$

From the Bode diagram of amplitude, or of phase, the average slope m' can be calculated as:

$$m' = \frac{\log\varepsilon}{\log\varepsilon + \log\eta} \quad (5)$$

Consequently, the circuit of Fig. 1a) represents an approach to D^{α}, $0 < \alpha < 1$, with $m' = \alpha$, based on a recursive pole/zero placement in the frequency domain. In fact, this method constitutes the so called

CRONE: Commande Robuste d'Ordre Non Entier, for implementations approximations of fractional order derivatives and integrals.

3. Evaluating Classical Expressions for the Static Electric Potential

It is well known that, for a homogeneous, linear and isotropic media, the electric potential φ at a point P by a single charge, a dipole and a quadrupole are [10, 11]:

$$\varphi = \frac{q}{4\pi\varepsilon_0}\frac{1}{r} + C, \quad \varphi = \frac{ql\cos\theta}{4\pi\varepsilon_0}\frac{1}{r^2} + C, r >> l, \quad \varphi = \frac{ql^2\left(3\cos^2\theta - 1\right)}{4\pi\varepsilon_0}\frac{1}{r^3} + C, r >> l \quad (6)$$

where ε_0 represents the permittivity, q the electric charge, r the radial distance and θ the corresponding angle with the axis.

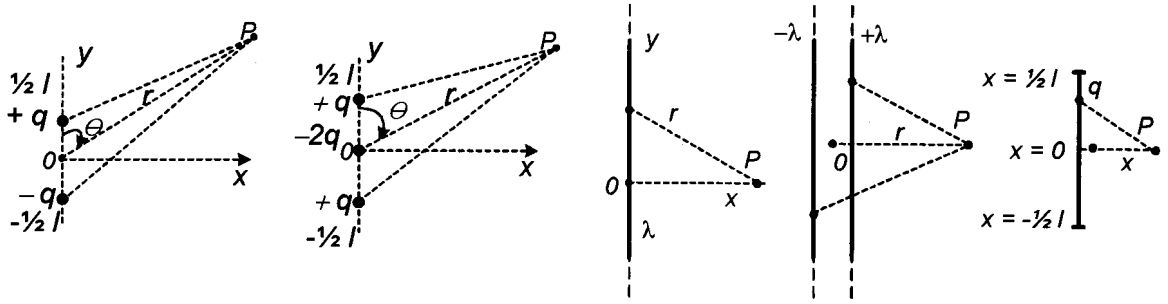
The electric potential φ at a point P (Fig. 2) for one very long straight filament carrying a charge λ per unit length, or for two opposite charged filaments are, respectively:

$$\varphi = -\frac{\lambda}{2\pi\varepsilon_0}\ln r + C, C \in \Re, \quad \varphi = \frac{\lambda l\cos\theta}{2\pi\varepsilon_0}\frac{1}{r} + C, r >> l \quad (7)$$

On the other hand, the potential resulting from a planar surface with charge density σ is given by:

$$\varphi = -\frac{\sigma}{2\varepsilon_0} r + C, C \in \Re \quad (8)$$

Analyzing expressions (6)-(8) we verify the relationship $\varphi \sim r^{-3}, r^{-2}, r^{-1}, \ln r, r$ that corresponds to the application of integer-order derivatives and integrals.



a) b) c) d) e)

Fig. 2. Electric potential of a) dipole b) quadrupole c) infinite line charge d) two opposite charged infinite filaments e) straight filament with finite length l and charge q

4. On the Implementation of Fractional Order Potential

The integer-order differential nature of the potential expressions motivated several authors [5] to propose its generalization in a FC perspective. Therefore, a fractional multipole produces at point P a potential $\varphi \sim r^{\alpha}$, $\alpha \in \Re$. Nevertheless, besides the abstract manipulation of mathematical expressions,

the truth is that there is no practical method, and physic interpretation, for establishing the fractional potential.

Inspired by the integer-order recursive approximation of fractional-order transfer functions presented previously, in this section we develop a numerical method for implementing a fractional order potential.

We start by re-evaluating the potential produced at point $P \equiv (x,y)$ by a straight filament with finite length l and charge q (Fig. 2 e):

$$\varphi = \frac{1}{4\pi\varepsilon_0}\frac{q}{l}\ln\left\{\left[y+\frac{1}{2}l+\sqrt{x^2+\left(y+\frac{1}{2}l\right)^2}\right]\Big/\left[y-\frac{1}{2}l+\sqrt{x^2+\left(y-\frac{1}{2}l\right)^2}\right]\right\}+C, C\in\Re \quad (9)$$

It is well-known that for $x\to\infty$ we have $\varphi\to\frac{q}{4\pi\varepsilon_0}\frac{1}{x}+C$ and, with $y = 0$, for $x\to 0$ we have $\varphi\to\frac{1}{2\pi\varepsilon_0}\frac{q}{l}\ln\left(\frac{1}{x}\right)+C$. Obviously these limit cases correspond to (6) and (7) respectively, that is, to a single charge and to an infinite filament.

Expression (9) changes smoothly between the two limit cases. Therefore, we can have an intermediate fractional-order relationship as long as we restrict to a limited working range. For example, for $0.1 < x < 0.3$ and $0.2 < x < 0.8$ we get the approximations $\varphi \approx 1.385x^{-0.532}$ and $\varphi \approx 1.031x^{-0.747}$, respectively, with squared coefficient of determination $R^2 = 0.996$ in both cases.

This means that standard integer-order potential relationships have a *global* nature while fractional-order potentials have a *local* nature possible to capture only in a restricted region. This conclusion leads to an implementation approach conceptually similar to the one described in section 2 that is, to an approximation scheme based on a recursive placement of integer-order functions. Nevertheless, in the present case we do not have the analytical formalism of Bode diagrams and, therefore, we decided to adopt a numerical approach.

In this line of thought, we developed a one-dimensional numerical algorithm that places recursively n charges q_i ($i = 0, 1, \ldots, n-1$) at the symmetrical positions $\pm x_i$ (with exception of $x_0 = 0$ that corresponds to the centre of the n-array of charges where there is a single charge q_0) and compares the resulting approximate potential φ_{app} with the desired reference potential φ_{ref} :

$$\varphi_{app} = \frac{q_0}{x}+\sum_{i=1}^{n-1}\frac{q_i}{4\pi\varepsilon_0}\left(\frac{1}{|x-x_i|}+\frac{1}{x+x_i}\right),\quad \varphi_{ref} = kx^{\alpha} \quad (10)$$

The optimization criteria minimizes the square error J yielding:

$$J = \sum_{k=1}^{m}\left(\ln\left|\frac{\varphi_{app}}{\varphi_{ref}}\right|\right)^2,\quad \min_i(J),\quad i = 0,1,\ldots,n-1 \quad (11)$$

where m is the number of sampling points along the x-axis.

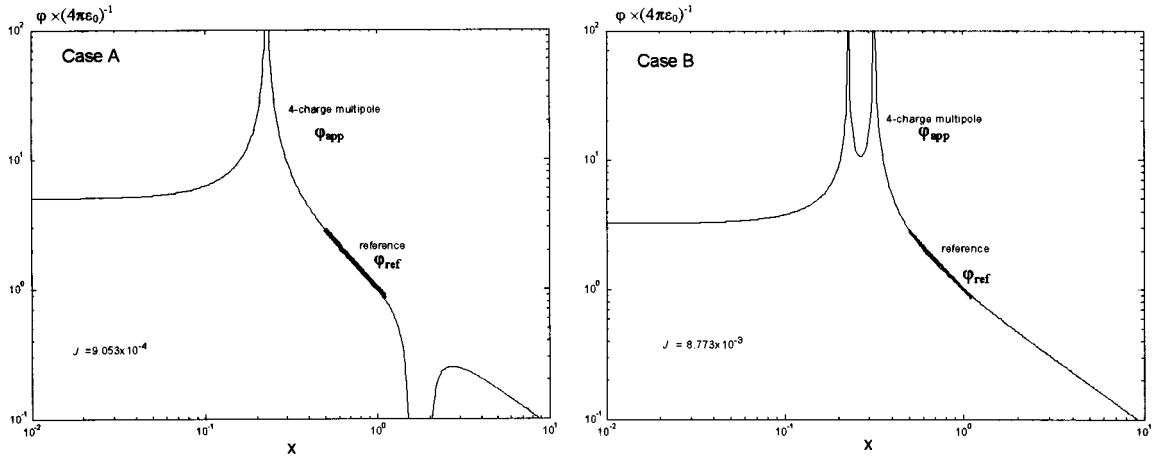


Fig. 3. Comparison of the electric potential φ_{app} and φ_{ref} *versus* x for $\varphi_{ref} = 1.0\, x^{-1.5}$, $0.5 < x < 1.1$ and a 4-charge approximation.

In the present case we consider a log-log perspective, similar to the one used in Bode diagrams, but its modification for a lin-lin case is straightforward. Moreover, in order to reduce the computational load, for a region $x_A < x < x_B$ we developed a two phase scheme for capturing the optimal values, namely (*i*) a first phase with a large sampling step $\Delta x = x_A\, r_1^{\,k}$ $(k = 0, 1, ...,)$ and (*ii*) a second phase with a smaller step $\Delta x = x'_A\, r_2^{\,k}$ $(k = 0, 1, ...)$ within the previously captured region $x'_A < x < x'_B$ for evaluating the optimal values with a larger precision. For example, Fig. 3 shows a 4-charge approximation for $\varphi_{ref} = 1.0\, x^{-1.5}$, $0.5 < x < 1.1$, leading to case A: $q_{1A} = -0.156$ and $q_{2A} = +0.598$ (with scale factor $\times(4\pi\varepsilon_0)^{-1}$), at $x_{1A} = \pm 1.729$ and $x_{2A} = \pm 0.2296$, respectively. The results show a good fit between the two functions. Nevertheless, for a given application, a superior precision may be required and, in that case, a larger number of charges must be used. In this line of thought, we study the precision of this method for different number of charges, namely for $n = 1$ up to $n = 7$ charges.

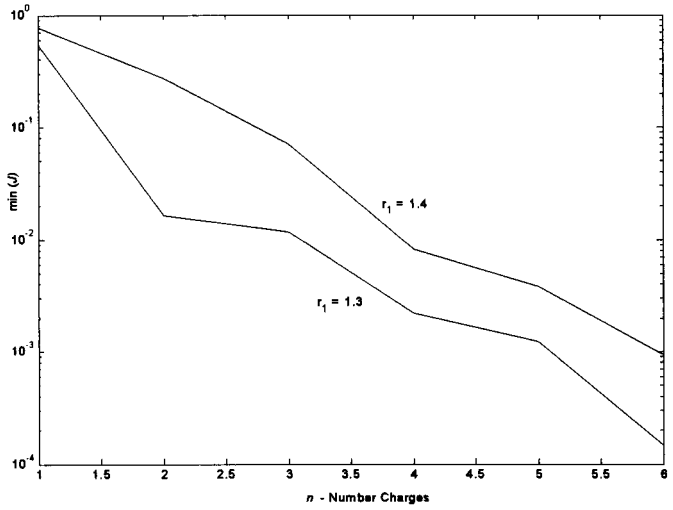


Fig. 4. Approximation error min (J) *vs.* number charges n for $r_1 = 1.3$ ($r_2 = 1.03$) and $r_1 = 1.4$ ($r_2 = 1.04$), $\varphi_{ref} = 1.0\, x^{-1.5}$, and $0.5 < x < 1.1$.

Figure 4 depicts min(J) *versus* n, for $r_1 = 0.3$ and $r_1 = 0.4$, and confirms that we have a better precision the larger the number of charges and the smaller the r_1. This chart can be approximated closely by the following expressions $\min(J) \approx 0.9167e^{-1.4416n}$ and $\min(J) \approx 3.4875e^{-1.3852n}$, respectively. We verify that the position of the charges varies significantly with the precision of the algorithm, namely with the increment r_1 of the numerical grid. Therefore, the pattern revealed by the charge is not clear and its comparison with a fractal recursive layout is still under investigation.

The experiments also reveal that it is possible to find more than one 'good' solution. For example, Fig. 3 shows a 4–charge approximation for $\varphi_{ref} = 1.0\,x^{-1.5}$, $0.5 < x < 1.1$, leading to case B: $q_{1B} = +0.1556$ and $q_{2B} = 0.3049$ (with scale factor $\times(4\pi\varepsilon_0)^{-1}$), at $x_{1B} = \pm 0.2296$ and $x_{2B} = \pm 0.3214$, respectively.

5. Conclusions

This paper addressed the problem of implementing a fractional-order electric potential. It was adopted an algorithm inspired on the Bode diagram recursive scheme. While in the Bode diagrams both numerical and analytical approaches are possible, in the present case only a numerical evaluation was implemented and the analytical counterpart remains to be investigated. In fact, this paper constitutes a first step towards the development of a systematic design technique and, consequently, several other aspects must be evaluated. Research on the approximation feasibility and convergence, error variation with the range and the number of charges, improvement when adopting an extended library of primitives rather than, merely, point charges and its extension to the three-dimensional space is presently under development.

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