

A new insight into complexity from the local fractional calculus view point: modelling growths of populations

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In this paper, we model the growths of populations by means of local fractional calculus. We formulate the local fractional rate equation and the local fractional logistic equation. The exact solutions of local fractional rate equation and local fractional logistic equation with the Mittag-Leffler function defined on Cantor sets are presented. The obtained results illustrate the accuracy and efficiency for modeling the complexity of linear and nonlinear population dynamics (PD). Copyright © 2015 John Wiley & Sons, Ltd.

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1. Introduction

The rate equation (RE) of the population dynamics (PD), structured by T. R. Malthus in 1798 [1, 2], is characterized by the ordinary differential equation (ODE):

$$\frac{dM(\tau)}{d\tau} = \kappa M(\tau), \quad (1)$$

where $M(\tau)$ represents the number of elements of the population in time τ , and κ ($\kappa \neq 0$) is the Malthus constant. With the initial population $M(0)$, the solution for the RE in time is $M(\tau) = M(0) e^{\kappa\tau}$, which expresses extinction or growth of the Malthus population.

The logistic equation (LE) of the PD, developed by P. R. Verhulst in 1838 [3, 4], is as follows:

$$\frac{dA(\tau)}{d\tau} = \kappa A(\tau) \left(1 - \frac{A(\tau)}{\gamma}\right), \quad (2)$$

where $A(\tau) = M(\tau)/M_{\max}$, for the population $M(\tau)$ and the value of the maximum attainable population M_{\max} , κ is the intrinsic growth parameter, and γ is the environmental carrying capacity.

The exact solution of the LE can be written in the form [4]:

$$A(\tau) = \frac{\gamma}{1 + (\gamma/A(0) - 1) e^{-\kappa\tau}}, \quad (3)$$

where $A(0) = M(0)/M_{\max}$, and $\gamma = \lim_{\tau \rightarrow \infty} A(\tau)$.

At the limit of environmental carrying capacity $\gamma = 1$, the LE becomes [5]

$$\frac{dA(\tau)}{d\tau} = \kappa A(\tau) (1 - A(\tau)), \quad (4)$$

and its exact solution is [5]

$$A(\tau) = \frac{A(0)}{A(0) + (1 - A(0))e^{-\kappa\tau}}, \quad (5)$$

where $A(0) = M(0)/M_{\max}$.

Fractional calculus (FC) was successfully used to describe the dynamics systems in engineering and applied science [6–12]. Recently, the applications of FC to the population growth models [5, 13–17] were proposed. The fractional-order RE and LE for fractional population dynamics in the sense of Caputo fractional derivative (FD) were determined by the expressions [5]

$$\frac{d^\eta M(\tau)}{d\tau^\eta} = \kappa M(\tau) \quad (0 < \eta < 1) \quad (6)$$

and

$$\frac{d^\eta A(\tau)}{d\tau^\eta} = \kappa A(\tau) (1 - A(\tau)) \quad (0 < \eta < 1), \quad (7)$$

respectively, where $M(\tau)$ is the number of the element of the fractional population in time τ , $A(\tau) = M(\tau)/M_{\max}$, κ ($\kappa \neq 0$) is the west constant of fractional population, and η is order of FD.

The solutions of the differentiable types is as follows [5]:

$$M(\tau) = M(0) E_\eta(\kappa\tau^\eta), \quad (8)$$

and

$$A(\tau) = \sum_{i=0}^{\infty} \left(\frac{A(0) - 1}{A(0)} \right)^i E_\eta(-n\kappa\tau^\eta), \quad (9)$$

where the Mittag-Leffler function (MLF) is defined by using the series

$$E_\eta(\xi) = \sum_{i=0}^{\infty} \frac{\xi^i}{\Gamma(1 + i\eta)}. \quad (10)$$

More recently, local fractional calculus (LFC) was presented to explain the complexity of dynamics systems in the fields of mathematical physics, such as the oscillator of free damped vibrations [18], heat transfer [19], diffusion on Cantor sets [20], Navier-Stokes flow [21], Laplace equation (LE) [22], signal processing [23], and others [24–28]. The purpose of this article is to present for modelling complexity of the growth of populations using LFC.

The structure of this article is designed as follows. In Section 2, the recent results for the local fractional derivative (LFD) are presented. In Section 3, the local fractional RE (LFRE) for the PD and its solution of non-differentiable type are discussed. In Section 4, the local fractional LE (LFLE) for the PD and its solution are proposed. In Section 5, the discussions for LFRE and LFLE are given. Finally, the conclusions are outlined in Section 6.

2. Preliminaries

In this section, the concept and properties of LFD and local fractional Laplace transform (LFLT), which is applied in this article, are presented.

Let $C_\varsigma(a, b)$ be a set of the non-differentiable functions (NDFs). The LFD of $\wp(\sigma)$ of order ς ($0 < \varsigma < 1$) at the point $\sigma = \sigma_0$ is defined as follows ([18–28]):

$$D^{(\varsigma)} \wp(\sigma_0) = \frac{d^\varsigma \wp(\sigma_0)}{d\sigma^\varsigma} = \lim_{\sigma \rightarrow \sigma_0} \frac{\Delta^\varsigma(\wp(\sigma) - \wp(\sigma_0))}{(\sigma - \sigma_0)^\varsigma}, \quad (11)$$

where

$$\Delta^\varsigma(\wp(\sigma) - \wp(\sigma_0)) \cong \Gamma(1 + \varsigma) [\wp(\sigma) - \wp(\sigma_0)], \quad (12)$$

with $\wp(\sigma_0) \in C_\varsigma(a, b)$.

The properties of the LFD are listed as follows [24, 25]:

1. $D^{(\varsigma)} [\Phi(\sigma) \pm \Theta(\sigma)] = D^{(\varsigma)} \Phi(\sigma) \pm D^{(\varsigma)} \Theta(\sigma)$, proved $\Phi(\sigma), \Theta(\sigma) \in C_\varsigma(a, b)$;
2. $D^{(\varsigma)} [\Phi(\sigma) / \Theta(\sigma)] = \{ [D^{(\varsigma)} \Phi(\sigma)] \Theta(\sigma) - \Phi(\sigma) [D^{(\varsigma)} \Theta(\sigma)] \} / \Theta^2(\sigma)$, provided $\Phi(\sigma), \Theta(\sigma) \in C_\varsigma(a, b)$ and $\Theta(\sigma) \neq 0$.

The Mittag-Leffler function defined on Cantor sets (MLFCS) with the fractal dimension ς is defined as follows [18,24,25]:

$$E_\varsigma(\sigma^\varsigma) = \sum_{i=0}^{\infty} \frac{\sigma^{i\varsigma}}{\Gamma(1 + i\varsigma)}. \quad (13)$$

The formulas of the LFD of NDFs [18, 24, 25] are listed in Table I.

Table I. The formulas of the LFD of the NDFs.	
NDFs	LFDs
$E_{\varsigma}(\sigma^{\varsigma})$ $\sigma^{\varsigma} / \Gamma(1 + \varsigma)$	$E_{\varsigma}(\sigma^{\varsigma})$ 1

LFD, local fractional derivative; NDF, non-differentiable function.

Table II. The operations of LFLT of the NDFs.	
$\wp(\sigma)$	$\wp_{\varsigma}(s)$
$E_{\varsigma}(a\sigma^{\varsigma})$ $\sigma^{\varsigma} / \Gamma(1 + \varsigma)$	$1 / (s^{\varsigma} - a)$ $1 / s^{2\varsigma}$

LFLT, local fractional Laplace transform; NDF, non-differentiable functions.

The LFLT of $\wp(\sigma)$, denoted by [24]

$$\Im_{\varsigma}[\wp(\sigma)] = \wp_{\varsigma}(s), \quad (14)$$

is applied to the LFD of the NDFs, namely,

$$\Im_{\varsigma}[\wp^{(\varsigma)}(\sigma)] = s^{\varsigma} \wp_{\varsigma}(s) - \wp(0). \quad (15)$$

The LFLT of the NDFs (see, for example, [24]) are shown in Table II.

3. The growth rate for population within local fractional derivative

The conventional and fractional-order REs are differentiable for the linear PD by using the ODEs.

The LFRE of the non-differentiable PD is characterized by using the local fractional ODE:

$$\frac{d^{\varsigma} M(\tau)}{d\tau^{\varsigma}} = \kappa M(\tau) \quad (0 < \varsigma < 1), \quad (16)$$

where $M(\tau)$ is the number of the fractal population in time τ , κ is the parameter of the fractal population, and ς is the fractal dimension.

Using the LFLT of Eq.(16), we have the following:

$$s^{\varsigma} M_{\varsigma}(s) - M(0) = \kappa M_{\varsigma}(s), \quad (17)$$

which yields

$$M_{\varsigma}(s) = \frac{M(0)}{s^{\varsigma} - \kappa}. \quad (18)$$

Thus, using the inverse LFLT, the non-differentiable solution for the LFRE is written in closed form:

$$M(\tau) = M(0) E_{\varsigma}(\kappa \tau^{\varsigma}). \quad (19)$$

4. The growth for population within local fractional derivative

Similarly to the Section 3, conventional and fractional-order LFLEs are differentiable for the nonlinear PD by using the nonlinear ODEs.

The LFLE of the fractal PD is formulated as follows:

$$\frac{d^{\varsigma} A(\tau)}{d\tau^{\varsigma}} = \kappa A(\tau) (1 - A(\tau)) \quad (0 < \varsigma < 1), \quad (20)$$

where $A(\tau) = M(\tau) / M_{\max}$ and κ is the intrinsic growth parameter for the fractal population.

The LFLE of the fractal PD may be determined by the following:

$$\frac{d^{\varsigma} A(\tau)}{d\tau^{\varsigma}} = \kappa A(\tau) \left(1 - \frac{A(\tau)}{\gamma}\right), \quad (21)$$

where $A(\tau) = M(\tau) / M_{\max}$, κ is the intrinsic growth parameter for the fractal population, and γ is the environmental carrying capacity, which satisfies with $\gamma = \lim_{\tau \rightarrow \infty} A(\tau)$.

With the help of Eq.(5), we can formulate a general solution in the form

$$A_*(\tau) = \frac{\psi_A}{\psi_B + \psi_C E_{\zeta}(-\kappa \tau^{\zeta})}, \quad (22)$$

where ψ_A , ψ_B , and ψ_C are the parameters for the fractal population dynamics.

Using the properties of the LFD (b), we find that

$$\begin{aligned} & \frac{d^{\zeta} A_*(\tau)}{d\tau^{\zeta}} \\ &= \frac{d^{\zeta}}{d\tau^{\zeta}} \left(\frac{\psi_A}{\psi_B + \psi_C E_{\zeta}(-\kappa \tau^{\zeta})} \right) \\ &= \frac{\psi_C \kappa E_{\zeta}(-\kappa \tau^{\zeta})}{[\psi_B + \psi_C E_{\zeta}(-\kappa \tau^{\zeta})]^2} \\ &= -\frac{\psi_A \psi_B \kappa E_{\zeta}(-\kappa \tau^{\zeta})}{[\psi_B + \psi_C E_{\zeta}(-\kappa \tau^{\zeta})]^2} + \frac{\psi_A \kappa E_{\zeta}(-\kappa \tau^{\zeta})}{\psi_B + \psi_C E_{\zeta}(-\kappa \tau^{\zeta})}, \end{aligned} \quad (23)$$

which leads to the following:

$$\frac{d^{\zeta} A_*(\tau)}{d\tau^{\zeta}} = \kappa A_*(\tau) \left(1 - \frac{\psi_B}{\psi_A} A_*(\tau) \right), \quad (24)$$

where

$$-\frac{\kappa \psi_B}{\psi_A} A_*^2(\tau) = -\frac{\psi_A \psi_B \kappa E_{\zeta}(-\kappa \tau^{\zeta})}{[\psi_B + \psi_C E_{\zeta}(-\kappa \tau^{\zeta})]^2}, \quad (25)$$

and

$$\kappa A_*(\tau) = \frac{\psi_A \kappa E_{\zeta}(-\kappa \tau^{\zeta})}{\psi_B + \psi_C E_{\zeta}(-\kappa \tau^{\zeta})}. \quad (26)$$

Making use of $\psi_A/\psi_B = 1$ and $A(0) = \psi_A/(\psi_B + \psi_C)$, the non-differentiable solution for Eq.(20) may be written in the form:

$$A(\tau) = \frac{A(0)}{A(0) + (1 - A(0)) E_{\zeta}(-\kappa \tau^{\zeta})}, \quad (27)$$

where $A(0) = M(0)/M_{\max}$.

In view of the expressions $\psi_A/\psi_B = \gamma$ and $A(0) = \psi_A/(\psi_B + \psi_C)$, the exact solution for Eq.(21) is of non-differentiable type:

$$A(\tau) = \frac{\gamma}{1 + (\gamma/A(0) - 1) E_{\zeta}(-\kappa \tau^{\zeta})}, \quad (28)$$

where $A(0) = M(0)/M_{\max}$.

5. Discussion

Using the limit of the MLFCS

$$\lim_{\zeta \rightarrow 1} E_{\zeta}(\kappa \tau^{\zeta}) = e^{-\kappa \tau}, \quad (29)$$

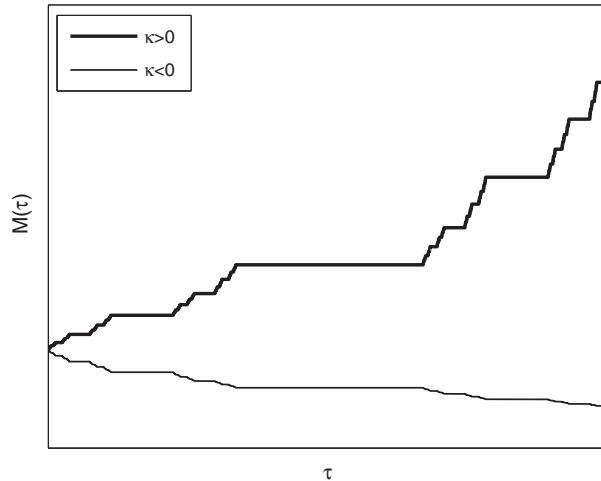


Figure 1. The non-differentiable solution for the local fractional rate equation when $\zeta = \ln 2/\ln 3$.

the solution of the RE for the PD is obtained [1,2]. We remark the non-differentiable solution for the LFRE as follows. Figure 1 shows that the solution of the extremely large fractal population, when $\kappa < 0$, is the extinction of Mittag-Leffler function defined on Cantor sets, and that the solution of the extremely small fractal population, when $\kappa > 0$, is the growth of the Mittag-Leffler function defined on Cantor sets. The RE within conventional derivative (CDRE) [1,2] presents the population in time; The fractional-order RE within fractional derivative (FDRE) [5] depicts the fractional population in time; The LFRE in our manuscript displays the fractal population in time. The CDRE, FDRE, and LFRE and their solutions (TSs) are listed in Table III. The solutions for the CDRE, FDRE, and LFRE, when $\kappa > 0$ and $M(0) = 1$, are represented in Figure 2.

Similarly, using Eq.(29), Eq.(27), and Eq.(28) are rewritten as Eq.(5) and Eq.(3), respectively. The LE within conventional derivative (CDLE) [1, 2] depicts the PD in time; The fractional-order LE within fractional derivative (FDLE) [5] exhibits the fractional PD in time; The LFLE in our manuscript shows the fractal PD in time. The exact solutions of the LFLE for nonlinear PD in different initial-values are illustrated in Figure 3.

The CDLE, FDLE, and LFLE and TSs are listed in Table IV.

Table III. REs of different types.	
REs	TSs
$\frac{dM(\tau)}{d\tau} = \kappa M(\tau)$	$M(\tau) = M(0) e^{\kappa \tau}$
$\frac{d^\eta M(\tau)}{d\tau^\eta} = \kappa M(\tau)$	$M(\tau) = M(0) E_\eta(\kappa \tau^\eta)$
$\frac{d^\zeta M(\tau)}{d\tau^\zeta} = \kappa M(\tau)$	$M(\tau) = M(0) E_\zeta(\kappa \tau^\zeta)$

RE, rate equation.

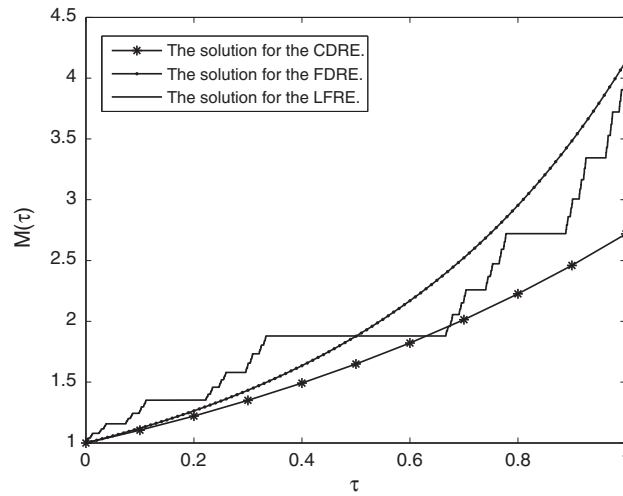


Figure 2. The solutions for the RE within conventional derivative, RE within fractional derivative, and local fractional rate equation when $\kappa > 0$ and $M(0) = 1$.

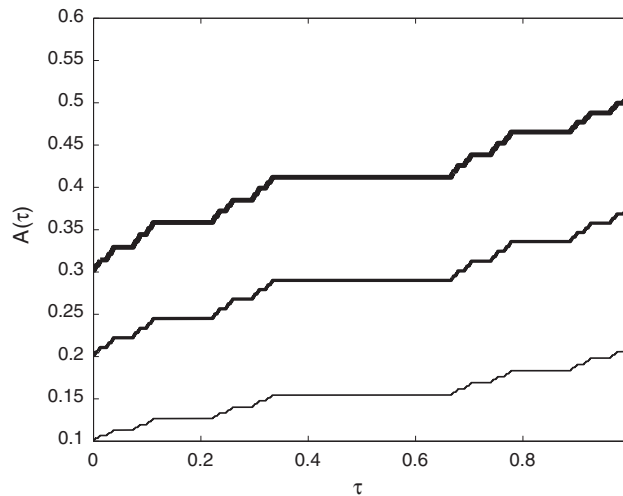


Figure 3. The closed solution of local fractional logistic equation when $\zeta = \ln 2 / \ln 3$.

Table IV. LEs of different types.	
LEs	TSs
$\frac{dA(\tau)}{d\tau} = \kappa A(\tau)(1 - A(\tau))$	$A(\tau) = \frac{A(0)}{A(0) + (1 - A(0))e^{-\kappa\tau}}$
$\frac{d^\eta A(\tau)}{d\tau^\eta} = \kappa A(\tau)(1 - A(\tau))$	$A(\tau) = \sum_{i=0}^{\infty} \left(\frac{A(0)-1}{A(0)} \right)^i E_\eta(-n\kappa\tau^\eta)$
$\frac{d^\varsigma A(\tau)}{d\tau^\varsigma} = \kappa A(\tau)(1 - A(\tau))$	$A(\tau) = \frac{A(0)}{A(0) + (1 - A(0))E_\varsigma(-\kappa\tau^\varsigma)}$

LE, logistic equation.

6. Conclusions

This work presented a new application of LFC to model the complexity of linear and nonlinear PD. The LFRE and LFLE of the growth in human population and their closed solutions with non-differentiable graphs were used to explain the PD. Comparative results of two models with CD, FD, and LFD were discussed. It is shown that LFC is a relevant tool to analyze the PD.

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